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Existence of positive solutions for singular fractional differential equations with infinite-point boundary conditions*

Limin Guo[a], **Lishan Liu**[a,c,1], **Yonghong Wu**[b,c]

[a]School of Mathematical Sciences, Qufu Normal University,
Qufu 273165, Shandong, China
guolimin811113@163.com; mathlls@163.com

[b]School of Statistics and Mathematics,
Zhongnan University of Economics and Law,
Wuhan 430073, China

[c]Department of Mathematics and Statistics, Curtin University,
Perth, WA6845, Australia
yhwu@maths.curtin.edu.au

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Abstract. In this paper, we investigate the existence of at least three positive solutions to a singular boundary value problem of Caputo's fractional differential equations with a boundary condition involving values at infinite number of points. Firstly, we establish Green's function and its properties. Then the existence of multiple positive solutions is obtained by Avery–Peterson's fixed point theorem. Finally, an example is given to demonstrate the application of our main results.

Keywords: Caputo's fractional differential equation, positive solution, Green's function, Avery–Peterson's fixed point theorem.

1 Introduction

In this paper, we consider the following infinite-point fractional differential equations boundary value problem:

$$
\begin{aligned}
&{}^cD_{0+}^{\alpha}u(t) + f\big(t, u(t), u'(t)\big) = 0, \quad 0 < t < 1,\\
&u(0) = u''(0) = 0, \qquad u'(1) = \sum_{j=1}^{\infty}\eta_j u(\xi_j),
\end{aligned}
\tag{1}
$$

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[1]Corresponding author.

where $2 < \alpha \leqslant 3$, $\eta_j \geqslant 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ ($j = 1, 2 \ldots$), $0 < \Delta = 1 - \sum_{j=1}^{\infty} \eta_j \xi_j$, $f(t, x, y)$ may be singular at $t = 0$, and ${}^cD_{0+}^{\alpha}$ is the standard Caputo derivative. The existence of multiple positive solutions is obtained for the boundary value problem under certain conditions. To our knowledge, this is the first attempt to investigate the boundary value problem of Caputo's fractional differential equations with a boundary condition involving values at infinite number of points. In this paper, we will study the existence of positive solutions to BVP (1), where $x \in C^2[0, 1]$ is said to be a positive solution of BVP (1) if and only if x satisfies (1) and $x(t) > 0$, $x'(t) > 0$ for any $t \in (0, 1]$.

Recently, boundary value problems for nonlinear fractional differential equations have attracted great research efforts worldwide, as they arise from the study of many important problems in various discipline areas such as fluid flows, electrical networks, rheology, biology and chemical physics. In practical applications, it is important to establish the conditions for the existence of positive solutions. Hence, many authors have investigated the existence of positive solutions for various fractional equation boundary value problems, and for details, the reader is referred to [2, 3, 4, 5, 10, 11, 12, 14, 15, 16, 20, 21, 22, 24] and the references therein. For some basic fixed point theorems, readers can refer to [6, 8]. In [23], the author considered the following fractional differential equation:

$$\begin{aligned} &D_{0+}^{\alpha} u(t) + g(t) f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u^{(i)}(1) = \sum_{i=1}^{\infty} \alpha_j u(\xi_j), \end{aligned}$$

where $2 < \alpha$, $n - 1 < \alpha \leqslant n$, $i \in [1, n-2]$ is a fixed integer, $\alpha_j \geqslant 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ ($j = 1, 2, \ldots$), f is allowed to have singularities with respect to both time and space variables. Various theorems were then established for the existence and multiplicity of positive solutions. In [17], the author discussed the existence and multiplicity of positive solutions of the following problem:

$$\begin{aligned} &D_{0+}^{\alpha} u(t) = a(t) f\bigl(t, u(t)\bigr), \quad t \in (0, 1), \\ &u(0) = u'(0) = 0, \qquad u(1) = \sum_{i=1}^{m} \beta_i u(\xi_i), \end{aligned}$$

where $2 < \alpha \leqslant 3$, $m \geqslant 1$ is integer, $\beta_i > 0$ for $1 \leqslant i \leqslant m$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $\sum_{i=1}^{m} \beta_i \xi_i^{\alpha-1} < 1$, $a(t) \in L[0, 1]$ is non-negative and not identically zero on any compact subset of $(0, 1)$, $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous and D_{0+}^{α} is the Riemann–Liouville differential fractional derivative of order α. Some results on the existence and multiplicity of positive solutions were obtained by the fixed point theorem. In [9], the authors investigated the existence of multiple positive solutions of the following fractional differential equation boundary value problem:

$$\begin{aligned} &{}^cD_{0+}^{\alpha} u(t) + f\bigl(t, u(t), u'(t), \ldots, u^{(i)}(t)\bigr) = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = \cdots = u^{(i-1)}(0) = u^{(i+1)}(0) = \cdots = u^{(n-1)}(0) = 0, \\ &u^{(i)}(1) = 0, \end{aligned}$$

where $n-1<\alpha\leqslant n, n\geqslant 2, \alpha-i>1, i\in N, 0\leqslant i\leqslant n-1, f(t,x_0,x_1,\dots,x_i)$ may be singular at $t=0$ and ${}^cD^{\alpha}_{0+}$ is the standard Caputo derivative. The authors obtained the existence result of at least three positive solutions for a two-point boundary value problem, in which the nonlinear terms contain derivatives up to order i by using Avery–Peterson's fixed point theorem.

