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Critical blow-up exponents for a nonlocal reaction-diffusion equation with nonlocal source and interior absorption*

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Abstract. This paper is concerned with a nonlocal reaction-diffusion equation with the nonlocal source and interior absorption with Dirichlet conditions or Neumann conditions. We investigate the critical blow-up exponents of the problem by constructing adequate supersolutions and subsolutions. Moreover, we show that the blowup rate is the same as the corresponding ODE under the appropriate hypotheses.

Keywords: nonlocal reaction-diffusion, blow-up rate, critical exponents.

1 Introduction

In this work we analyze some features of the blow-up phenomenon arising in nonlocal diffusion problems associated to the nonlocal Laplacian equation. More precisely, we will study the Neumann problem in $\Omega \times (0,T)$

$$\begin{aligned} &u_t = \int_{\Omega} J(x-y)\big(u(y,t)-u(x,t)\big)\,\mathrm{d}y + \int_{\Omega} u^q\,\mathrm{d}x - ku^p, \quad x\in\Omega,\ t\in(0,T),\\ &u(x,0)=u_0(x), \quad x\in\Omega, \end{aligned} \tag{1}$$

and the Dirichlet problem in $\mathbb{R}^N \times (0,T)$,

$$\begin{aligned} &u_t = \int_{\mathbb{R}^N} J(x-y)\big(u(y,t)-u(x,t)\big)\,\mathrm{d}y + \int_{\Omega} u^q\,\mathrm{d}x - ku^p, \quad x\in\Omega,\ t\in(0,T),\\ &u(x,t)=0, \quad x\in\mathbb{R}^N\setminus\Omega,\ t\in(0,T), \qquad u(x,0)=u_0(x), \quad x\in\Omega, \end{aligned} \tag{2}$$

where $\Omega\ (\supset B_1)$ is a bounded connected C^1 domain, B_1 is the the unit ball, $k,p,q>0$, the kernel $J\in C(\mathbb{R}^N)$ verifies $J>0$ in B_1, $J=0$ in $\mathbb{R}^N\setminus B_1$, $J(-z)=J(z)$ with

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$\int_{B_1} J(z)\,\mathrm{d}z = 1$, and the initial datum $u_0(x)$ is a nonnegative, nontrivial, bounded and continuous function. In model (2), we prescribe the values of u outside Ω, which is the analogous of prescribing the so called Dirichlet boundary conditions for the classical heat equation. However, the boundary data is not understood in the usual sense (see [1, 8]).

Nonlocal evolution equations of the form

$$\frac{\partial}{\partial t}u(x,t) = J * u - u(x,t) = \int\limits_{\mathbb{R}^N} J(x-y)u(y,t)\,\mathrm{d}y - u(x,t) \tag{3}$$

have been widely used to model the dispersal of a species recently (see [2,3,4,5,7,9,10,11] and references therein). More precisely, as stated in [11], if $u(x,t)$ and $J(x-y)$ are thought to be the density of a species at the point x at time t and the probability distribution of jumping from location y to location x, respectively, then $\int_{\mathbb{R}^N} J(x-y)u(y,t)\,\mathrm{d}y$ is the rate at which individuals are arriving to position x from all other places, and $-u(x,t) = -\int_{\mathbb{R}^N} J(x-y)u(x,t)\,\mathrm{d}y$ is the rate at which they are leaving location x to travel to all other sites. It is well known that equation (3) and the classical heat equation $u_t = \Delta u$ have some similar properties, such as the maximum principle and perturbations propagate with infinite speed [11]. Lately, the blow-up problem for a nonlocal diffusion equation with a reaction term

$$u_t = \int\limits_{\Omega} J(x-y)\big(u(y,t) - u(x,t)\big)\,\mathrm{d}y + u^p(x,t) \tag{4}$$

were considered in [16] with Neumann boundary conditions and nonnegative and nontrivial initial data. They found the a critical exponent $p = 1$, namely, if $p > 1$, the corresponding solution to (4) (with Neumann boundary conditions) blows up. Conversely, if $p \leqslant 1$, every solution to (4) is global. More recently, Zhou et al. [21] investigated a nonlocal problem of the following form:

$$u_t = \int\limits_{\Omega} J(x-y)\big(u(y,t) - u(x,t)\big)\,\mathrm{d}y + \delta \mathrm{e}^u(x,t) \tag{5}$$

with Neumann boundary conditions and nonnegative and nontrivial initial data. They showed the local existence and uniqueness of the solution to (5). Furthermore, under appropriate hypotheses, they gave the estimates of the blow-up rate and obtained that the blow-up set is a single point $x = 0$ for radially symmetric solution with a single maximum at the origin. It is noted that, by invoking the regularizing effect, Souplet and Wang et al. [17, 18] studied the blow-up properties of solutions for the Dirichlet (or Neumann) boundary value problem of the reaction-diffusion equation

$$u_t = \mathrm{d}\Delta u + \int\limits_{\Omega} u^q(x,t)\,\mathrm{d}x - ku^p$$

with $p, q \geqslant 1$ and $k, d > 0$. For other related results, which concerned the blow-up or extinction of solutions for reaction-diffusion equation with the nonlocal source and

