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On comparison of the estimators of the Hurst index and the diffusion coefficient of the fractional Gompertz diffusion process*

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Abstract. We study some estimators of the Hurst index and the diffusion coefficient of the fractional Gompertz diffusion process and prove that they are strongly consistent and most of them are asymptotically normal. Moreover, we compare the asymptotic behavior of these estimators with the aid of computer simulations.

Keywords: fractional Gompertz diffusion process, Hurst index, diffusion coefficient.

1 Introduction

Many applications make use of processes that are described by stochastic differential equations (SDEs). Recently, much attention has been paid to SDEs driven by the fractional Brownian motion (fBm) and to the problems of statistical estimation of model parameters. Statistical aspects of the models driven by the fBm have been studied in many articles. Especially much attention has been paid to the estimation of the parameters of drift. We focus on estimators of the Hurst index and the diffusion coefficient. Recently, some new estimators of the Hurst index and of the diffusion coefficient have been proposed (see [1, 2, 13, 14]). This paper aims to compare them using discrete observations of the sample paths of the solution of the SDE.

As the test process, we will consider the fractional Gompertz diffusion process (fGd)

$$X_t = \int_0^t (\alpha X_s - \beta X_s \ln X_s)\,\mathrm{d}s + \sigma \int_0^t X_s\,\mathrm{d}B_s^H, \quad X_0 = x_0 > 0,\ 0 \leqslant t \leqslant T, \tag{1}$$

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where $\alpha, \beta \neq 0$, and σ are real parameters and B^H is a fBm with the Hurst index $H \in (1/2, 1)$. Almost all sample paths of B^H have bounded p-variation for each $p > 1/H$ on $[0, T]$ for every $T > 0$. The second integral in (1) is the pathwise Riemann–Stieltjes integral with respect to the process having finite p-variation.

The reasons we have chosen fGd as the test process are as follows. Firstly, it is a nonlinear process. To equation (1) it is possible to apply a pathwise approach and use a chain rule for the composition of a smooth function and a function of bounded p-variation with $1 < p < 2$. This approach allows to easily obtain the unique explicit solution of equation (1) for $H \in (1/2, 1)$ in the class of processes, almost all sample paths of which have bounded p-variation with $1 < p < 2$. Secondly, the structure of the increments of fGd allows us to apply a wider class of estimators without imposing additional restrictions on the process. The normalization of quadratic variation by the square of the process value at a fixed point allows us to derive the asymptotic normality of these estimators. The application of this approach allows to consider similar statistics for the equations with time-dependent coefficients. Moreover, in case of the standard Brownian motion, i.e., for $H = 1/2$, this process plays an important role in the modeling of population growth.

Dung [7] proved that a class of fractional geometric mean reversion processes expressed by a fractional SDE of the form

$$X_t = \int_0^t (\alpha_s X_s - \beta_s X_s \ln X_s)\,\mathrm{d}s + \int_0^t \sigma_s X_s\,\mathrm{d}W_s^H, \quad X_0 = x_0 > 0,\ 0 \leqslant t \leqslant T,$$

where W^H is a fractional Brownian motion of the Liouville form, has a unique solution. It follows from his results that, if the coefficients in the equation above are constant, its solution will be of the form

$$X_t = \exp\left\{ \mathrm{e}^{-\beta t} \ln x_0 + \alpha \int_0^t \mathrm{e}^{-\beta(t-s)}\,\mathrm{d}s + \sigma \int_0^t \mathrm{e}^{-\beta(t-s)}\,\mathrm{d}W_s^H \right\}. \tag{2}$$

In the Appendix, it will be shown that equation (1) has the solution of the same form even without the assumption required by Dung.

In case of the fractional Ornstein–Uhlenbeck process and the geometric Brownian motion, a comparison of various estimators of the Hurst index was presented in [11]. The estimators based on quadratic variations were compared to some of the other known estimators. It should be noted that these estimators are not asymptotically normal. Moreover, only one of the estimators considered in the aforementioned paper is included in the comparison presented in this article.

A reader interested in the existence of the solution of the Gompertz diffusion process with respect to the standard Brownian motion and the estimation of its parameters is encouraged to read [15, 9, 8] and the references therein.

The structure of the paper is as follows. Section 2 presents the estimators considered in the rest of the paper. Section 3 contains the numerical comparison of the estimators' performance. Sections 4–6 are dedicated to proofs of strong consistency of the considered estimators in case of the fractional Gompertz diffusion process. In the Appendix, the existence and uniqueness of the solution of equation (1) is proved.

2 Estimators

In the rest of the paper, we will deal with the problem of estimating the Hurst index and the diffusion coefficient of the fractional Gompertz diffusion process based on discrete observations of its sample paths. The estimation of the trend parameters α and β, although not included in the present paper, can be performed using the least squares method. Using the change of variable $Z_t = \ln X_t$, equation (1) can be reduced to the fractional Vasicek model, to which the least squares method is then applied (see, e.g., [17]).

2.1 Hurst index estimators

Let $\pi_n = \{\tau_k^{m_n},\ k = 0, \ldots, m_n\}$, $n \geqslant 1$, $\mathbb{N} \ni m_n \uparrow \infty$, be a sequence of partitions of the interval $[0, T]$. If partition π_n is uniform, then $\tau_k^{m_n} = kT/m_n$ for all $k \in \{0, \ldots, m_n\}$. If $m_n \equiv n$, we write t_k^n instead of $\tau_k^{m_n}$. Let $(X_t)_{t\in[0,T]}$ be a stochastic process and

$$\Delta^{(1)} X\bigl(\tau_k^{m_n}\bigr) = X\bigl(\tau_k^{m_n}\bigr) - X\bigl(\tau_{k-1}^{m_n}\bigr),$$
$$\Delta^{(2)} X\bigl(\tau_k^{m_n}\bigr) = X\bigl(\tau_{k+1}^{m_n}\bigr) - 2X\bigl(\tau_k^{m_n}\bigr) + X\bigl(\tau_{k-1}^{m_n}\bigr),$$

$k = 1, \ldots, m_n - (i-1)$, $i = 1, 2$. Denote

$$V_{m_n,T}^{(i)} = \sum_{k=1}^{m_n-(i-1)} \left(\frac{\Delta^{(i)} X(\tau_k^{m_n})}{X(\tau_{k-1}^{m_n})}\right)^2, \quad i = 1, 2,$$

and

$$W_{n,k} = \sum_{j=-k_n+1}^{k_n-1} \bigl(\Delta^{(2)} X_{s_j^n + t_k^n}\bigr)^2 = \sum_{j=-k_n+1}^{k_n-1} \bigl(X_{s_{j+1}^n + t_k^n} - 2X_{s_j^n + t_k^n} + X_{s_{j-1}^n + t_k^n}\bigr)^2,$$

where $s_j^n = jT/m_n$, $m_n = nk_n$, and $k_n = n^2$.

Theorem 1. *Assume that X is a solution of the fractional Gompertz SDE and $1/2 < H < 1$. Then*

$$\widehat{H}_n^{(j)} \xrightarrow{\text{a.s.}} H, \quad j = 1, 2, 3, 4,$$

and

$$2\ln 2\,\sqrt{n}\bigl(\widehat{H}_n^{(1)} - H\bigr) \xrightarrow{\text{d}} \mathcal{N}\bigl(0, \sigma_*^2\bigr), \qquad 2\sqrt{n}\,\ln\frac{n}{T}\bigl(\widehat{H}_n^{(2)} - H\bigr) \xrightarrow{\text{d}} N\bigl(0; \sigma_H^2\bigr),$$

$$\sqrt{n}\bigl(\widehat{H}_n^{(3)} - H\bigr) \xrightarrow{\text{d}} \mathcal{N}\left(0, \sigma_\ell^2\left(\mathbf{r}, \frac{\mathbf{z}}{2\sqrt{\mathbf{r}}}\right)\right),$$

with

$$\sigma_*^2(H) = \frac{3}{2}\sigma^2(H) - 2\sigma_{1,2}(H)$$

and known variances σ_H^2*,* $\sigma_{1,2}(H)$*,* $\sigma_\ell^2(\mathbf{r}, (\mathbf{z}/\sqrt{\mathbf{r}})/2)$ *defined in Section* 4.2*, where*

$$\widehat{H}_n^{(1)} = \frac{1}{2} - \frac{1}{2\ln 2}\ln\frac{V_{2n,T}^{(2)}}{V_{n,T}^{(2)}}, \tag{3}$$

$$\widehat{H}_n^{(2)} = \frac{1}{2} + \frac{1}{2\ln k_n}\ln\left(\frac{2}{n}\sum_{k=2}^{n}\frac{(\Delta^{(2)}X(t_k^n))^2}{W_{n,k-1}}\right), \tag{4}$$

$$\widehat{H}_n^{(3)} = -\frac{1}{2}\sum_{j=1}^{\ell} z_j \ln\frac{V_{n_j,T}^{(2)}}{n_j - 1}, \quad n_j = r_j n,\ r_j \in \mathbb{N},\ j = 1, \dots, \ell,\ \ell \neq 1, \tag{5}$$

$$z_i = \frac{y_i}{\sum_{i=1}^{\ell} y_i^2}, \quad \textit{and} \quad y_i = \ln r_i - \frac{1}{\ell}\sum_{i=1}^{\ell}\ln r_i,$$

$$\widehat{H}_n^{(4)} = \frac{1}{0.1468}\left(\frac{1}{n^4 - 2}\sum_{k=1}^{n^4-2}\frac{|\Delta^{(2)}X(t_k^{n^4}) + \Delta^{(2)}X(t_{k+1}^{n^4})|}{|\Delta^{(2)}X(t_k^{n^4})| + |\Delta^{(2)}X(t_{k+1}^{n^4})|} - 0.5174\right). \tag{6}$$

Remark 1. The estimators $\widehat{H}_n^{(i)}$, $i = 1, 2, 3, 4$, were considered in [14,13,2] and [1]. The estimator $\widehat{H}_n^{(2)}$ can be used to estimate the Hurst index of the generic form of the SDE with an additional restriction on the diffusion coefficient.

