

Existence of solutions for second-order integral boundary value problems*

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Abstract. In this paper, using a new comparison result and monotone iterative method, we consider the existence of solution of integral boundary value problem for second-order differential equation. To obtain corresponding results, we also discuss second-order differential inequalities. The interesting point is that the one-sided Lipschitz constant is related to the first eigenvalues corresponding to the relevant operators.

Keywords: integral boundary value problem, monotone iterative method, Fredholm theorem.

1 Introduction

We will devote to considering the existence of solution of the following integral boundary value problem for second-order differential equation, using the method of upper and lower solutions and its associated monotone iterative technique

$$\begin{aligned} -x''(t) &= f(t, x(t)), \quad t \in (0, 1), \\ x(0) &= \int_0^1 x(s) \, dA(s), \quad x(1) = \int_0^1 x(s) \, dB(s), \end{aligned} \quad (1)$$

where $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$; A and B are right continuous on $[0, 1)$, left continuous at $t = 1$; and nondecreasing on $[0, 1]$, with $A(0) = B(0) = 0$, $\int_0^1 u(s) \, dA(s)$ and $\int_0^1 u(s) \, dB(s)$ denote the Riemann–Stieltjes integrals of u with respect to A and B , respectively.

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The theory of integral boundary value problems for differential equations is an important and significant branch of nonlinear analysis [1, 6, 10–12, 18–20, 22, 23, 25–28, 30–32]. It is worth mentioning that integral boundary value problems for differential equations appear often in investigations connected with applied mathematics and physics such as heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics [8, 9, 21]. One of the basic problems considered in the theory of integral boundary value problems for differential equations is to establish convenient conditions guaranteeing the existence of solutions of those equations.

To obtain existence results for differential equations, someone used the monotone iterative method [2, 5, 14]. There is a vast literature devoted to the applications of this method to differential equations with different boundary conditions, for details, see [4, 7, 15, 16, 24, 29]. In [3], Alberto Cabada and Susana Lois successfully investigated different maximum and anti-maximum principles for the operator $L[M]u = -u'' + Mu$ with separated boundary conditions. Motivated by [3], in this paper, we first present a new comparison theorem for the operator $-u'' - \lambda u$ with integral boundary value condition, and then, by using the monotone iterative technique, we investigate the extremal solutions of (1). We should note that the constant λ is related to the first eigenvalues corresponding to the relevant operators.

Throughout this paper, we always suppose that

(H1) $\kappa_1 > 0, \kappa_4 > 0, \kappa = \kappa_1\kappa_4 - \kappa_2\kappa_3 > 0$, where

$$\begin{aligned}\kappa_1 &= 1 - \int_0^1 (1-t) dA(t), & \kappa_2 &= \int_0^1 t dA(t), \\ \kappa_3 &= \int_0^1 (1-t) dB(t), & \kappa_4 &= 1 - \int_0^1 t dB(t).\end{aligned}$$

2 Preliminaries and lemmas

Let X be the Banach space $C[0, 1]$ with $\|x\| = \sup_{t \in [0, 1]} |x(t)|$. Define a set $P \subset X$ by

$$P = \{x \in X: x(t) \geq 0, t \in [0, 1]\}.$$

It can be easily verified that P is indeed a cone in X .

For $\sigma \in X$ and $\mu_1, \mu_2, \lambda \in \mathbb{R}$. Consider the following linear integral boundary value problems

$$\begin{aligned}-x''(t) &= \lambda x(t) + \sigma(t), & t &\in (0, 1), \\ x(0) &= \int_0^1 x(s) dA(s) + \mu_1, & x(1) &= \int_0^1 x(s) dB(s) + \mu_2.\end{aligned}\tag{2}$$