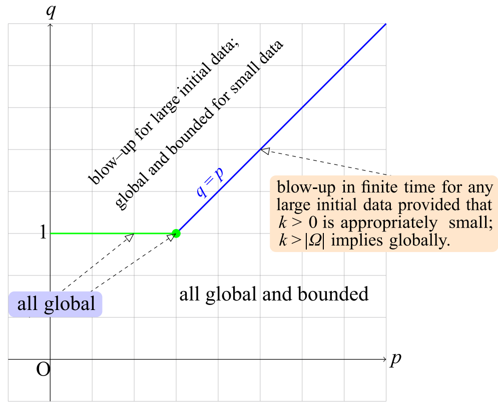

Figure 1

interior absorption, one can see [12, 15, 19, 20, 22] and references therein. In this paper, under the appropriate hypotheses $k, p, q > 0$, we discuss systems (2) and (1) and obtain the critical blow-up exponents and the blow-up rate based on the first principal eigenvalue of the nonlocal heat equation, and thus avoid using the regularizing effect, since there is no regularizing effect in general [6]. Unfortunately, due to the lack of compactness of the minimizer sequence [10], our method cannot be used to solve the blow-up for the nonlocal p-Laplacian equation with a reaction term, which was discussed by Ferreira and Pérez-Llanos [10].

Motivated by the above works, the purpose of this paper is to analyze the blow-up phenomenon for problems (1) and (2), that is, we want to show that problems (1) and (2) share many important properties with the corresponding reaction diffusion equation

$$\begin{aligned} &u_t = \Delta u + \int_\Omega u^q \,\mathrm{d}x - ku^p, \quad x \in \Omega,\ t \in (0, T), \\ &\frac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega,\ t \in (0, T), \qquad u(x, 0) = u_0(x), \quad x \in \Omega, \end{aligned} \tag{6}$$

and

$$\begin{aligned} &u_t = \Delta u + \int_\Omega u^q \,\mathrm{d}x - ku^p, \quad x \in \Omega,\ t \in (0, T), \\ &u(x, t) = 0, \quad x \in \partial\Omega,\ t \geqslant 0, \qquad u(x, 0) = u_0(x), \quad x \in \Omega, \end{aligned} \tag{7}$$

such as blow-up condition and blow-up rates [17, 18]. Through the main points, we can obtain that there exists a critical curvilinear line $q^* = \max(p, 1)$ such that the (q, p)-parameter plane is divided into three parts, with the bottom part corresponding to all global and bounded solution and the top part of the line corresponding to blow-up for large initial data (global and bounded for small data). Moreover, there exists a critical point on this line such that the line is also divided into three parts, which exhibit different features of blow-up phenomenon (see Fig. 1).

Before giving our main results, we will give some preliminary lemmas, which plays a crucial role in the following proofs.

Let $\psi(x)$ satisfy

$$\begin{aligned} &\int_{\mathbb{R}^N} J(x-y)\big(\psi(y)-\psi(x)\big)\,\mathrm{d}y = -\lambda\psi(x), \quad x \in \Omega, \\ &\psi(x) = 0, \quad x \notin \bar{\Omega}. \end{aligned} \tag{8}$$

Due to Theorem 2.1 of [14], problem (8) admits a unique eigenvalue $\lambda_1 := \lambda_1(\Omega)$ associated to a positive eigenfunction $\psi \in C(\bar{\Omega})$. In the following, we will always assume

$$m = \min_{\bar{\Omega}} \psi(x) > 0 \quad \text{and} \quad M = \max_{\bar{\Omega}} \psi(x) > 0. \tag{9}$$

Lemma 1. (*See* [18, *Lemma* 3.1].) *Let $f(t)$ satisfy the following ODE problem*:

$$\begin{aligned} &f'(t) = f^q(t)|\Omega| - kf^p(t), \quad t > 0, \\ &f(0) = f_0 > 0, \quad p, q \geqslant 1. \end{aligned} \tag{10}$$

(i) *If $q < p$, then $f(t)$ exists globally.*
(ii) *If $p = q = 1$, or $p = q > 1$ and $|\Omega| < k$, then $f(t)$ exists globally*; *if $p = q > 1$ and $|\Omega| > k$, then $f(t)$ blows up in finite time.*
(iii) *If $q > p$, then there exists $f^* > 0$ such that $f(t)$ exists globally when $f_0 < f^*$ and $f(t)$ blows up in finite time when $f_0 > f^*$.*

Now our main results can be stated as follows.

Theorem 1 [Global existence of Dirichlet problem].

(i) *If $q \leqslant 1$, then every solution to problem* (2) *is global.*
(ii) *If $p > q > 1$, then every solution to problem* (2) *is global.*
(iii) *If $p = q > 1$ and $|\Omega| \leqslant k$, then every solution to problem* (2) *is global.*
(iv) *If $q > p > 1$, then problem* (2) *has global solutions for any conveniently small initial data.*
(v) *If $q = p > 1$ and $|\Omega| > k$, then problem* (2) *has global solutions for any conveniently small initial data.*

Theorem 2 [Blow-up and blow-up rate of Dirichlet problem].