2.2 Diffusion coefficient estimators

In this section, we describe four estimators of the diffusion coefficient. The application of the fourth is not explicitly justified, however, this can be performed. It was proposed in [2] for the fractional geometric Brownian motion. The aforementioned paper shows it to be a weakly consistent estimator of the diffusion coefficient σ^2.

Theorem 2. *Assume that* X *is a solution of the fractional Gompertz SDE,* $1/2 < H < 1$*, and* $\widehat{H}_n = H + o_\omega(\phi(n))$*, where* o_ω *is defined in Section* 5*. If* $\phi(n) = o(\ln^{-1} n)$*, then*

$$\widehat{\sigma}_{1,n}^2 = \frac{n^{2\widehat{H}_n - 1}}{T^{2\widehat{H}_n}} V_{n,T}^{(1)} \xrightarrow{\text{a.s.}} \sigma^2,$$

$$\widehat{\sigma}_{2,n}^2 = \frac{n^{2\widehat{H}_n - 1}}{T^{2\widehat{H}_n}(4 - 2^{2\widehat{H}_n})} V_{n,T}^{(2)} \xrightarrow{\text{a.s.}} \sigma^2,$$

$$\widehat{\sigma}_{3,n}^2 = \frac{\sum_{k=1}^{n}(\Delta^{(1)}X(t_k^n))^2}{(\frac{T}{n})^{2\widehat{H}_n}\sum_{k=1}^{n} X^2(t_{k-1}^n)} \xrightarrow{\text{a.s.}} \sigma^2.$$

If $\phi(n) = o(n^{-1/2}\ln^{-1} n)$*, then*

$$\sqrt{n}\big(\widehat{\sigma}_{2,n}^2 - \sigma^2\big) \xrightarrow{\text{d}} \mathcal{N}\big(0; \sigma^4\sigma^2(H)\big) \quad \textit{for } 0 < H < 1,$$

where variance $\sigma^2(H)$ *are defined in Section* 4.2*.*

For the purposes of comparison, we shall also consider

$$\widehat{\sigma}_{4,n} = \frac{\exp(\widehat{B})}{4 - 2^{2\widehat{H}_n^{(3)}}}, \quad \widehat{B} = \frac{1}{2}\left(\frac{1}{\ell}\sum_{i=1}^{\ell} \ln \frac{V_{n_i,T}^{(2)}}{n_i - 1}\right) + \widehat{H}_n^{(3)}\left(\frac{1}{\ell}\sum_{i=1}^{\ell} \ln n_i\right),$$

where n_i and $\widehat{H}_n^{(3)}$ are defined in Theorem 1.

Remark 2. The estimators $\widehat{\sigma}_{i,n}^2$, $i = 1, 2$, are similar to the estimators used in the book [2] for the evaluation of the diffusion coefficient σ of the solutions of linear SDE when H is known. The estimator $\widehat{\sigma}_{3,n}^2$ is used to estimate the diffusion coefficient of the fractional Ornstein–Uhlenbeck process when H is known (see [17]). We have shown that this restriction can be lifted.

3 Modeling of the estimators

The goal of this section is to describe the numerical simulations that were performed in order to compare the behavior of the estimators considered in this paper.

The sample paths of the fractional Brownian motion, which were further used to construct the sample paths of the fractional Gompertz diffusion process, were simulated using the Wood–Chan circulant matrix embedding method [16]. The values of the constants involved in these simulations were, unless explicitly stated otherwise, $X_0 = 3$, $\alpha = 0.5$, $\beta = 2$, and $\sigma = 1.5$. We considered these sample paths on the unit interval, hence, $T = 1$. The number of replicates was 300 in all of the considered cases. In what follows, we present the dependencies of the estimators both on the true parameter values (H, σ) and on the sample size (n). We have also checked for possible dependencies of the estimators of the Hurst index and the diffusion coefficient on the values of the other parameters of the considered equation, namely, the drift coefficients α and β and the initial condition X_0. No such dependencies of significant impact have been observed.

3.1 Modeling of the Hurst index estimators

Figures 1 and 2 display, respectively, the dependence of the four estimators of the Hurst index H on its true value and on the sample size (length of the sample path) n. In Fig. 1, the same sample sizes $n = 2^{10}$ were used for all of the considered estimators, which does suggest that the estimators $\widehat{H}_n^{(4)}$ and $\widehat{H}_n^{(2)}$ would be a priori less efficient. However, in practical applications, the sample size is usually fixed, hence, the motivation was to see what kind of performance the considered estimators would show given the exact same number of observations. In Fig. 2, the value of the Hurst index was chosen as $H = 0.75$. The values of r_j were taken to be powers of 2 (more precisely, $r_j = 2^{j-1}$, $j = 1, \dots, l$) and, further, the values of n_i were taken as $n_i = n/r_i$, where n denotes the (fixed) maximum sample size length. The value of l was (arbitrarily) taken to be 4, as simulation results suggested that both considerably smaller (e.g., 2) and considerably larger (e.g., $\log_2 n - 1$) values yielded inferior performance. It does appear plausible that for much bigger sample sizes, it might be beneficial to increase this value further,

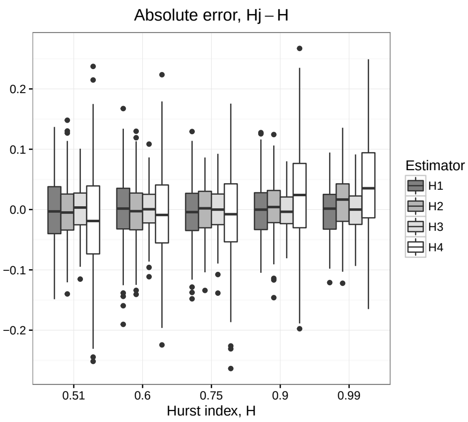

Figure 1. Dependence of the absolute error on H.

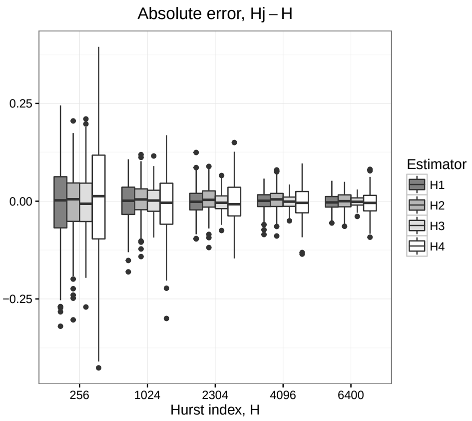

Figure 2. Dependence of the absolute error on n.

however, in this study, sample sizes exceeding 6400 points were not considered. It can be seen that the performance of the estimator $\widehat{H}_n^{(4)}$ is slightly lacking compared to that of the other estimators, which, despite imposing rather different requirements on the sample sizes, show similar precision.

3.2 Modeling of the diffusion coefficient estimators

In order to calculate the estimators $\widehat{\sigma}^2_{1,n}$, $\widehat{\sigma}^2_{2,n}$, and $\widehat{\sigma}^2_{3,n}$, we need to supply them with the estimated values of the Hurst index. In Figs. 3 and 4 presented below, the diffusion coefficient estimator $\widehat{\sigma}^2_{i,n}$, using the Hurst index estimator $\widehat{H}_n^{(j)}$, is denoted as 'si_hj', $i, j = 1, 2, 3$. The estimator $\widehat{\sigma}^2_{4,n}$ is denoted as 's4'. The graphs present the relative differences, namely, $(\widehat{\sigma}_{i,n} - \sigma)/\sigma$. In Fig. 3, the sample size was chosen to be $n = 2^{10}$

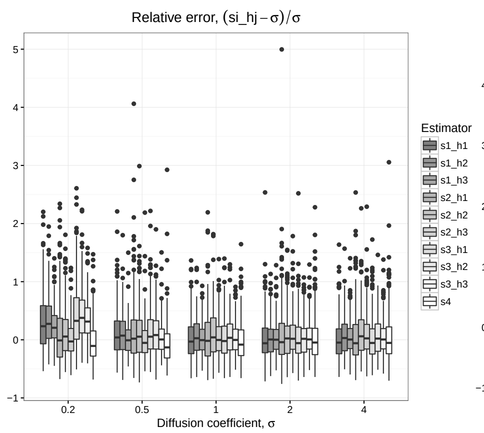

Figure 3. Dependence of the relative error on σ.