To study (2), consider the operator $T : X \rightarrow X$ defined by

$$(Tx)(t) = \int_0^1 k(t, s)x(s) \, ds + \kappa^{-1}(1-t, t) \begin{pmatrix} \kappa_4 & \kappa_2 \\ \kappa_3 & \kappa_1 \end{pmatrix} \begin{pmatrix} \int_0^1 dA(t) \int_0^1 k(t, s)x(s) \, ds \\ \int_0^1 dB(t) \int_0^1 k(t, s)x(s) \, ds \end{pmatrix}$$

and the function

$$\rho(t) = \kappa^{-1}(1-t, t) \begin{pmatrix} \kappa_4 & \kappa_2 \\ \kappa_3 & \kappa_1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix},$$

where $k(t, s)$ is given by

$$k(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Then if (H1) holds, by [27, 28], $x \in C^2[0, 1]$ is a solution of (2) if and only if $x \in X$ is a solution of the equation

$$(I - \lambda T)x = T\sigma + \rho. \quad (3)$$

For the function $k(t, s)$, it is easy to know that

$$t(1-t)s(1-s) \leq k(t, s) \leq s(1-s), \quad t, s \in [0, 1]. \quad (4)$$

Take

$$e(t) = \begin{cases} t(1-t), & A(t) = B(t) \equiv 0, \\ (1-t), & A(t) \neq 0, B(t) \equiv 0, \\ t, & A(t) \equiv 0, B(t) \neq 0, \\ 1, & A(t) \neq 0, B(t) \neq 0. \end{cases}$$

For sake of simplicity, we only prove the following Lemma 2 in the case that $A(t) \equiv 0$ and $B(t) \neq 0$ hold. Similar arguments applies when the other condition hold with cones $K_1 = \{x \in P : x(t) \geq t(1-t)\|x\|, t \in [0, 1]\}$, $K_2 = \{x \in P : x(t) \geq \gamma_2(1-t) \times \|x\|, t \in [0, 1]\}$, $K_3 = \{x \in P : x(t) \geq \gamma_3\|x\|, t \in [0, 1]\}$, respectively, where

$$\gamma_2 = \frac{\int_0^1 t(1-t) \, dA(t)}{\kappa_1 + \int_0^1 dA(t)}, \quad 0 < \gamma_3 = \frac{\nu}{\rho} < 1,$$

$$\rho = 1 + \frac{1}{\kappa} \left((\kappa_4 + \kappa_3) \int_0^1 dA(t) + (\kappa_2 + \kappa_1) \int_0^1 dB(t) \right),$$

$$\nu = \min_{t \in [0, 1]} \frac{1}{\kappa} \left(((1-t)\kappa_4 + t\kappa_3) \int_0^1 t(1-t) \, dA(t) + ((1-t)\kappa_2 + t\kappa_1) \int_0^1 t(1-t) \, dB(t) \right).$$

Now, the operator T can be simplified as

$$(Tx)(t) = \int_0^1 k(t, s)x(s) \, ds + \frac{t}{\kappa_4} \int_0^1 dB(t) \int_0^1 k(t, s)x(s) \, ds. \quad (5)$$

Define a set $K \subset X$ by

$$K = \{x \in P: x(t) \geq \gamma t \|x\|, t \in [0, 1]\},$$

where $\gamma = \int_0^1 t(1-t) \, dB(t) / (\kappa_4 + \int_0^1 dB(t))$. It can be easily verified that K is indeed a cone in X and $K \subset P$.

Lemma 1. $T(P) \subset K$ and the map $T : K \rightarrow K$ is completely continuous.

Proof. Inequality (4) and the definition of T imply that $T(P) \subset K$. The completely continuity of the integral operator T is well known. This completes the proof. \square

Definition 1. (See [13].) Let e be a fixed nonzero element in the positive cone P of the Banach space X . The linear operator T is said to be increasing if $T(P) \subset P$. The linear operator T is said to be e -bounded if, for every nonzero $x \in P$, a natural number n and two positive numbers α, β can be found such that $\alpha e \leq T^n x \leq \beta e$.

Lemma 2. The operator T defined by (5) is a e -bounded operator, in which e is given by $e(t) = t$.