(i) *If $q = p > 1$ and k is small enough such that $0 < kM^p < \int_\Omega \psi^q(x)\,\mathrm{d}x$ holds, then the solution of problem* (2) *blows up in finite time for initial data sufficiently large, where M and $\psi(x)$ are given by* (9) *and* (8), *respectively. Moreover, let $k < |\Omega|$ and u be a solution to problem* (2), *which blows up at time T. Then,*

$$\lim_{t\to T}(T-t)^{1/(p-1)}\big|u(t)\big|_\infty = \big[(p-1)\big(|\Omega|-k\big)\big]^{-1/(p-1)}. \tag{11}$$

(ii) *If $q > p > 1$, then the solution of problem* (2) *blows up in finite time for any conveniently large initial data. Moreover, let u be a solution to problem* (2)*, which blows up at time T. Then,*

$$\lim_{t\to T}(T-t)^{1/(q-1)}\big|u(t)\big|_\infty = \big[(q-1)\big|\Omega\big|\big]^{-1/(q-1)}. \tag{12}$$

Theorem 3 [Global existence and blow-up for Neumann problem].

(i) *If $p > q \geqslant 1$, then problem* (1) *exists globally.*

(ii) *If $p = q = 1$, or $p = q > 1$ and $|\Omega| \leqslant k$, then problem* (1) *exists globally; if $p = q > 1$ and $|\Omega| > k$, then problem* (1) *blows up in finite time provided that $u_0(x)$ is not identically zero.*

(iii) *If $q > p \geqslant 1$, then problem* (1) *has global solution when $u_0(x) < f^*$, and the solution of problem* (1) *blows up in finite time when $u_0(x) > f^*$.*

(iv) *If $q < 1$, then every solution to problem* (1) *is global.*

Theorem 4 [Blow-up rate of Neumann problem]. *Let $q > p > 1$ or $q = p > 1$, and $|\Omega| > k$ and u be a solution to problem* (1)*, which blows up at time T. Then*

$$\lim_{t\to T}(T-t)^{1/(q-1)}\big|u(t)\big|_\infty = \big[(q-1)\big(|\Omega|-k\big)\big]^{-1/(q-1)}. \tag{13}$$

Remark 1.

(i) From Theorem 1 we derive: if $\min\{p,q\} \leqslant 1$, then the nonnegative solution of (2) cannot blow up for any continuous and nonnegative initial data.

(ii) By Theorem 3, if $\min\{p,q\} \leqslant 1$, the nonnegative solution of (1) cannot blow up for any continuous and nonnegative initial data.

(iii) It follows from Theorems 2 and 4 that the diffusion term (nonlocal source and interior absorption) plays no role and the blow-up rate is given by the ODE $u_t = |\Omega|u^q$ $(q > p > 1)$ and $u_t = (|\Omega| - k)u^p$ $(q = p > 1)$.

The rest of the paper is organized as follows. In Section 2, we prove the existence of local solutions for problems (1) and (2) and show a comparison principle for the solutions. In Section 3, we deal with the conditions ensuring blow-up versus global existence of solutions to problems (1) and (2). Also in Section 3, the blow-up rate of solutions of problems (1) and (2) are given.

2 Local existence of solutions and main properties

Without loss of generality, we only prove the existence and uniqueness of the solution of problem (2). And we will use the Banach fixed point theorem, which comes from [10] and [16] after some modification.

Let $t_0 > 0$ be fixed and consider the Banach space $X_{t_0} = C([0,t_0]; C(\bar\Omega))$ with the norm

$$\|w\|_{X_{t_0}} = \max_{0\leqslant t\leqslant t_0}\big\|w(\cdot,t)\big\|_{L^\infty(\bar\Omega)} = \max_{0\leqslant t\leqslant t_0}\max_{\bar\Omega}\big|w(x,t)\big|.$$

We will obtain the solution of problem (2) as a fixed point of the operator $F : B_0 \to B_0$ defined by

$$F_{w_0}(w)(x,t) = w_0(x) + \int_0^t \int_{\mathbb{R}^N} J(x-y)\big(w(y,s) - w(x,s)\big)\,\mathrm{d}y\,\mathrm{d}s + \int_0^t \int_\Omega \big|w(x,s)\big|^{q-1} w(x,s)\,\mathrm{d}x\,\mathrm{d}s - k\int_0^t \big|w(x,s)\big|^{p-1} w(x,s)\,\mathrm{d}s,$$

where $p, q > 0$ and $B_0 := B(u_0, 2\|u_0\|_{L^\infty})$. The following lemma is the main ingredient of the proof of the main results.

Lemma 2. *The operator F_{u_0} is well defined and mapping B_0 into B_0. Moreover, let $w, z \in B_0$. Then there exists a positive constant $C = C(J, \Omega, u_0, p, q, k)$ such that*

$$\big\|F_{u_0}(w)(x,t) - F_{u_0}(z)(x,t)\big\|_{X_{t_0}} \leqslant Ct\|w - z\|_{X_{t_0}}. \tag{14}$$

Thus, F_{u_0} is a strict contraction in the ball B_0 provided t_0 is small enough.