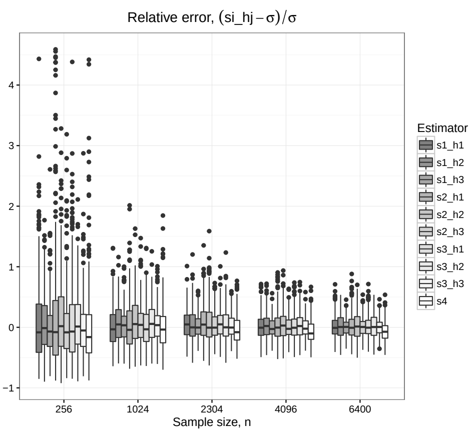

Figure 4. Dependence of the relative error on n.

for all of the considered estimators. In Fig. 4, the value of the diffusion coefficient was chosen as $\sigma = 1$. It can be seen that the performance of all the considered estimators is roughly similar. The convergence rate of $\widehat{\sigma}^2_{4,n}$ appears slower, although it seems to perform better for the values of σ close to zero. For the other estimators, it appears that using $\widehat{H}_n^{(3)}$ yields better numerical characteristics.

4 Preliminaries

4.1 Variation

Let $p > 0$, $-\infty < a < b < \infty$ be fixed and $\mathcal{K} = \{\{x_0, \dots, x_n\} \mid a = x_0 < x_1 < \dots < x_n = b, n \geqslant 1\}$ denotes the set of all possible partitions of $[a, b]$. For any $f : [a, b] \to \mathbb{R}$, define

$$v_p\bigl(f; [a,b]\bigr) = \sup_{\varkappa \in \mathcal{K}} \sum_{k=1}^{n} \bigl|f(x_k) - f(x_{k-1})\bigr|^p, \qquad V_p\bigl(f; [a,b]\bigr) = v_p^{1/p}\bigl(f; [a,b]\bigr).$$

If $v_p(f; [a, b]) < \infty$, f is said to have a bounded p-variation on $[a, b]$.

In the rest of the paper, $\mathcal{W}_p([a, b])$ denotes the class of functions on $[a, b]$ with bounded p-variation and $C\mathcal{W}_p([a, b]) = \{f \in \mathcal{W}_p([a, b]) \mid f \text{ is continuous}\}$. In case of a fixed interval $[a, b]$, we abbreviate the notations and write $v_p(f)$, $V_p(f)$, etc. instead of $v_p(f; [a, b])$, $V_p(f; [a, b])$.

Below we list several facts used in the sequel. For details, we refer the reader to [6].

- $p \geqslant 1 \ \Rightarrow\ f \mapsto V_p(f)$ is a semi-norm on $\mathcal{W}_p$.
- $f = \text{const} \ \Leftrightarrow\ V_p(f) = 0$.
- $f \in \mathcal{W}_p \ \Rightarrow\ \sup_{x\in[a,b]} |f(x)| < \infty$.
- $p \geqslant 1,\ f \in \mathcal{W}_p \ \Rightarrow\ f \in \mathcal{W}_q$ for all $q \geqslant p$.
- $f, g \in \mathcal{W}_p \ \Rightarrow\ fg \in \mathcal{W}_p$.
- Let $f \in \mathcal{W}_q$ and $h \in \mathcal{W}_p$ with $p, q \in (0, \infty)$: $1/p + 1/q > 1$. Then an integral $\int_a^b f\,\mathrm{d}h$ exists as the Riemann–Stieltjes integral provided f and h have no common discontinuities. If the integral exists, the Love–Young inequality

$$\left|\int_a^b f\,\mathrm{d}h - f(y)\bigl[h(b) - h(a)\bigr]\right| \leqslant C_{p,q} V_q(f) V_p(h) \tag{7}$$

holds for all $y \in [a, b]$, where $C_{p,q} = \zeta(p^{-1} + q^{-1})$ and $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$. Moreover,

$$V_p\left(\int_a^{\cdot} f\,\mathrm{d}h\right) \leqslant C_{p,q} V_{q,\infty}(f) V_p(h),$$

where $V_{q,\infty}(f) = V_q(f) + \sup_{a\leqslant x\leqslant b} |f(x)|$. Note that $f \mapsto V_{q,\infty}(f)$ is a norm on $\mathcal{W}_q$, $q \geqslant 1$.

- $h \in C\mathcal{W}_p \;\Rightarrow\; g(y) = \int_a^y f\,\mathrm{d}h$, $y \in [a,b]$, is continuous.
- Let ϕ be a locally Lipschitz function and let $f \in \mathcal{W}_p([a,b])$. Then the composite function $\phi \circ f$ has bounded p-variation, that is, $\phi \circ f \in \mathcal{W}_p([a,b])$, where $(\phi \circ f)(x) = \phi(f(x))$.

The chain rule is based on the Riemann–Stieltjes integrals.

Theorem 3 [Chain rule]. (*See* [6].) *Let $p \in [1;2)$ and $f = (f_1,\dots,f_d)$: $[a,b] \to \mathbb{R}^d$ be such a function that for each $k = 1,\dots,d$, $f_k \in C\mathcal{W}_p([a,b])$. Let $g : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function with locally Lipschitz partial derivatives g_k', $k = 1,\dots,d$. Then each $g_l' \circ f$ is Riemann–Stieltjes-integrable with respect to f_k and*

$$(g \circ f)(b) - (g \circ f)(a) = \sum_{k=1}^{d} \int_a^b (g_k' \circ f)\,\mathrm{d}f_k.$$

Proposition 1 [Substitution rule]. (*See* [6].) *Let $f, g, h \in C\mathcal{W}_p([a,b])$, $1 \leqslant p < 2$. Then*

$$\int_a^b f(x)\,\mathrm{d}\left(\int_a^x g(y)\,\mathrm{d}h(y)\right) = \int_a^b f(x)g(x)\,\mathrm{d}h(x).$$

4.2 Several results on fBm

Recall that the fBm $(B_t^H)_{t\in[0,T]}$ with the Hurst index $H \in (0,1)$ is a real-valued continuous centered Gaussian process with the covariance given by

$$\mathbf{E}\big(B_t^H B_s^H\big) = \frac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big).$$

In order to consider the strong consistency and asymptotic normality of the given estimators, we need several facts about B^H (see [3, 2, 4, 10]).

Limit results. For consideration of the asymptotic properties of the estimators $\widehat{H}_n^{(i)}$, $i = 1,2$, we shall use the following results. Let

$$\widehat{V}_{n,T}^{(i)B^H} = \frac{n^{2H-1}}{c_i} \sum_{k=1}^{n-1} \big(T^{-H} \Delta^{(i)} B^H\big(t_k^n\big)\big)^2, \quad H \neq \frac{1}{2},\ i = 1,2,$$

where $c_1 = 1, c_2 = 4 - 2^{2H}$. Then

$$\widehat{V}_{n,T}^{(i)B^H} \overset{\text{a.s.}}{\to} 1 \quad \text{as } n \to \infty,\ i = 1,2;$$

$$\sqrt{n}\big(\widehat{V}_{n,T}^{(1)B^H} - 1\big) \overset{\mathrm{d}}{\to} \mathcal{N}\big(0, \sigma_*^2(H)\big), \quad H \in (0, 3/4),$$

$$\sqrt{n}\begin{pmatrix} \widehat{V}_{n,T}^{(2)B^H} - 1 \\ \widehat{V}_{2n,T}^{(2)B^H} - 1 \end{pmatrix} \overset{\mathrm{d}}{\to} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2(H) & \sigma_{1,2}(H) \\ \sigma_{1,2}(H) & \sigma^2(H)/2 \end{pmatrix}\right)$$

with

$$\sigma_*^2(H) = 2\left(1 + 2\sum_{j=1}^{\infty} \widehat{\rho}_H^2(j)\right), \qquad \sigma^2(H) = 2\left(1 + 2\sum_{j=1}^{\infty} \rho_H^2(j)\right),$$

$$\sigma_{1,2}(H) = \sum_{j\in\mathbb{Z}} \widetilde{\rho}_H^2(j),$$

$$\begin{aligned}
\widehat{\rho}_H(j) &= -\frac{1}{2}\bigl[-|j-1|^{2H} + 2|j|^{2H} - |j+1|^{2H}\bigr],\\
\rho_H(j) &= \frac{1}{2(4-2^{2H})}\bigl(-6|j|^{2H} - \bigl(|j-2|^{2H} + |j+2|^{2H}\bigr)\\
&\quad + 4\bigl(|j-1|^{2H} + |j+1|^{2H}\bigr)\bigr),\\
\widetilde{\rho}_H(j) &= \frac{1}{2(4-2^{2H})2^{H}}\bigl(-|j-2|^{2H} + 2|j-1|^{2H} + |j|^{2H}\\
&\quad - 4|j+1|^{2H} + |j+2|^{2H} + 2|j+3|^{2H} - |j+4|^{2H}\bigr).
\end{aligned}$$