Proof. For all $x \in P \setminus \{\theta\}$, take $\alpha(x) = (1/\kappa_4) \int_0^1 dB(t) \int_0^1 k(t, s)x(s) \, ds$ and $\beta(x) = \int_0^1 x(s) \, ds + (1/\kappa_4) \int_0^1 dB(t) \int_0^1 k(t, s)x(s) \, ds$. It follows from (5) that

$$\alpha(x)t \leq (Tx)(t) \leq \beta(x)t, \quad t \in [0, 1].$$

So, T is a e -bounded operator. This completes the proof. \square

By Lemma 2 and Krein–Rutman theorem [13], we know that the operator T defined by (5), the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.

Remark 1. Let φ^* be the positive eigenfunction of T corresponding to λ_1 , thus, $\lambda_1 T\varphi^* = \varphi^*$. Then by Definition 1, there exist $\alpha(\varphi^*), \beta(\varphi^*) > 0$ such that

$$\alpha e(t) \leq (T\varphi^*)(t) = \frac{1}{\lambda_1} \varphi^*(t) \leq \beta e(t).$$

Hence, we obtained that T is φ^* -bounded operator.

We present a new comparison result, which is crucial for our discussion.

Lemma 3. *Supposed that $\lambda < \lambda_1$. If $u \in C^2[0, 1]$ satisfies*

$$\begin{aligned} -u''(t) &\geq \lambda u(t), \quad t \in (0, 1), \\ u(0) &\geq \int_0^1 u(s) dA(s), \quad u(1) \geq \int_0^1 u(s) dB(s), \end{aligned}$$

then $u(t) \geq 0, t \in [0, 1]$.

Proof. Set $\sigma(t) = -u''(t) - \lambda u(t)$, $\mu_1 = u(0) - \int_0^1 u(s) dA(s)$, $\mu_2 = u(1) - \int_0^1 u(s) dB(s)$. Then $\mu_1 \geq 0, \mu_2 \geq 0, \sigma(t) \geq 0$ and $\rho(t) \geq 0, t \in [0, 1]$. Moreover, by (3), Lemma 2 and Remark 1, there exists $\beta > 0$ such that

$$u(t) \geq \lambda(Tu)(t) \geq -|\lambda|(T|u|)(t) \geq -|\lambda|\beta\varphi^*(t). \quad (6)$$

Thus, there exists $\delta > 0$ such that

$$u(t) + \delta\varphi^*(t) \geq 0, \quad t \in [0, 1]. \quad (7)$$

When $\lambda = 0$, the result is obviously true by (6). So, in the rest of this proof, we assume that $\lambda \neq 0$. We prove the result from two cases:

Case 1. $\lambda > 0$. Let $T_1x = \lambda Tx, x \in X$. Then $T_1 : X \rightarrow X$ is a bounded linear operator and $T_1(P) \subset P$. Moreover, we have $r(T_1) = \lambda r(T) = \lambda/\lambda_1 < 1$. Therefore, the inverse operator $(I - T_1)^{-1}$ exists and

$$(I - T_1)^{-1} = I + T_1 + T_1^2 + \cdots + T_1^n + \cdots.$$

It follows from $T_1(P) \subset P$ that $(I - T_1)^{-1}(P) \subset P$. So, we have $u = (I - T_1)^{-1}(I - T_1)u \geq 0$ by (6).

Case 2. $\lambda < 0$. Suppose the contrary. Therefore, by (7), there exists a smallest positive number δ , says δ_0 , such that $u(t) + \delta_0\varphi^*(t) \geq 0, t \in [0, 1]$. Now, $\tilde{u}(t) := u(t) + \delta_0\varphi^*(t)$ satisfies

$$\begin{aligned} -\tilde{u}''(t) &\geq \lambda u + \lambda_1\delta_0\varphi^*(t) > \lambda u + \lambda\delta_0\varphi^*(t) = \lambda\tilde{u}(t), \quad t \in (0, 1), \\ \tilde{u}(0) &\geq \int_0^1 \tilde{u}(s) dA(s), \quad \tilde{u}(1) \geq \int_0^1 \tilde{u}(s) dB(s). \end{aligned}$$