Proof. Since the convolution in space with the function J is uniformly continuous, it is easy to see that $F_{u_0}(w)$ is continuous as the function of x. We first prove that the operator F_{u_0} maps B_0 into B_0. For any $(x,t) \in \overline{\Omega} \times [0, t_0]$, we have

$$\begin{aligned}
&\big|F_{u_0}(w)(x,t) - u_0(x)\big| \\
&\quad\leqslant \left|\int_0^t \int_{\mathbb{R}^N} J(x-y)\big(w(y,s) - w(x,s)\big)\,\mathrm{d}y\,\mathrm{d}s\right| \\
&\qquad + \int_0^t \int_\Omega \big|w(x,s)\big|^q\,\mathrm{d}x\,\mathrm{d}s + k\int_0^t \big|w(x,s)\big|^p\,\mathrm{d}s \\
&\quad\leqslant 2\|J\|_{L^\infty}|\Omega|\|w\|_{X_{t_0}} + k\int_0^t \big|w(x,s)\big|^p\,\mathrm{d}s + \int_0^t \int_\Omega \big|w(x,s)\big|^q\,\mathrm{d}x\,\mathrm{d}s \\
&\quad\leqslant \big(2\|J\|_{L^\infty}|\Omega|\|w\|_{X_{t_0}} + \|w\|^q_{X_{t_0}}|\Omega| + k\|w\|^p_{X_{t_0}}\big)t,
\end{aligned}$$

which assures that $F_{u_0}(w)$ is continuous at $t = 0$. And for any $w \in B_0$, we conclude $F_{u_0}(w) \in B_0$. Thus, F_{u_0} maps B_0 into B_0.

On the other hand, for any $(x, t_1), (x, t_2) \in \overline{\Omega} \times [0, t_0]$, taking into account that w vanishes outside Ω, we have

$$\begin{aligned}
&\big|F_{u_0}(w)(x,t_2) - F_{u_0}(w)(x,t_1)\big| \\
&\quad\leqslant \left|\int_{t_1}^{t_2} \int_{\mathbb{R}^N} \big(w(y,s) - w(x,s)\big)\,\mathrm{d}y\,\mathrm{d}s\right| + \int_{t_1}^{t_2} \int_\Omega \big|w(x,s)\big|^q\,\mathrm{d}x\,\mathrm{d}s + k\int_{t_1}^{t_2} \big|w(x,s)\big|^p\,\mathrm{d}s
\end{aligned}$$

$$\leqslant \int_{t_1}^{t_2}\int_{\mathbb{R}^N} J(x-y)\big|w(y,s)-w(x,s)\big|\,\mathrm{d}y\,\mathrm{d}s + \left|\int_{t_1}^{t_2}\int_{\Omega} \big|w(x,s)\big|^q\,\mathrm{d}x\,\mathrm{d}s\right|$$
$$+ k\left|\int_{t_1}^{t_2} \big|w(x,s)\big|^p\,\mathrm{d}s\right|$$
$$\leqslant \big(2\|J\|_{L^\infty}|\Omega|\|w\|_{X_{t_0}} + \|w\|_{X_{t_0}}^q|\Omega| + k\|w\|_{X_{t_0}}^p\big)|t_2-t_1|$$
$$\leqslant \max\big\{2\|J\|_{L^\infty}, |\Omega|, k\big\}\big(\|w\|_{X_{t_0}} + \|w\|_{X_{t_0}}^q + \|w\|_{X_{t_0}}^p\big)|t_2-t_1|,$$

which shows that $F_{u_0}(w)$ is continuous in time for any $t \in [0,t_0]$.

To prove the estimate (14), we argue as follows: for any $(x,t) \in \overline{\Omega} \times [0,t_0]$, it holds

$$\big|F_{u_0}(w)(x,t) - F_{u_0}(z)(x,t)\big|$$
$$\leqslant \int_0^t\int_{\mathbb{R}^N} \big|J(x-y)\big[w(y,s)-z(y,s)-\big(w(x,s)-z(x,s)\big)\big]\big|\,\mathrm{d}y\,\mathrm{d}s$$
$$+ \int_0^t\int_{\Omega} \big|w^q(x,s)-z^q(x,s)\big|\,\mathrm{d}x\,\mathrm{d}s + k\int_0^t \big|w^p(x,s)-z^p(x,s)\big|\,\mathrm{d}s$$
$$\leqslant 2\|J\|_{L^\infty}|\Omega|\|w-z\|_{X_{t_0}}t + \max\big\{q|\Omega|, kp\big\}\big(\xi^{p-1}+\xi^{q-1}\big)\|w-z\|_{X_{t_0}}t$$
$$= Ct\|w-z\|_{X_{t_0}}, \tag{15}$$

where $C = \max\{q|\Omega|, kp\}(\xi^{p-1}+\xi^{q-1}) + \|J\|_{L^\infty}$ and $\xi \leqslant \max\{\|w\|_{X_{t_0}}, \|z\|_{X_{t_0}}\}$. The arbitrariness of $(x,t) \in \overline{\Omega} \times [0,t_0]$ gives the desired estimate (14).