In order to prove the asymptotic normality of the estimator $\widehat{H}_n^{(3)}$, we need the following result obtained in [2]. Let $n_i = r_i n$, $i = 1, \dots, \ell$, where r_i, $n \in \mathbb{N}$, and z_i, $i = 1, \dots, \ell$, are defined in Theorem 1. Then

$$\frac{1}{2}\sum_{i=1}^{\ell} \frac{z_i}{\sqrt{r_i}}\sqrt{n_i}\bigl(\widehat{V}_{n_i,T}^{(i)B^H} - 1\bigr) \xrightarrow{\mathrm{d}} \mathcal{N}\left(0, \sigma_{2,\ell}^2\left(\mathbf{r}, \frac{\mathbf{z}}{2\sqrt{\mathbf{r}}}\right)\right),$$

where $\mathbf{r} = (r_1, \dots, r_\ell)$, $\mathbf{z} = (z_1, \dots, z_\ell)$,

$$\sigma_{2,\ell}^2(\mathbf{k}, \mathbf{d}) = \sum_{i=1}^{\ell}\sum_{j=1}^{\ell} d_i d_j \rho_2(k_i, k_j), \quad \mathbf{k} = (k_1, \dots, k_\ell) \in \mathbb{N}^\ell,\ \mathbf{d} = (d_1, \dots, d_\ell) \in \mathbb{R}^\ell,$$

$$\begin{aligned}
\rho_2(k_i, k_j) &= \frac{1}{\sqrt{k_i k_j}} \sum_{p=1}^{+\infty} c_{2p,2}^2 \cdot (2p)! \left(\sum_{s=0}^{k_i-1}\sum_{r=-\infty}^{+\infty} \rho_{k_i,k_j}^{2p}(k_i r + k_j s)\right),\\
c_{2p,2} &= \frac{1}{(2p)!}\prod_{i=0}^{p-1}(2-2i),\\
\rho_{b,c}(x) &= \frac{1}{2(4-2^{2H})}(bc)^{-H}\bigl[-|x|^{2H} + 2|x-b|^{2H} - |x-2b|^{2H}\\
&\quad + 2|x+c|^{2H} - 4|x+c-b|^{2H} + 2|x+c-2b|^{2H}\\
&\quad - |x+2c|^{2H} + 2|x+2c-b|^{2H} - |x+2c-2b|^{2H}\bigr].
\end{aligned}$$

If $k_i = k_j$, then

$$\rho_2(k, k) = 2\sum_{r=-\infty}^{+\infty} \rho_H^2(r) = 2\left(1 + 2\sum_{j=1}^{\infty} \rho_H^2(j)\right).$$

Variation of B^H. It is known that almost all sample paths of B^H are locally Hölder of order strictly less than H, $0 < H < 1$. To be more precise, for all $0 < \varepsilon < H$ and $T > 0$, there exists a nonnegative random variable $G_{\varepsilon,T}$ such that $\mathbf{E}(|G_{\varepsilon,T}|^p) < \infty$ for all $p \geqslant 1$ and

$$\sup_{s,t\in[0;T]} \left|B_t^H - B_s^H\right| \leqslant G_{\varepsilon,T}|t-s|^{H-\varepsilon} \quad \text{a.s.} \tag{8}$$

Thus, $B^H \in C\mathcal{W}_{H_\varepsilon}([0,T])$, $H_\varepsilon = 1/(H-\varepsilon)$.

The rate of convergence of the Hurst index.

Theorem 4. (*See* [12, *Thm.* 3].) *For any* $t \in [0;T]$, *define* $r_{nt} = [tn/T]$, $\rho_{nt} = (r_{nt}/n)T$ *and*

$$\widehat{V}_{nt}^{(2)B^H} = \frac{n^{2H-1}}{T^{2H-1}(4-2^{2H})} \sum_{k=i}^{r_{nt}} \left(\Delta^{(2)} B^H\left(t_k^n\right)\right)^2.$$

Then

$$\sup_{t\in[0;T]} \left|\widehat{V}_{nt}^{(2)B^H} - \rho_{nt}\right| = O_\omega\left(n^{-1/2}\ln^{1/2} n\right), \tag{9}$$

where O_ω *is defined in Section* 5.

5 Properties of the increments of the Gompertz diffusion process

The fractional Gompertz diffusion process X has the explicit solution given by

$$X_t = \exp\left\{ e^{-\beta t}\ln x_0 + \frac{\alpha}{\beta}\left(1-e^{-\beta t}\right) + \sigma\int_0^t e^{-\beta(t-s)}\,dB_s^H \right\}, \quad 0 \leqslant t \leqslant T.$$

Moreover, it is unique in $C\mathcal{W}_{1/(H-\varepsilon)}([0,T])$ for all $\varepsilon \in (0, H-1/2)$. The proof of this can be found in the Appendix. Now we will consider the structure of increments of the Gompertz diffusion process.

To avoid cumbersome expressions, we introduce the symbols O_ω and o_ω. Let (Y_n) be a sequence of r.v.s, ς is an a.s. nonnegative r.v. and $(a_n) \subset (0,\infty)$ vanishes. $Y_n = O_\omega(a_n)$ means that $|Y_n| \leqslant \varsigma\cdot a_n$; $Y_n = o_\omega(a_n)$ means that $|Y_n| \leqslant \varsigma\cdot b_n$ with $b_n = o(a_n)$. In particular, $Y_n = o_\omega(1)$ corresponds to the sequence (Y_n), which tends to 0 a.s. as $n \to \infty$.

Lemma 1. *Suppose that* X *satisfies* (1), $\varepsilon \in (0, H-1/2)$, *and partition* π_n *of the interval* $[0,T]$ *is uniform. Then the following relations hold*:

$$\Delta X_{\tau_k^{m_n}} = X_{\tau_{k-1}^{m_n}}\left[\sigma\Delta B_{\tau_k^{m_n}}^H + O_\omega(d_n)\right] = X_{\tau_{k-1}^{m_n}} O_\omega\left(d_n^{H-\varepsilon}\right), \quad k = 1,\dots,m_n, \tag{10}$$

$$\Delta^{(2)} X_{\tau_k^{m_n}} = X_{\tau_{k-1}^{m_n}}\left[\sigma\Delta^{(2)} B_{\tau_k^{m_n}}^H + O_\omega\left(d_n^{2(H-\varepsilon)}\right)\right], \quad k = 2,\dots,m_n, \tag{11}$$

where $d_n = \tau_k^{m_n} - \tau_{k-1}^{m_n}$ *and* $d_n \to 0$ *as* $n \to \infty$. *Moreover,* $\mathbf{E}O_\omega(1) < \infty$.

Proof. For the sake of simplicity, we will omit the index m_n for the points $\tau_k^{m_n}$. Let the sample path $t \mapsto X_t$ be continuous. We first prove (10). Note that

$$\Delta X_{\tau_k} = X_{\tau_k} - X_{\tau_{k-1}} \quad \text{and} \quad X_{\tau_k} = X_{\tau_{k-1}} \exp\{\Delta Y_{\tau_k}\},$$

where

$$Y_t = \mathrm{e}^{-\beta t} \ln x_0 + \frac{\alpha}{\beta}\bigl(1 - \mathrm{e}^{-\beta t}\bigr) + \sigma \int_0^t \mathrm{e}^{-\beta(t-s)}\, \mathrm{d}B_s^H.$$

It is clear that

$$\Delta Y_{\tau_k} = \mathrm{e}^{-\beta\tau_{k-1}}\bigl(\mathrm{e}^{-\beta(\tau_k-\tau_{k-1})} - 1\bigr)\left(\ln x_0 - \frac{\alpha}{\beta} + \sigma \int_0^{\tau_{k-1}} \mathrm{e}^{\beta s}\, \mathrm{d}B_s^H\right) + \sigma \mathrm{e}^{-\beta\tau_k} \int_{\tau_{k-1}}^{\tau_k} \bigl[\mathrm{e}^{\beta s} - \mathrm{e}^{\beta\tau_k}\bigr]\, \mathrm{d}B_s^H + \sigma \int_{\tau_{k-1}}^{\tau_k} \mathrm{d}B_s^H.$$

From the Chain rule it follows that

$$\int_0^t \mathrm{e}^{\beta s}\, \mathrm{d}B_s^H = \mathrm{e}^{\beta t} B_t^H - \beta \int_0^t \mathrm{e}^{\beta s} B_s^H\, \mathrm{d}s.$$

Thus,

$$\left|\int_0^{\tau_{k-1}} \mathrm{e}^{\beta s}\, \mathrm{d}B_s^H\right| = \left|\mathrm{e}^{\beta\tau_{k-1}} B_{\tau_{k-1}}^H - \beta \int_0^{\tau_{k-1}} \mathrm{e}^{\beta s} B_s^H\, \mathrm{d}s\right| \leqslant \mathrm{e}^{|\beta|T}\bigl(|\beta|T + 1\bigr) \sup_{t\leqslant T} \bigl|B_t^H\bigr|.$$

Provided

$$\mathrm{e}^{-\beta(\tau_k-\tau_{k-1})} = 1 + O(d_n),$$

it follows that

$$Z_{k-1} := \mathrm{e}^{-\beta\tau_{k-1}}\bigl(\mathrm{e}^{-\beta(\tau_k-\tau_{k-1})} - 1\bigr)\left[\ln x_0 - \frac{\alpha}{\beta} + \sigma \int_0^{\tau_{k-1}} \mathrm{e}^{\beta s}\, \mathrm{d}B_s^H\right] = O_\omega(d_n).$$