We first show that $\tilde{u}(t) > 0, t \in (0, 1)$. In fact, if there exists $t_0 \in (0, 1)$ such that $\tilde{u}(t_0) = \min_{t \in [0, 1]} \tilde{u}(t) = 0$, we have $0 \geq -\tilde{u}''(t_0) > \lambda\tilde{u}(t_0) = 0$, which is a contradiction. Next, we prove that $\tilde{u}(t) \geq \delta_1 e(t)$ for some $\delta_1 > 0$ and $t \in [0, 1]$. Otherwise, we can find $\{t_n\}_{n=1}^\infty \subset [0, 1]$ such that $\tilde{u}(t_n) \leq e(t_n)/n$. So, we have $\lim_{n \rightarrow \infty} \tilde{u}(t_n) = 0$. Notice that $\tilde{u}(t) > 0, t \in (0, 1)$, we necessarily have, by passing to a subsequence if needed, $\lim_{n \rightarrow \infty} t_n = 0$ or/and $\lim_{n \rightarrow \infty} t_n = 1$. Considering the continuity of \tilde{u} , we

have $\tilde{u}(0) = 0$ or/and $\tilde{u}(1) = 0$, and hence $\tilde{u}'(0) > 0$ or/and $\tilde{u}'(1) < 0$ by Theorem 4 in [17, Chap. 1]. So, the inequality $\tilde{u}(t) \geq \delta_1 e(t)$ holds for some $\delta_1 > 0$ and $t \in [0, 1]$. By Remark 1 and (6), we can find $\tilde{\delta} > 0$ such that $\tilde{u}(t) \geq \tilde{\delta} \varphi^*(t)$ and hence $u(t) + \delta_0 \varphi^*(t) \geq -\tilde{\delta}/(|\lambda|\beta)u(t)$, that is, $u(t) + \delta_0(1 + \tilde{\delta}/(|\lambda|\beta))^{-1} \varphi^*(t) \geq 0$ contradicting with the minimality of δ_0 . Therefore, the case $\min_{t \in [0,1]} u(t) < 0$ does not occur. This completes the proof. \square

Lemma 4. For $\sigma \in X$ and $\lambda < \lambda_1$, the linear integral boundary value problems (2) has an unique solution in X .

Proof. To obtain the required results, we only need to prove that the operator equation

$$(I - \lambda T)x = T\sigma + \rho \quad (8)$$

has an unique fixed point in X . From Lemma 3 operator equation $(I - \lambda T)x = \theta$ has only a zero solution. Then by Lemma 1 and the Fredholm alternative theorem for linear compact operator, the operator equation (8) has an unique solution in X for any given $\sigma \in X$ and $\rho \in X$. This completes the proof. \square

3 Main results

In this section, on the basis of Lemma 3 and Lemma 4, using the monotone iterative technique, we shall show an existence theorem of extremal solutions of (1).

Definition 2. $u_0 \in X$ is called a lower solution of the differential equation (1) if

$$\begin{aligned} -u_0''(t) &\leq f(t, u_0(t)), \quad t \in (0, 1), \\ u_0(0) &\leq \int_0^1 u_0(t) dA(t), \quad u_0(1) \leq \int_0^1 u_0(t) dB(t). \end{aligned}$$

Analogously, $v_0 \in X$ is called a upper solution of the differential equation (1) if

$$\begin{aligned} -v_0''(t) &\geq f(t, v_0(t)), \quad t \in (0, 1), \\ v_0(0) &\geq \int_0^1 v_0(t) dA(t), \quad v_0(1) \geq \int_0^1 v_0(t) dB(t). \end{aligned}$$

In what follows, we assume that

$$u_0(t) \leq v_0(t), \quad t \in [0, 1]$$

and define the order interval

$$[u_0, v_0] = \{x \in X: u_0(t) \leq x(t) \leq v_0(t), t \in [0, 1]\}.$$

Theorem 1. Assume the following conditions hold:

(H2) u_0, v_0 are lower and upper solutions of (1), respectively, such that $u_0(t) \leq v_0(t)$ on $[0, 1]$.