Finally, choosing t_0 such that $Ct_0 < 1$, (14) ensures that $F_{u_0}(w)$ is a strict contraction in the ball $B_0 \subset X_{t_0}$ and thus completes the proof of this Lemma. □

Employing the above Lemmas, we derive

Theorem 5. *For every $u_0 \in C(\bar{\Omega})$, problem* (2) *admits a unique solution $u \in C([0,T); C(\bar{\Omega}))$. Moreover, if the maximal existence time $T < \infty$, then the solution blows up in $L^\infty(\bar{\Omega})$-norm, that is,*

$$\limsup_{t\to T} \big\|u(x,t)\big\|_{L^\infty(\bar{\Omega})} = +\infty. \tag{16}$$

Proof. It follows from Lemma 2 that F_{u_0} is a strict contraction in B_0 for t_0 small enough. By the Banach' theorem fixed point theorem there exists only one fixed point of F_{u_0} in B_0. This proves the existence and uniqueness of solution of (2) in the time interval $[0,t_0]$. To continue we may take $u(x,t_0)$ as initial data and obtain a unique solution of (2) in the time interval $[0,t_1]$. If $\|u\|_{X_{t_1}} < \infty$, taking as initial datum $u(\cdot,t_1) \in C(\bar{\Omega})$ and arguing as before, it is possible to extend the solution up to some interval $[0,t_2)$ for certain $t_2 > t_1$. Hence, we can conclude that if the maximal time of the existence of the solution, T, is finite, then the solution blows up in $L^\infty(\bar\Omega)$-norm, that is,

$$\limsup_{t\to T}\big\|u(x,t)\big\|_{L^\infty(\bar\Omega)} = +\infty.$$

Otherwise, the solution of problem (2) is global. □

Definition 1. A nonnegative function $\bar u \in C^1([0,T);C(\bar\Omega))$ is a supersolution of problem (2) (respectively of (1)) if it satisfies

$$\begin{aligned}
&\frac{\partial}{\partial t}\bar u(x,t) \geqslant \int\limits_A J(x-y)\big(\bar u(y,t)-\bar u(x,t)\big)\,\mathrm{d}y + \int\limits_\Omega \bar u^q\,\mathrm{d}x - k\bar u^p, \quad (x,t)\in Q_T\\
&\bar u(x,0)\geqslant u_0(x), \quad x\in\Omega,
\end{aligned} \tag{17}$$

where $Q_T := \Omega\times(0,T)$. $A=\mathbb{R}^N$ in case problem (2) (respectively $A=\Omega$ for problem (1)). The subsolution is defined similarly by reversing the inequalities. Furthermore, if u is a supersolution as well as a subsolution, then we call it a solution of problem (2) or (1).

Lemma 3. *Let $\overline{u}$ be a supersolution of problem* (2) (*or* (1))*. Then if $u_0 \geqslant 0$, we have $\overline{u}(x,t)\geqslant 0$ for $(x,t)\in\overline{\Omega}\times[0,T)$.*

Proof. Without loss of generality, we only prove the solution of problem (2) satisfying $\overline{u}(x,t)\geqslant 0$ for $(x,t)\in\overline{\Omega}\times[0,T)$. By an approximation procedure we restrict ourselves to consider strict inequalities for the supersolution. Indeed, we can take $\bar u(x,t)+\delta t+\delta$ ($\delta>0$) as a strict supersolution and take limit as $\delta\to 0$ at the end.

Arguing by contradiction, we assume that there exist a first time t_0 and some point $x_0\in\Omega$ at which $\bar u(x_0,t_0)=0$, and then $\overline{u}(y,t_0)\geqslant 0$ for all $y\in\Omega$. Therefore, we derive

$$\begin{aligned}
\frac{\partial}{\partial t}\bar u(x_0,t_0) &> \int\limits_\Omega J(x_0-y)\big(\bar u(y,t_0)-\bar u(x_0,t_0)\big)\,\mathrm{d}y\\
&\quad - \int\limits_{\mathbb{R}^N\setminus\Omega} J(x_0-y)\bar u(x_0,t_0)\,\mathrm{d}y + \int\limits_\Omega \bar u^q(y,t_0)\,\mathrm{d}y - k\bar u^p(x_0,t_0)\\
&= \int\limits_\Omega J(x_0-y)\bar u(y,t_0)\,\mathrm{d}y + \int\limits_\Omega \bar u^q(y,t_0)\,\mathrm{d}y \geqslant 0,
\end{aligned}$$

which contradicts with $(\partial/\partial t)\bar u(x_0,t_0)\leqslant 0$. □

Applying Lemma 3, we can get

Lemma 4. *Let $\bar u, \underline{u}$ be super and subsolutions to* (2) (*or* (1))*, respectively. Then $\bar u(x,t)\geqslant \underline{u}(x,t)$ for every $(x,t)\in\Omega\times[0,T)$.*

3 The proof of main results

Once the existence of the solutions to problem (2) and (1) is established, we begin to analyze the critical blow-up exponents for nonnegative solutions. As the first step, we discuss the existence of global solutions to problem (2).