Further,

$$\begin{aligned}\left|\int_{\tau_{k-1}}^{\tau_k} \bigl[\mathrm{e}^{\beta s} - \mathrm{e}^{\beta\tau_k}\bigr]\, \mathrm{d}B_s^H\right| &\leqslant C_{1,H_\varepsilon} V_1\bigl(\mathrm{e}^{\beta\cdot}; [\tau_{k-1}, \tau_k]\bigr) V_{H_\varepsilon}\bigl(B^H; [\tau_{k-1}, \tau_k]\bigr) \\ &\leqslant C_{1,H_\varepsilon} \mathrm{e}^{2|\beta|T} |\beta| (\tau_k - \tau_{k-1}) V_{H_\varepsilon}\bigl(B^H; [\tau_{k-1}, \tau_k]\bigr) \\ &\leqslant C_{1,H_\varepsilon} G_{\varepsilon,T} \mathrm{e}^{2|\beta|T} |\beta| (\tau_k - \tau_{k-1})^{1+H-\varepsilon} \\ &= O_\omega\bigl(d_n^{1+H-\varepsilon}\bigr)\end{aligned}$$

since $|\mathrm{e}^x - 1| \leqslant |x|\mathrm{e}^{|x|}$ for all $x \in \mathbb{R}$. Consequently,

$$\begin{aligned} X_{\tau_k} &= X_{\tau_{k-1}} \exp\bigl\{Z_{k-1} + O_\omega\bigl(d_n^{1+H-\varepsilon}\bigr) + \sigma\Delta B^H_{\tau_k^n}\bigr\} \\ &= X_{\tau_{k-1}}\bigl[1 + Z_{k-1} + O_\omega\bigl(d_n^{1+H-\varepsilon}\bigr) + \sigma\Delta B^H_{\tau_k^n} + O_\omega\bigl(d_n^{2(H-\varepsilon)}\bigr)\bigr] \end{aligned} \tag{12}$$

and

$$\Delta X_{\tau_k} = X_{\tau_{k-1}}\bigl[Z_{k-1} + O_\omega\bigl(d_n^{2(H-\varepsilon)}\bigr) + \sigma\Delta B^H_{\tau_k^n}\bigr] = X_{\tau_{k-1}} O_\omega\bigl(d_n^{H-\varepsilon}\bigr). \tag{13}$$

Since (see Section 4.2 and [5])

$$\mathbf{E}\Bigl(\sup_{t\leqslant T}\bigl|B_t^H\bigr|\Bigr)^p < \infty \quad \text{and} \quad \mathbf{E}|G_{\varepsilon,T}|^p < \infty$$

for all $p \geqslant 1$, then $\mathbf{E}O_\omega(1) < \infty$.

Next, we prove (11). Taking into account (12) and (13) we get

$$\begin{aligned} \Delta^{(2)} X_{\tau_k} &= X_{\tau_k}\bigl[Z_k + O_\omega\bigl(d_n^{2(H-\varepsilon)}\bigr) + \sigma\Delta B^H_{\tau_{k+1}^n}\bigr] \\ &\quad - X_{\tau_{k-1}}\bigl[Z_{k-1} + O_\omega\bigl(d_n^{2(H-\varepsilon)}\bigr) + \sigma\Delta B^H_{\tau_k^n}\bigr] \\ &= X_{\tau_{k-1}}\bigl[1 + Z_{k-1} + O_\omega\bigl(d_n^{2(H-\varepsilon)}\bigr) + \sigma\Delta B^H_{\tau_k^n}\bigr] \\ &\quad \times \bigl[Z_k + O_\omega\bigl(d_n^{2(H-\varepsilon)}\bigr) + \sigma\Delta B^H_{\tau_{k+1}^n}\bigr] \\ &\quad - X_{\tau_{k-1}}\bigl[Z_{k-1} + O_\omega\bigl(d_n^{2(H-\varepsilon)}\bigr) + \sigma\Delta B^H_{\tau_k^n}\bigr] \\ &= X_{\tau_{k-1}}\Bigl[(Z_k - Z_{k-1}) + O_\omega\bigl(d_n^{2(H-\varepsilon)}\bigr) + \sigma\Delta^{(2)} B^H_{\tau_k^n}\Bigr] \\ &\quad + X_{\tau_{k-1}} O_\omega\bigl(d_n^{2(H-\varepsilon)}\bigr). \end{aligned}$$

Since

$$\begin{aligned} Z_k - Z_{k-1} &= \bigl(\mathrm{e}^{-\beta\tau_k} - \mathrm{e}^{-\beta\tau_{k-1}}\bigr)\bigl(\mathrm{e}^{-\beta d_n} - 1\bigr)\Biggl[\ln x_0 - \frac{\alpha}{\beta} + \sigma\int_0^{\tau_{k-1}} \mathrm{e}^{\beta s}\,\mathrm{d}B_s^H\Biggr] \\ &\quad + \sigma\mathrm{e}^{-\beta\tau_k}\bigl(\mathrm{e}^{-\beta d_n} - 1\bigr)\int_{\tau_{k-1}}^{\tau_k} \mathrm{e}^{\beta s}\,\mathrm{d}B_s^H \\ &= O_\omega\bigl(d_n^{2(H-\varepsilon)}\bigr), \end{aligned}$$

then

$$\begin{aligned} \Biggl|\int_{\tau_{k-1}}^{\tau_k} \mathrm{e}^{\beta s}\,\mathrm{d}B_s^H\Biggr| &\leqslant C_{1,H_\varepsilon} V_{1,\infty}\bigl(\mathrm{e}^{\beta\cdot}; [\tau_{k-1}, \tau_k]\bigr) V_{H_\varepsilon}\bigl(B^H; [\tau_{k-1}, \tau_k]\bigr) \\ &\leqslant 2C_{1,H_\varepsilon}\mathrm{e}^{|\beta|T} V_{H_\varepsilon}\bigl(B^H; [\tau_{k-1}, \tau_k]\bigr) \\ &\leqslant 2C_{1,H_\varepsilon} G_{\varepsilon,T}\mathrm{e}^{|\beta|T}|\beta|(\tau_k - \tau_{k-1})^{H-\varepsilon} \\ &= O_\omega\bigl(d_n^{H-\varepsilon}\bigr). \end{aligned}$$

Thus,

$$\Delta^{(2)} X_{\tau_k} = X_{\tau_{k-1}}\bigl[\sigma\Delta^{(2)} B^H_{\tau_k^n} + O_\omega\bigl(d_n^{2(H-\varepsilon)}\bigr)\bigr]. \qquad \square$$

6 Proofs of the main theorems

6.1 Proof of Theorem 1

(i) The convergence of the statistics $\widehat{H}_n^{(1)}$ and $\widehat{H}_n^{(2)}$ considered in Theorem 1 follows from Lemma 1. Indeed, the asymptotics of the increments of the solution X of equation (1) are the same as the asymptotics of the increments of the solution of the equation with polynomial drift in [14]. Thus, in order to establish the convergence of the estimator $\widehat{H}_n^{(1)}$, it suffices to repeat the proof of Theorem 2 in [14]. Further, note that hypotheses (H) and (H_1) in [13] are satisfied for the solution of equation (1), i.e.,

$$\begin{aligned}\Delta X_{\tau_k^{m_n}} &= O_\omega\bigl(d_n^{H-\varepsilon}\bigr), \quad k=1,\dots,m_n,\\ \Delta^{(2)} X_{\tau_k^{m_n}} &= \sigma X_{\tau_{k-1}^{m_n}}\Delta^{(2)} B^H_{\tau_k^{m_n}} + O_\omega\bigl(d_n^{2(H-\varepsilon)}\bigr), \quad k=2,\dots,m_n.\end{aligned}$$

It follows from Lemma 1 and the a.s. continuity of $t\mapsto X_t$. Thus, it suffices to apply Theorem 2.2 in [13].