(H3) There exists $\lambda < \lambda_1$ such that

$$f(t, y) - f(t, x) \geq \lambda(y - x),$$

whenever $u_0(t) \leq x \leq y \leq v_0(t)$, $t \in [0, 1]$.

Then there exist monotone sequences $\{u_n(t)\}$, $\{v_n(t)\}$, which converge uniformly to the extremal solutions of (1) in the order interval $[u_0, v_0]$, respectively.

Proof. For all $\xi \in [u_0, v_0]$, consider (2) with

$$\sigma(t) = f(t, \xi(t)) - \lambda\xi(t), \quad \mu_1 = \mu_2 = 0.$$

By Lemma 4, problem (2) has a unique solution $x \in X$. Denote an operator $S : [u_0, v_0] \rightarrow X$ by $x = S\xi$. Then the operator S has the following properties:

(i) $u_0 \leq Su_0, Sv_0 \leq v_0$.

Let $u_1 = Su_0$, $p(t) = u_1(t) - u_0(t)$. By (H2) and (H3), we have that

$$-p''(t) \geq \lambda p(t), \quad t \in (0, 1),$$

$$p(0) \geq \int_0^1 p(s) dA(s), \quad p(1) \geq \int_0^1 p(s) dB(s),$$

which implies by virtue of Lemma 3 that $p(t) \geq 0$ for all $t \in [0, 1]$, i.e., $u_0 \leq Su_0$. Similar argument show that $v_0 \leq Sv_0$.

(ii) S is nondecreasing in $[u_0, v_0]$.

Let $\xi_1, \xi_2 \in [u_0, v_0]$ be such that $\xi_1 \leq \xi_2$. Suppose that $p = S\xi_2 - S\xi_1$. By (H2) and (H3), we have

$$-p''(t) \geq \lambda p(t), \quad t \in (0, 1),$$

$$p(0) = \int_0^1 p(s) dA(s), \quad p(1) = \int_0^1 p(s) dB(s),$$

which implies by virtue of Lemma 3 that $p(t) \geq 0$ for all $t \in [0, 1]$, i.e., S is nondecreasing. This together with (i) implies that $S : [u_0, v_0] \rightarrow [u_0, v_0]$.

Now let $u_n = Su_{n-1}$, $v_n = Sv_{n-1}$, $n = 1, 2, 3, \dots$. Following (i) and (ii), we have

$$u_0 \leq u_1 \leq \dots \leq u_{n-1} \leq u_n \leq \dots \leq \dots \leq v_n \leq v_{n-1} \leq \dots \leq v_1 \leq v_0. \quad (9)$$

Using the standard arguments, it is easy to show that $\{u_n\}$ and $\{v_n\}$ are uniformly bounded and equicontinuous in $[u_0, v_0]$. By (9) and the Arzela–Ascoli theorem, we have

$$\lim_{n \rightarrow \infty} u_n(t) = u^*(t), \quad \lim_{n \rightarrow \infty} v_n(t) = v^*(t)$$

uniformly on $t \in [0, 1]$, and u^*, v^* satisfy (1). Moreover, $u^*, v^* \in [u_0, v_0]$. Thus, u^* and v^* are solutions of (1) in $[u_0, v_0]$.