3.1 Blow-up versus global existence for the Dirichlet problem

In this section, we will use super and subsolution techniques to derive some conditions on the existence or nonexistence of global solution.

Proof of Theorem 1. The idea of the proof is to construct the suitable supersolution of problem (2). Indeed, assume that $u(x,t)$ is the solution of (2) with the initial datum $u_0(x)$. Suppose that φ satisfies

$$\begin{aligned} &-\int_{\mathbb{R}^N} J(x-y)\big(\varphi(y)-\varphi(x)\big)\,\mathrm{d}y = 1, \quad x \in \Omega, \\ &\varphi(x) = 0, \quad x \notin \bar{\Omega}. \end{aligned} \tag{18}$$

Due to Theorem 1 of [13], there is a unique positive solution of the problem (18). Now, choose γ conveniently small such that

$$\gamma > \gamma^p\biggl(\int_{\Omega} \varphi^p(x)\,\mathrm{d}x - k\varphi^p(x)\biggr) \quad \text{if } p>1 \text{ and } |\Omega|>k. \tag{19}$$

For the above γ, we choose initial datum $u_0(x)$ such that $\max_{x\in\bar{\Omega}} u_0(x) \leqslant \gamma m$, where m is given by (9). Let

$$\bar{u} = \begin{cases} \max\{(\|u_0\|_{L^\infty}+1)/m,\ (\int_\Omega \psi^q\,\mathrm{d}x/(\lambda_1 m))^{1/(1-q)}\}\psi(x) & \text{if } q<1, \\ (\|u_0\|_{L^\infty}+1)\mathrm{e}^{2|\Omega|t} & \text{if } q=1, \\ \max\{\|u_0\|_{L^\infty}+1,\ (|\Omega|/k)^{1/(p-q)}\} & \text{if } p>q>1, \\ \|u_0\|_{L^\infty}+1 & \text{if } p=q>1 \text{ and } |\Omega|\leqslant k, \\ (|\Omega|/k)^{1/(p-q)} & \text{if } q>p>1 \text{ and } (|\Omega|/k)^{1/(p-q)} \geqslant \|u_0\|_{L^\infty}. \\ \gamma\varphi(x) & \text{if } q=p>1 \text{ and } |\Omega|>k, \end{cases}$$

where ψ is given by (8). Then we can prove that $\bar{u}$ is a global supersolution of (2). To this end, if $q<1$, it holds that

$$\begin{aligned} P\bar{u} &= \bar{u}_t(x,t) - \biggl[\int_{\mathbb{R}^N} J(x-y)\big(\bar{u}(y,t)-\bar{u}(x,t)\big)\,\mathrm{d}y + \int_{\Omega} \bar{u}^q\,\mathrm{d}x - k\bar{u}^p\biggr] \\ &\geqslant \lambda_1\biggl(\frac{\int_\Omega \psi^q\,\mathrm{d}x}{\lambda_1 m}\biggr)^{1/(1-q)}\biggl(1 - \frac{\lambda_1 m}{\int_\Omega \psi^q\,\mathrm{d}x}\frac{\int_\Omega \psi^q(x)\,\mathrm{d}x}{\lambda_1\psi}\biggr)\psi(x) \geqslant 0, \end{aligned}$$

which implies that problem (2) has global solution. The assertion can be proved similarly for the other cases, and thus the proof of this theorem is completed. □

The following two lemmas play an important role in the proof of Theorems 2 and 4.

Lemma 5. *Let u be a solution to* (2) (*or* (1))*, which blows up at time T.*

(i) *If $q > p > 1$, then*

$$\max_{x\in\Omega} u(x,t) \geqslant \bigl[(q-1)|\Omega|\bigr]^{-1/(q-1)}(T-t)^{-1/(q-1)}. \tag{20}$$

(ii) *If $q = p > 1$ and $0 < k < |\Omega|$, then*

$$\max_{x\in\Omega} u(x,t) \geqslant \bigl[(p-1)\bigl(|\Omega|-k\bigr)\bigr]^{-1/(p-1)}(T-t)^{-1/(p-1)}. \tag{21}$$