(ii) Now we prove the convergence of the statistic $\widehat{H}_n^{(3)}$. The proof presented below follows the outline of the proof of Theorem 3.18 in [2]. By Lemma 1 we get

$$\begin{aligned}&\left(\frac{n^H}{\sigma T^H\sqrt{4-2^{2H}}}\right)^2\frac{V_{n,T}^{(2)}}{n-1}\\ &\quad=\left(\frac{n^H}{T^H\sqrt{4-2^{2H}}}\right)^2\frac{1}{n-1}\sum_{i=1}^{n-1}\left[\bigl(\Delta_n^{(2)}B_i^H\bigr)^2+O_\omega\bigl(n^{-3(H-\varepsilon)}\bigr)\right]\\ &\quad=\frac{n}{n-1}\widehat{V}_{n,T}^{(2)B^H}+\frac{1}{4-2^{2H}}O_\omega\bigl(n^{-H+3\varepsilon}\bigr)\xrightarrow{\text{a.s.}}1.\end{aligned}\tag{14}$$

Assume that $3\varepsilon<H-1/2$. By (14) and Theorem 4 we get

$$\begin{aligned}\ln\frac{V_{n,T}^{(2)}}{n-1} &= -2H\ln\frac{n}{T}+2\ln\bigl(\sigma\sqrt{4-2^{2H}}\bigr)+\ln\frac{n}{n-1}\\ &\quad+\ln\left[\bigl(\widehat{V}_{n,T}^{(2)B^H}-1\bigr)+1+\frac{n-1}{n(4-2^{2H})}O_\omega\bigl(n^{-H+3\varepsilon}\bigr)\right]\\ &= -2H\ln\frac{n}{T}+2\ln\bigl(\sigma\sqrt{4-2^{2H}}\bigr)+\ln\frac{n}{n-1}\\ &\quad+\ln\bigl[O_\omega(n^{-1/2}\ln^{1/2}n)+1+O_\omega\bigl(n^{-H+3\varepsilon}\bigr)\bigr]\\ &= -2H\ln\frac{n}{T}+2\ln\bigl(\sigma\sqrt{4-2^{2H}}\bigr)+O_\omega\bigl(n^{-1/2}\ln^{1/2}n\bigr).\end{aligned}$$

Thus,

$$\begin{aligned}\widehat{H}_n^{(3)} &= -\frac{1}{2}\sum_{i=1}^{\ell}z_i\left[-2H\ln\frac{n_i}{T}+2\ln\bigl(\sigma\sqrt{4-2^{2H}}\bigr)\right]\\ &\quad+O_\omega\bigl(n^{-1/2}\ln^{1/2}n\bigr).\end{aligned}$$

We will notice the following properties:

$$\sum_{i=1}^{\ell} y_i = 0, \qquad \sum_{i=1}^{\ell} z_i y_i = 1, \qquad \sum_{i=1}^{\ell} z_i = \frac{\sum_{i=1}^{\ell} y_i}{\sum_{i=1}^{\ell} y_i^2} = 0.$$

Using those, we get

$$\begin{aligned}
\widehat{H}_n^{(3)} &= H \sum_{i=1}^{\ell} z_i \ln(r_i n) - \ln\bigl(\sigma\sqrt{4-2^{2H}}\bigr) \sum_{i=1}^{\ell} z_i + O_\omega\bigl(n^{-1/2}\ln^{1/2} n\bigr) \\
&= H \sum_{i=1}^{\ell} z_i \ln(r_i n) + O_\omega\bigl(n^{-1/2}\ln^{1/2} n\bigr) \\
&= H \sum_{i=1}^{\ell} z_i \bigl[y_i - y_i + \ln(r_i n)\bigr] + O_\omega\bigl(n^{-1/2}\ln^{1/2} n\bigr) \\
&= H + H \sum_{i=1}^{\ell} z_i \left[\ln n + \frac{1}{\ell}\sum_{i=1}^{\ell} \ln r_i\right] + O_\omega\bigl(n^{-1/2}\ln^{1/2} n\bigr) \\
&= H + O_\omega\bigl(n^{-1/2}\ln^{1/2} n\bigr).
\end{aligned} \tag{15}$$

So the estimator $\widehat{H}_n^{(3)}$ is strongly consistent.

Now we prove the asymptotic normality of the estimator $\widehat{H}_n^{(3)}$. From (14) and (15) it follows that

$$\widehat{H}_n^{(3)} = H - \frac{1}{2}\sum_{i=1}^{\ell} z_i \bigl(\widehat{V}_{n_i,T}^{(2)B^H} - 1\bigr) + O_\omega\bigl(n^{-H+3\varepsilon}\bigr).$$

Thus,

$$\begin{aligned}
\sqrt{n}\bigl(\widehat{H}_n^{(3)} - H\bigr) = {}& -\frac{1}{2}\sum_{i=1}^{\ell} \frac{z_i}{\sqrt{r_i}} \left[\frac{1}{\sqrt{r_i n}} \sum_{k=1}^{r_i n-1} \left(\left(\frac{(r_i n)^H}{T^H\sqrt{4-2^{2H}}} \Delta^{(2)} B^H\bigl(t_k^{r_i n}\bigr)\right)^2 - 1\right)\right] \\
&+ O_\omega\bigl(n^{1/2-H+3\varepsilon}\bigr),
\end{aligned}$$

and we obtain the asymptotic normality of the estimator $\widehat{H}_n^{(3)}$ by the application of the limit results from Section 4.2.

(iii) It remains to determine the convergence of $\widehat{H}_n^{(4)}$. Denote

$$\begin{aligned}
R^{2,n}(X) &= \frac{1}{n^4-2} \sum_{k=1}^{n^4-2} \frac{|\Delta^{(2)} X(\tau_k^{m_n}) + \Delta^{(2)} X(\tau_{k+1}^{m_n})|}{|\Delta^{(2)} X(\tau_k^{m_n})| + |\Delta^{(2)} X(\tau_{k+1}^{m_n})|}, \\
\Lambda_2(H) &= \mathbf{E} \frac{|\Delta^{(2)} B_1^H + \Delta^{(2)} B_2^H|}{|\Delta^{(2)} B_1^H| + |\Delta^{(2)} B_2^H|},
\end{aligned}$$

where $\Delta^{(2)} B_j^H = B^H(j+1) - 2B^H(j) + B^H(j-1)$, $j = 1, 2$. This statistic was introduced in [1]. Further on, we will require the following lemma, which is a simple modification of Lemma 3.1 in [1]. In this lemma, we have lifted the requirement for the random variables Z_1 and Z_2 to be independent. This became possible due to the application of less precise estimators of the partial derivatives.

Lemma 2. *Let $\psi(x_1, x_2) = (|x_1 + x_2|)/(|x_1| + |x_2|)$, $x_1, x_2 \in \mathbb{R}$, and let (Z_1, Z_2) be a Gaussian vector with zero mean and variance $\mathbf{E}Z_i^2 = 1$, $i = 1, 2$. Then for any r.v. ξ_i, $i = 1, 2$, with finite second moments, we have*

$$\mathbf{E}\bigl|\psi(Z_1 + \xi_1, Z_2 + \xi_2) - \psi(Z_1, Z_2)\bigr| \leqslant 23 \max_{i=1,2} \sqrt[3]{\mathbf{E}\xi_i^2}. \tag{16}$$

Let us proceed to the following claim.

Proposition 2. *Let X be the solution of the fractional Gompertz SDE observed at times $\tau_k^{m_n} = (k/n^4)T$, $k = 0, 1, \ldots, n^4$. Then*

$$R^{2,n}(X) \overset{\text{a.s.}}{\to} \Lambda_2(H) \quad \textit{as } n \to \infty \textit{ for } H \in (1/2, 1).$$

Proof. For the sake of simplicity, we will omit the index m_n for the points $\tau_k^{m_n}$ and denote $d_n = T/n^4$. From Lemma 1 it follows that

$$\Delta^{(2)} X_{\tau_k} + \Delta^{(2)} X_{\tau_{k+1}} = \sigma X_{\tau_k}\bigl[\Delta^{(2)} B^H_{\tau_k} + \Delta^{(2)} B^H_{\tau_{k+1}} + \zeta_1 + \zeta_2\bigr]$$

for every $\varepsilon \in (0, H - 1/2)$, where

$$\begin{aligned} \zeta_1 &= O_\omega\bigl(d_n^{2(H-\varepsilon)}\bigr), \\ \zeta_2 &= O_\omega\bigl(d_n^{H-\varepsilon}\bigr)\bigl[\sigma\Delta^{(2)} B^H_{\tau_{k+1}} + O_\omega\bigl(d_n^{2(H-\varepsilon)}\bigr)\bigr] = O_\omega\bigl(d_n^{2(H-\varepsilon)}\bigr). \end{aligned}$$

Therefore

$$R^{2,n}(X) = \frac{1}{n^4 - 2} \sum_{k=1}^{n^4-2} \frac{|Z_1 + Z_2 + \xi_1 + \xi_2|}{|Z_1 + \xi_1| + |Z_2 + \xi_2|},$$

where

$$\begin{aligned} Z_1 &= \frac{1}{d_n^H \sqrt{4 - 2^{2H}}}\, \Delta^{(2)} B^H_{\tau_k}, &\qquad \xi_1 &= \frac{\zeta_1}{d_n^H \sqrt{4 - 2^{2H}}}, \\ Z_2 &= \frac{1}{d_n^H \sqrt{4 - 2^{2H}}}\, \Delta^{(2)} B^H_{\tau_{k+1}}, &\qquad \xi_2 &= \frac{\zeta_2}{d_n^H \sqrt{4 - 2^{2H}}}, \end{aligned}$$

and

$$\mathbf{E}Z_1^2 = \frac{n^{8H}}{T^{2H}(4 - 2^{2H})}\, \mathbf{E}\bigl(\Delta^{(2)} B^H_{\tau_k}\bigr)^2 = 1.$$

Let us apply Lemma 2. From the inequality (16) it follows that

$$\begin{aligned} \mathbf{E}\bigl|R^{2,n}(X) - R^{2,n}\bigl(B^H\bigr)\bigr| &= \left(\frac{d_n^{2(H-2\varepsilon)}}{4 - 2^{2H}}\right)^{1/3} \sqrt[3]{\mathbf{E}O_\omega(1)} \\ &= d_n^{2(H-2\varepsilon)/3} \sqrt[3]{\mathbf{E}O_\omega(1)}. \end{aligned}$$