Next, we prove that u^* and v^* are extremal solutions of (1) in $[u_0, v_0]$. In fact, we assume that x is any solution of (1). That is,

$$\begin{aligned} -x''(t) &= f(t, x(t)), \quad t \in (0, 1), \\ x(0) &= \int_0^1 x(t) \, dA(t), \quad x(1) = \int_0^1 x(t) \, dB(t). \end{aligned}$$

By (H2), (H3) and Lemma 3, it is easy by induction to show that

$$u_n \leq x \leq v_n, \quad n = 1, 2, 3, \dots \quad (10)$$

Now, letting $n \rightarrow \infty$ in (10), we have $u^* \leq x \leq v^*$. That is, u^* and v^* are extremal solutions of (1) in $[u_0, v_0]$. This completes the proof. \square

4 Example

Consider the following problem:

$$\begin{aligned} -x''(t) &= \frac{1}{6}(t - x(t))^3 + t^2 \sin \frac{x(t)}{4}, \quad t \in (0, 1), \\ x(0) &= 0, \quad x(1) = \int_0^1 x(s) \, ds. \end{aligned} \quad (11)$$

Obviously,

$$f(t, x) = \frac{1}{6}(t - x)^3 + t^2 \sin \frac{x}{4}, \quad A(t) = 0, \quad B(t) = t.$$

Let $u_0(t) = 0$, $v_0(t) = \pi t$. Then it is easy to verify that

$$\kappa_1 = 1, \quad \kappa_2 = 0, \quad \kappa_3 = \kappa_4 = \frac{1}{2}, \quad \kappa = \kappa_1 \kappa_4 - \kappa_2 \kappa_3 = \frac{1}{2},$$

$$-u_0''(t) = 0 \leq \frac{t^3}{6} = \frac{1}{6}(t - u_0(t))^3 + t^2 \sin \frac{u_0(t)}{4}, \quad t \in (0, 1),$$

$$u_0(0) = 0, \quad u_0(1) = 0 = \int_0^1 u_0(s) \, ds$$

and

$$-v_0''(t) = 0 \geq \frac{t^3}{12}(3\pi - 2(\pi - 1)^3) \geq \frac{1}{6}(t - v_0(t))^3 + t^2 \sin \frac{v_0(t)}{4}, \quad t \in (0, 1),$$

$$v_0(0) = 0, \quad v_0(1) = \pi > \frac{\pi}{2} = \int_0^1 v_0(s) \, ds.$$

Conditions (H1) and (H2) hold.

Let λ_1 be the first eigenvalue of the linear operator T given by (5) with $B(t) = t$, and φ^* be an eigenfunction corresponding to eigenvalue λ_1 . Thus, we have

$$\begin{aligned}\varphi^{*''} + \lambda_1 \varphi^*(t) &= 0, \quad t \in (0, 1), \\ \varphi^*(0) &= 0, \quad \varphi^*(1) = \int_0^1 \varphi^*(s) \, ds.\end{aligned}$$

By ordinary method, we get $\varphi^*(t) = c \sin \sqrt{\lambda_1} t$ for some $c \in R$ and $\lambda_1 \in (0, \pi^2)$, where λ_1 is the unique positive solutions of the equation

$$\sqrt{\lambda_1} \sin \sqrt{\lambda_1} + \cos \sqrt{\lambda_1} - 1 = 0, \quad \lambda_1 \in (0, \pi^2).$$

Set $h(\lambda) = \sqrt{\lambda} \sin \sqrt{\lambda} + \cos \sqrt{\lambda} - 1$. Noting that $h(4\pi^2/9) = (2\pi/3) \sin(2\pi/3) + \cos(2\pi/3) - 1 = (2\pi/3)(\sqrt{3}/2) - (1/2) - 1 > 0$ and $h(\pi^2) = -2 < 0$, we have

$$\lambda_1 > \frac{4\pi^2}{9}.$$

In addition, for $0 \leq x \leq y \leq \pi t$, $t \in [0, 1]$, we have

$$f(t, y) - f(t, x) \geq -\frac{(\pi - 1)^2}{2}(y - x) \geq -\frac{4\pi^2}{9}(y - x).$$

So, conditions (H3) holds. Therefore, (11) satisfies all conditions of Theorem 1. By Theorem 1, there exist monotone iterative sequences $\{u_n\}$, $\{v_n\}$, which converge uniformly on $[0, 1]$ to the extremal solutions of (11) in $[u_0, v_0]$.

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