Proof. (i) Without loss of generality, we only prove case (i) for problem (2). The main idea we use is a recent method introduced by Pérez-Llanos and Rossi in [16]. For any fixed $t \in (0,T)$, let $x_0 \in \bar{\Omega}$ be such that $\max_{x\in\bar{\Omega}} u(\cdot,t) = u(x_0,t)$. Thanks to the fact that the blowing-up solutions to problem (2) verify that $u(x,t) \geqslant 0, u \not\equiv 0$ in Ω and $u(x,t) \equiv 0, x \notin \Omega$, we derive from (2) that $\max_{x\in\mathbb{R}^N} u(\cdot,t) = \max_{x\in\Omega} u(\cdot,t) = u(x_0,t)$. Consequently, at this point, the following estimate follows:

$$u_t(x_0,t) = \int\limits_{\mathbb{R}^N} J(x_0-y)u(y,t)\,\mathrm{d}y + \int\limits_{\Omega} u^q(x,t)\,\mathrm{d}x - ku^p(x_0,t) - u(x_0,t), \tag{22}$$

which, together with $u(y,t) \leqslant u(x_0,t)$ (for any $y \in \mathbb{R}^N$), implies that

$$\begin{aligned} u_t(x_0,t) &= \int\limits_{\mathbb{R}^N} J(x_0-y)u(y,t)\,\mathrm{d}y + \int\limits_{\Omega} u^q(x,t)\,\mathrm{d}x - ku^p(x_0,t) - u(x_0,t) \\ &\leqslant u(x_0,t) + \int\limits_{\Omega} u^q(x,t)\,\mathrm{d}x - ku^p(x_0,t) - u(x_0,t) \\ &\leqslant u^q(x_0,t)|\Omega|. \end{aligned}$$

Integrating the above inequality from (t,T) and taking into account that $q > 1$, we derive

$$u(x_0,t) \geqslant \bigl[(q-1)|\Omega|\bigr]^{-1/(q-1)}(T-t)^{-1/(q-1)}, \tag{23}$$

that is,

$$\max_{x\in\Omega} u(x,t) \geqslant \bigl[(q-1)|\Omega|\bigr]^{-1/(q-1)}(T-t)^{-1/(q-1)}. \tag{24}$$

Using the same arguments as in the proof of case (i), we can derive the result of case (ii). We complete the proof of the Lemma. □

Lemma 6. *Assuming u is a solution to* (2) *(or* (1)*), which blows up at time T.*

(i) *If* $q > p > 1$, *then*

$$\max_{x\in\Omega} u(x,t) \leqslant \bigl[(q-1)|\Omega|\bigr]^{-1/(q-1)}(T-t)^{-1/(q-1)}. \tag{25}$$

(ii) *If* $q = p > 1$ *and* $0 < k < |\Omega|$, *then*

$$\max_{x\in\Omega} u(x,t) \leqslant \bigl[(p-1)\bigl(|\Omega|-k\bigr)\bigr]^{-1/(p-1)}(T-t)^{-1/(p-1)}. \tag{26}$$

Proof. (i) Without loss of generality, we only prove case (i) for problem (2). For (ii), it is very similar, we omitted it. For any $t < \tilde{T} < T$, we have that $\bar{u} = [(q-1)|\Omega|]^{-1/(q-1)} \times (\tilde{T}-t)^{-1/(q-1)}$ is a supersolution of (2) provided that

$$|u_0|_\infty \leqslant \bigl[(q-1)|\Omega|\bigr]^{-1/(q-1)}(\tilde{T}-t)^{-1/(q-1)},$$

where u_0 is the initial data of (2). In fact,

$$\begin{aligned} P\bar{u} &= \bar{u}_t(x,t) - \Biggl[\int\limits_{\mathbb{R}^N} J(x-y)\bigl(\bar{u}(y,t)-\bar{u}(x,t)\bigr)\,\mathrm{d}y + \int\limits_{\Omega} \bar{u}^q\,\mathrm{d}x - k\bar{u}^p\Biggr] \\ &\geqslant \frac{1}{q-1}\bigl[(q-1)|\Omega|\bigr]^{-1/(q-1)}(\tilde{T}-t)^{-q/(q-1)} \\ &\quad - \bigl[(q-1)|\Omega|\bigr]^{-q/(q-1)}(\tilde{T}-t)^{-q/(q-1)}|\Omega| \\ &= (q-1)^{-q/(q-1)}\bigl[|\Omega|\bigr]^{-1/(q-1)}(\tilde{T}-t)^{-q/(q-1)} \\ &\quad - (q-1)^{-q/(q-1)}\bigl[|\Omega|\bigr]^{-1/(q-1)}(\tilde{T}-t)^{-q/(q-1)} = 0. \end{aligned}$$

Therefore, by comparison principle, we derive $u(x,t) \leqslant \bar{u} = [(q-1)|\Omega|]^{-1/(q-1)} \times (\tilde{T}-t)^{-1/(q-1)}$ for any $(x,t) \in \Omega \times (0,T)$. This implies that

$$\max_{x\in\Omega} u(x,t) \leqslant \bigl[(q-1)|\Omega|\bigr]^{-1/(q-1)}(\tilde{T}-t)^{-1/(q-1)}. \tag{27}$$

Now, letting $\tilde{T} \to T$ in (27), we have

$$\max_{x\in\Omega} u(x,t) \leqslant \bigl[(q-1)|\Omega|\bigr]^{-1/(q-1)}(T-t)^{-1/(q-1)}. \tag{28}$$

The proof is now completed. □

Proof of Theorem 2. The proof is divided into two different cases.