Then the Chebyshev's inequality yields

$$\mathbf{P}\big(\big|R^{2,n}(X)-R^{2,n}(B^H)\big|>n^{-\beta}\big)\leqslant n^{\beta}d_n^{2(H-2\varepsilon)/3}\sqrt[3]{\mathbf{E}O_\omega(1)}$$
$$<T^{2(H-2\varepsilon)/3}n^{\beta-8(H-2\varepsilon)/3}\sqrt[3]{\mathbf{E}O_\omega(1)}$$

for $\varepsilon\in(0,(H-1/2)/2)$, $0<\beta<1/3$ and

$$\sum_{n=1}^{\infty}\mathbf{P}\big(\big|R^{2,n}(X)-R^{2,n}\big(B^H\big)\big|>n^{-\beta}\big)\leqslant\sqrt[3]{\mathbf{E}O_\omega(1)}\sum_{n=1}^{\infty}n^{\beta-8(H-2\varepsilon)/3}<\infty.$$

According to the Borel–Cantelli lemma,

$$\mathbf{P}\Big(\limsup_{n\to\infty}\big\{\big|R^{2,n}(X)-R^{2,n}\big(B^H\big)\big|>n^{-\beta}\big\}\Big)=0,$$

which implies that $R^{2,n}(X)\overset{\text{a.s.}}{\to}R^{2,n}(B^H)$, $n\to\infty$.

The convergence $R^{2,n}(B^H)\overset{\text{a.s.}}{\to}\Lambda_2(H)$, $n\to\infty$ is established in [1] and holds for $H\in(0;1)$. Clearly, provided $R^{2,n}(X)\overset{\text{a.s.}}{\to}R^{2,n}(B^H)$ and $R^{2,n}(B^H)\overset{\text{a.s.}}{\to}\Lambda_2(H)$, $n\to\infty$, it follows that $R^{2,n}(X)\overset{\text{a.s.}}{\to}\Lambda_2(H)$, $n\to\infty$, which completes the proof. □

The estimator $\widehat{H}_n^{(4)}$ based on $R^{2,n}(X)$ can be obtained using the approximation formula provided in [1, Remark 4.3].

6.2 Proof of Theorem 2

The proof of the convergence of $\widehat{\sigma}^2_{2,n}$ is analogous to that of $\widehat{c}^2_n$ in [14]. Let us prove that $\widehat{\sigma}^2_{1,n}\overset{\text{a.s.}}{\to}\sigma^2$ as $n\to\infty$. Suppose that $d_n=T/n$. From Lemma 1 it follows that

$$d_n^{-2H}n^{-1}V_{n,T}^{(1)}=\sigma^2d_n^{-2H}n^{-1}\sum_{i=1}^{n}\big(\Delta B^H_{t_k^n}\big)^2+d_n^{-2H}O_\omega\big(d_n^{1+H-\varepsilon}\big)$$
$$=\sigma^2\widehat{V}_{n,T}^{(1)B^H}+O_\omega\big(d_n^{1-H-\varepsilon}\big).$$

Since

$$\widehat{V}_{n,T}^{(1)B^H}\overset{\text{a.s.}}{\to}1\quad\text{and}\quad\frac{n^{2(H-\widehat{H}_n)}}{T^{2(H-\widehat{H}_n)}}=\exp\biggl\{o_\omega(\phi(n))\ln\biggl(\frac{n}{T}\biggr)^2\biggr\}\to1,\tag{17}$$

it can be concluded that

$$\widehat{\sigma}^2_{1,n}=\frac{n^{2\widehat{H}_n-1}}{T^{2\widehat{H}_n}}V_{n,T}^{(1)}\overset{\text{a.s.}}{\to}\sigma^2.$$

Further, let us prove that $\widehat{\sigma}^2_{3,n}\overset{\text{a.s.}}{\to}\sigma^2$. Denote $d_n=T/n$. By (17) it suffices to show that

$$\widetilde{\sigma}^2_n=\frac{\sum_{k=1}^n(\Delta^{(1)}X_{t_k^n})^2}{d_n^{2H}\sum_{k=1}^nX^2_{t_{k-1}^n}}\overset{\text{a.s.}}{\to}\sigma^2.$$

Notice that

$$\frac{\sum_{k=1}^{n}\big(\Delta^{(1)}X_{t_k^n}\big)^2}{d_n^{2H}\sum_{k=1}^{n}X_{t_{k-1}^n}^2} = \frac{d_n^{1-2H}\sum_{k=1}^{n}\big(\Delta^{(1)}X_{t_k^n}\big)^2}{d_n\sum_{k=1}^{n}X_{t_{k-1}^n}^2}$$

and

$$d_n^{1-2H}\sum_{k=1}^{n}\big(\Delta^{(1)}X_{t_k^n}\big)^2 = \sigma^2 d_n^{1-2H}\sum_{k=1}^{n}X_{t_{k-1}^n}^2\big(\Delta B_{t_k^n}^H\big)^2 + O_\omega\big(d_n^{1-H-\varepsilon}\big). \tag{18}$$

In order to estimate (18), observe that

$$d_n^{1-2H}\sum_{k=1}^{n}X_{t_{k-1}^n}^2\big(\Delta B_{t_k^n}^H\big)^2 = \int_0^T X_t^2\,\mathrm{d}\widehat{V}_{nt}^{(1)B^H}$$

and (see [12, Thm. 7])

$$d_n^{1-2H}\sum_{k=1}^{n}X_{t_{k-1}^n}^2\big[\big(\Delta B_{t_k^n}^H\big)^2 - \mathbf{E}\big(\Delta B_{t_k^n}^H\big)^2\big] = \int_0^T X_t^2\,\mathrm{d}\big(V_{nt}^{(1)B^H} - \mathbf{E}V_{nt}^{(1)B^H}\big) \overset{\text{a.s.}}{\to} 0.$$

Since

$$d_n^{1-2H}\sum_{k=1}^{n}X_{t_{k-1}^n}^2\mathbf{E}\big(\Delta B_{t_k^n}^H\big)^2 = d_n\sum_{k=1}^{n}X_{t_{k-1}^n}^2 \overset{\text{a.s.}}{\to} \int_0^T X_t^2\,\mathrm{d}t,$$

then

$$\widetilde{\sigma}_n^2 \overset{\text{a.s.}}{\to} \sigma^2.$$

6.3 The convergence rate of $\widehat{H}_n^{(i)}$, $i = 1, 2, 3$

Theorem 2 makes use of the conditions $\widehat{H}_n = H + o_\omega(\phi(n))$, $\phi(n) = o(\ln^{-1} n)$ for strong consistency. Let us show that this indeed holds for $\widehat{H}_n^{(i)}$, $i = 1, 2, 3$.

The convergence rate of $\widehat{H}_n^{(1)}$. From Lemma 1 and the proof of Theorem 2 in [14] it follows that

$$\widehat{H}_n^{(1)} = \widetilde{H}_n + O_\omega\big(n^{-H+3\varepsilon}\big),$$

where

$$\widetilde{H}_n = \frac{1}{2} - \frac{1}{2\ln 2}\ln\frac{\widehat{V}_{2n,T}^{(2)B^H}}{2^{2H-1}\widehat{V}_{n,T}^{(2)B^H}} = H - \frac{1}{2\ln 2}\ln\frac{\widehat{V}_{2n,T}^{(2)B^H}}{\widehat{V}_{n,T}^{(2)B^H}}. \tag{19}$$

It suffices to consider the convergence rate of the logarithmic term in equation (19). Using Theorem 4, we get

$$\ln\frac{\widehat{V}_{2n,T}^{(2)B^H}}{\widehat{V}_{n,T}^{(2)B^H}} = \ln\frac{1 + O_\omega((2n)^{-1/2}\ln^{1/2}(2n))}{1 + O_\omega(n^{-1/2}\ln^{1/2} n)} = \ln\big(1 + o_\omega\big(n^{-1/2}\ln n\big)\big)$$
$$= o_\omega\big(n^{-1/2}\ln n\big).$$

Then the statistic $\widetilde{H}_n$ has the convergence rate of $o_\omega(n^{-1/2}\ln n)$. Consequently, $\widehat{H}_n^{(1)}$ satisfies the required condition if $\varepsilon < (H-1/2)/3$.

The convergence rate of $\widehat{H}_n^{(2)}$. Denote

$$S_{n,T} := \frac{2}{nk_n^{2H-1}} \sum_{k=2}^{n} \frac{(\Delta^{(2)} X_{t_k^n})^2}{W_{n,k-1}}.$$

Then

$$\widehat{H}_n^{(2)} = H + \frac{\ln S_{n,T}}{2\ln k_n} = H + \frac{\ln S_{n,T}}{4\ln n}.$$

Proceeding along the lines of the proof of Theorem 2.2 from [13], it can be concluded that

$$\begin{aligned} S_{n,T} &= \frac{\widetilde{V}_{n,T}^{B^H} + O_\omega(n^{-(H-3\varepsilon)})}{1 + O_\omega(k_n^{-1/2}\ln^{1/2} n) + O_\omega(m_n^{2\varepsilon} n^{-(H-\varepsilon)})} \\ &= \frac{\widetilde{V}_{n,T}^{B^H} + O_\omega(n^{-(H-3\varepsilon)})}{1 + O_\omega(n^{-1}\ln^{1/2} n) + O_\omega(n^{-(H-7\varepsilon)})}. \end{aligned}$$

If $\varepsilon < (H-1/2)/7$, then

$$S_{n,T} = \frac{1 + O_\omega(n^{-1/2}\ln^{1/2} n)}{1 + O_\omega(n^{-(H-7\varepsilon)})} = 1 + O_\omega\big(n^{-1/2}\ln n\big).$$

Hence, $\widehat{H}_n^{(2)} = H + o_\omega(1/\ln n)$ if $\varepsilon < (H-1/2)/7$.

The convergence rate of $\widehat{H}_n^{(3)}$ was obtained in the proof of Theorem 1.

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Appendix

A.1 Auxiliary results

Firstly, we consider a non-random integral equation

$$x_t = x_0 + \int_0^t (\alpha x_s - \beta x_s \ln x_s)\,\mathrm{d}s + \sigma \int_0^t x_s\,\mathrm{d}h_s, \quad x_0 \geqslant 0,\ \beta \neq 0,\ 0 \leqslant t \leqslant T, \tag{A.1}$$

where $h \in C\mathcal{W}_p([0,T])$, $1 < p < 2$, and prove two auxiliary theorems used in the sequel.

Theorem A.1. *The function*

$$x_t = \exp\left\{ \mathrm{e}^{-\beta t} \ln x_0 + \frac{\alpha}{\beta}\bigl(1 - \mathrm{e}^{-\beta t}\bigr) + \sigma \int_0^t \mathrm{e}^{-\beta(t-s)}\,\mathrm{d}h_s \right\}, \quad t \in [0,T], \tag{A.2}$$

is an element of $C\mathcal{W}_p([0,T])$, $1 < p < 2$, *and satisfies equation* (A.1).

Proof. We show that $x \in C\mathcal{W}_p([0,T])$, $1 < p < 2$. Let

$$z_t = \mathrm{e}^{-\beta t} \ln x_0 + \frac{\alpha}{\beta}\bigl(1 - \mathrm{e}^{-\beta t}\bigr) + \sigma \int_0^t \mathrm{e}^{-\beta(t-s)}\,\mathrm{d}h_s.$$

It is evident that $z \in C\mathcal{W}_p([0,T])$, $1 < p < 2$. Thus, by the property of composition of functions (see Section 4.1) we get $x \in C\mathcal{W}_p([0,T])$, $1 < p < 2$.

Now we verify that the function (A.2) satisfies (A.1). This statement can be checked by the application of the Chain rule and the Substitution rule. Namely, let $F(t,x,y) = \exp\{\mathrm{e}^{-\beta t}(\ln x_0 + \alpha x + \sigma y)\}$ and denote

$$A_t = \int_0^t \mathrm{e}^{\beta s}\,\mathrm{d}s, \qquad C_t = \int_0^t \mathrm{e}^{\beta s}\,\mathrm{d}h_s.$$

Note that $x_t = F(t, A_t, C_t)$ and

$$\begin{aligned} F(t, A_t, C_t) = F(0;0;0) &+ \int_0^t \partial_t F(s, A_s, C_s)\,\mathrm{d}A_s + \int_0^t \partial_x F(s, A_s, C_s)\,\mathrm{d}C_s \\ &+ \int_0^t \partial_y F(s, A_s, C_s)\,\mathrm{d}s. \end{aligned} \tag{A.3}$$

It follows from (A.3) and Proposition 1

$$\begin{aligned} x_t &= x_0 - \beta \int_0^t x_s \ln x_s\,\mathrm{d}s + \alpha \int_0^t x_s \mathrm{e}^{-\beta s}\,\mathrm{d}A_s + \sigma \int_0^t x_s \mathrm{e}^{-\beta s}\,\mathrm{d}C_s \\ &= x_0 - \beta \int_0^t x_s \ln x_s\,\mathrm{d}s + \alpha \int_0^t x_s\,\mathrm{d}s + \sigma \int_0^t x_s\,\mathrm{d}h_s \end{aligned}$$

since $\mathrm{d}A_s = \mathrm{e}^{\beta s}\,\mathrm{d}s$, $\mathrm{d}C_s = \mathrm{e}^{\beta s}\,\mathrm{d}h_s$. □

Theorem A.2. *The integral equation* (A.1) *has a unique solution in* $C\mathcal{W}_p([0,T])$, $1 < p < 2$.

Proof. We have already shown that at least one solution $x \in C\mathcal{W}_p([0,T])$ exists. Assume it is not unique and $y \in C\mathcal{W}_p([0,T])$ is a different one.

Further, one can find a set of points $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n = T$, which satisfies

$$V_p\big(h; [\tau_{k-1}, \tau_k]\big) \leqslant \frac{1}{4|\sigma|C_{p,p}}$$

for all k. Assume we have proved that $x_{\tau_{k-1}} = y_{\tau_{k-1}}$.

Using the well-known inequality $\ln(1+x) \leqslant x$, $x > 0$, we get

$$|\ln x_s - \ln y_s| = \left|\ln\left(1 + \frac{y_t - x_t}{x_t}\right)\right| \leqslant \left|\frac{y_t - x_t}{x_t}\right| \leqslant L_{x,T}|y_t - x_t|$$

and

$$|\ln x_s| \leqslant \left|\ln\Big(\max_{0\leqslant t\leqslant T} x_s\Big)\right| =: \widehat{L}_{x,T},$$

where $L_{x,T} = (\min_{0\leqslant t\leqslant T}|x_t|)^{-1} > 0$. Then

$$\begin{aligned}
&V_{p,\infty}\big(x - y; [\tau_{k-1}, \tau_k]\big) \\
&\quad = V_{p,\infty}\big(x - y - (x_{\tau_{k-1}} - y_{\tau_{k-1}}); [\tau_{k-1}, \tau_k]\big) \\
&\quad \leqslant 2|\alpha| \int\limits_{\tau_{k-1}}^{\tau_k} |x_t - y_t|\,\mathrm{d}t + 2|\beta| \int\limits_{\tau_{k-1}}^{\tau_k} |x_t \ln x_t - y_t \ln y_t|\,\mathrm{d}t \\
&\qquad + 2|\sigma|C_{p,p}V_{p,\infty}\big(x - y; [\tau_{k-1}, \tau_k]\big)V_p\big(h; [\tau_{k-1}, \tau_k]\big) \\
&\quad \leqslant 2\big(|\alpha| + |\beta|\widehat{L}_{x,T} + |\beta|L_{x,T}\big) \int\limits_{\tau_{k-1}}^{\tau_k} |x_t - y_t|\,\mathrm{d}t \\
&\qquad + 2|\sigma|C_{p,p}V_{p,\infty}\big(x - y; [\tau_{k-1}, \tau_k]\big)V_p\big(h; [\tau_{k-1}, \tau_k]\big)
\end{aligned}$$

and

$$\begin{aligned}
&V_{p,\infty}\big(x - y; [\tau_{k-1}, \tau_k]\big) \\
&\quad \leqslant 4\big(|\alpha| + |\beta|\widehat{L}_{x,T} + |\beta|L_{x,T}\big) \int\limits_{\tau_{k-1}}^{\tau_k} |x_t - y_t|\,\mathrm{d}t \\
&\quad \leqslant 4\big(|\alpha| + |\beta|\widehat{L}_{x,T} + |\beta|L_{x,T}\big) \int\limits_{\tau_{k-1}}^{\tau_k} V_{p,\infty}\big(x - y; [\tau_{k-1}, t]\big)\,\mathrm{d}t.
\end{aligned}$$

Therefore by Gronwall's inequality $V_{p,\infty}(x - y; [\tau_{k-1}, \tau_k]) = 0$, and we can conclude that $x = y$ on $[\tau_{k-1}, \tau_k]$. Since $x_{\tau_0} = x_0 = y_{\tau_0}$, the claim of the theorem follows from the repetitive application of the reasoning explained above. □

A.2 The solution of SDE

Since almost all sample paths of B^H, $1/2 < H < 1$, are continuous and have bounded $H_\varepsilon = 1/(H-\varepsilon)$-variation, $\varepsilon \in (0, H-1/2)$, the pathwise Riemann–Stieltjes integral $\int_0^t X_s \,\mathrm{d}B_s^H$ exists for $X \in C\mathcal{W}_{H_\varepsilon}([0,T])$. So SDE (1) is well defined for almost all ω, and the obtained result for a non-random integral equation can be applied to an equation driven by fBm.

Theorem A.3. *Suppose that $X_0 > 0$ and $m \geqslant 2$. The stochastic process*

$$X_t = \exp\left\{ \mathrm{e}^{-\beta t} \ln x_0 + \frac{\alpha}{\beta}\left(1 - \mathrm{e}^{-\beta t}\right) + \sigma \int_0^t \mathrm{e}^{-\beta(t-s)} \,\mathrm{d}B_s^H \right\}, \quad \beta \neq 0,\ 0 \leqslant t \leqslant T,$$

for almost all ω belongs to $C\mathcal{W}_{H_\varepsilon}([0,T])$ and is the unique solution of (1).

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