(i) $q = p > 1$. Assume that k is small enough such that

$$kM^p < \int\limits_{\Omega} \psi^q(x)\,\mathrm{d}x.$$

Let $w(x,t) = g(t)\psi(x)$, where $g(t)$ satisfies the following ODE problem:

$$g'(t) + \frac{kM^p - \int_\Omega \psi^p(x)\,\mathrm{d}x}{M} g^p(t) + \lambda_1 g(t) = 0, \quad t \in (0,T),$$
$$g(0) = g_0 > 0 \tag{29}$$

with $g_0 > 1$ and large enough. By $p > 1$ and hypothesis (29) we have that $g(t)$ is not decreasing, and there exists $0 < T^* < +\infty$ such that $\lim_{t\to T^*} g(t) = +\infty$. Hence, we can infer that $w(x,t)$ is the subsolution of (2) provided that $g_0 \leqslant \min_{x\in\bar\Omega} u_0(x)/M$. In fact, with the help of $q = p$, we readily find that

$$Pw = w_t(x,t) - \biggl[\int\limits_{R^N} J(x-y)\bigl(w(y,t) - w(x,t)\bigr)\,\mathrm{d}y + \int\limits_\Omega w^p\,\mathrm{d}x - kw^p\biggr]$$
$$\leqslant \psi(x)\biggl[g'(t) + \frac{kM^p - \int_\Omega \psi^p(x)\,\mathrm{d}x}{M} g^p(t) + \lambda_1 g(t)\biggr] = 0.$$

Thanks to Lemma 4 and taking u_0 conveniently large, we derive that $u(x,t) \geqslant w(x,t)$ for any $x \in \Omega$ and $t < T^*$. Since w blows up in finite time T^*, we have that u blows up in finite time $\tilde{T} \leqslant T^*$.

(ii) $q > p > 1$. Suppose that $u(x,t)$ is the solution of (2) with the initial datum $u_0(x)$ and

$$B = \biggl[\frac{kM^p}{2\int_\Omega \psi^q(x)\,\mathrm{d}x}\biggr]^{1/(q-p)}.$$

By $q > p > 1$ we can choose $A > 0$ appropriately large such that

$$\lambda_1 M + A^p M^p - A^q \int\limits_\Omega \psi^p(x)\,\mathrm{d}x \leqslant 0,$$
$$Am \geqslant \biggl[\frac{kM^p}{2\int_\Omega \psi^q(x)\,\mathrm{d}x}\biggr]^{1/(q-p)} = B.$$

Setting $w(x,t) = A\psi(x)$. Then by the arguments as those in the proof of the first case of Theorem 1, we can get $w(x,t)$ is the subsolution of (2) provided that $\min_{x\in\bar\Omega} u_0(x)/M \geqslant A$. Invoking Lemma 4 to (2), we derive

$$u(x,t) \geqslant A\psi(x) \geqslant \biggl[\frac{kM^p}{2\int_\Omega \psi^q(x)\,\mathrm{d}x}\biggr]^{1/(q-p)}.$$

Therefore, $u(x,t)$ satisfies

$$u_t - \int\limits_{\mathbb{R}^N} J(x-y)\bigl(u(y,t) - u(x,t)\bigr)\,\mathrm{d}y - \frac{kM^p}{2\int_\Omega \psi^p(x)\,\mathrm{d}x}\int\limits_\Omega u^p\,\mathrm{d}x + ku^p \geqslant 0,$$

that is, u is the supersolution of

$$v_t - \int\limits_{\mathbb{R}^N} J(x-y)\big(v(y,t)-v(x,t)\big)\,\mathrm{d}y - \frac{kM^p}{2\int_\Omega \psi^p(x)\,\mathrm{d}x} \int\limits_\Omega v^p\,\mathrm{d}x + kv^p = 0. \tag{30}$$

On the other hand, according to the results of the above (the case $p = q$), we can conclude that the solution of problem (30) blows up. Thus, it follows from the comparison principle that the solution $u(x,t)$ of (2) blows up in finite time.

Now we discuss the blow-up rate, that is, the speed at which solution blows up.

Applying Lemmas 5 and 6, we can derive the blow-up rate. □

In the following, we deal with blow-up versus global existence of solutions to problem (1).

3.2 Blow-up versus global existence for the Neumann problem

In this section, we analyze the blow-up phenomenon arising in the Neumann problem (1).

Proof of Theorem 3. The idea of the proof is to construct the suitable supersolution of problem (1). Indeed, assume that $u(x,t)$ is the solution of (1) with the initial datum $u_0(x)$.

The proof is divided into two different cases.

(i) $q < 1$. Let $z(t)$ be the solution of

$$\begin{aligned} &z'(t) = z^q(t)|\Omega|, \quad t > 0,\\ &z(0) = z_0 > 0, \quad q < 1. \end{aligned} \tag{31}$$

Obviously, $z(t)$ is a supersolution of problem (1). Therefore, by the comparison principle, every solution to problem (1) is global.

(ii) Other cases. Let $f(t)$ be the solution of (10). It is clear that $f(t)$ can be taken as a comparison function of problem (1). Hence, we can get the conclusion. □

Proof of Theorem 4. Employing Lemmas 5 and 6, we can easily derive the blow-up rate. □

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