Motivated by the results above, in this paper, we investigate the existence of positive solutions for a class of singular fractional differential equations subject to infinite-point boundary conditions. Compared with previous work in the field, our work presented in this paper has several new features. Firstly, values at infinite points are involved in the boundary conditions of the boundary value problem (1). Secondly, our study is on singular nonlinear differential boundary value problems, that is, $f(t,u,v)$ is allowed to be singular at $t=0$. Thirdly, the nonlinear term involves the first order derivative. Fourthly, the main tool used in this paper is Avery–Peterson's fixed point theorem.

2 Preliminaries and lemmas

For the convenience of the reader, we first present some basic definitions and lemmas, which are to be used in the proof of our results and can also be found in the recent literature such as [18,19]. Firstly, we let $E=C^1[0,1]$ be the Banach space with the maximum norm

$$\|u\|=\max\bigl\{\|u\|_0,\|u'\|_0\bigr\},$$

where $\|u\|_0=\max_{t\in[0,1]}|u(t)|$, $\|u'\|_0=\max_{t\in[0,1]}|u'(t)|$. We also list below a condition to be used later in the paper.

(H0) $f:(0,1]\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$, and there exists a constant $0<\sigma<1$ such that $t^\sigma f(t,x_0,x_1)$ is continuous on $[0,1]\times\mathbb{R}^+\times\mathbb{R}^+$, in which $\mathbb{R}^+=[0,+\infty)$.

Definition 1. (See [18, 19].) The Riemann–Liouville fractional integral of order $\alpha>0$ of a function $y:(0,\infty)\to\mathbb{R}^1=(-\infty,+\infty)$ is given by

$$I^{\alpha}_{0+}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,\mathrm{d}s,$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2. (See [18, 19].) The Riemann–Liouville fractional derivative of order $\alpha>0$ of a continuous function $y:(0,\infty)\to\mathbb{R}^1$ is given by

$$D^{\alpha}_{0+}y(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_0^t\frac{y(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s,$$

where $n=[\alpha]+1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 3. (See [18,19].) The Caputo fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}^1$ is given by

$$ {}^{c}D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}} \,\mathrm{d}s, $$

where α is fractional number, $n = [\alpha] + 1$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 1. (*See* [18,19].) *Assume that* $u \in C^n[0,1]$*, then*

$$ I_{0+}^{\alpha}\, {}^{c}D_{0+}^{\alpha} u(t) = u(t) - C_1 - C_2 t - \cdots - C_n t^{n-1}, $$

where n *is the least integer greater than or equal to* α*,* $C_i \in \mathbb{R}^1$ $(i = 1,2,\dots,n)$*.*

Lemma 2. (*See* [7, *Thm.* 1.2.7].) *Let* $H \subset C^1[J,E]$*, then* H *is a relatively compact set if and only if*:

(a) H' *is equicontinuous and* $H'(t)$ *is a relatively compact set for any* $t \in J$ *on* E;
(b) *There exists* $t_0 \in J$ *such that* $H(t_0)$ *is a relatively compact set on* E.

Lemma 3. *Given* $y \in C(0,1) \cap L^1[0,1]$*, then the solution of the BVP*

$$ \begin{gathered} {}^{c}D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \qquad u'(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j) \end{gathered} \tag{2} $$

can be expressed by

$$ u(t) = \int\limits_0^1 G(t,s) y(s) \,\mathrm{d}s, \quad t \in [0,1], \tag{3} $$

where

$$ G(t,s) = \frac{1}{\Delta\Gamma(\alpha)} \begin{cases} tP(s)(1-s)^{\alpha-2} - \Delta(t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ tP(s)(1-s)^{\alpha-2}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{4} $$

in which

$$ P(s) = \alpha - 1 - \sum_{s \leqslant \xi_j} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\alpha-1} (1-s), \qquad \Delta = 1 - \sum_{j=1}^{\infty} \eta_j \xi_j, $$

and, obviously, $G(t,s)$ *is continuous on* $[0,1] \times [0,1]$*.*

Proof. By means of Lemma 1, we can reduce (2) to an equivalent integral equation

$$ u(t) = -I_{0+}^{\alpha} y(t) + C_1 + C_2 t + C_3 t^2 $$

for $C_1, C_2, C_3 \in \mathbb{R}$. Consequently, we get

$$u'(t) = -I_{0^+}^{\alpha-1} y(t) + C_2 + 2C_3 t,$$
$$u''(t) = -I_{0^+}^{\alpha-2} y(t) + 2C_3.$$

From $u(0) = u''(0) = 0$, $u'(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j)$ we have $C_1 = C_3 = 0$, but $C_2 \neq 0$, and thus,

$$u(t) = C_2 t - I_{0+}^{\alpha} y(t).$$

On the other hand, $u'(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j)$, and so combining with

$$u'(1) = C_2 - I_{0^+}^{\alpha-1} y(1),$$

we get

$$\begin{aligned} C_2 &= \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)(1-\sum_{j=1}^{\infty} \eta_j \xi_j)} y(s)\, \mathrm{d}s - \sum_{j=1}^{\infty} \eta_j \int_0^{\xi_j} \frac{(\xi_j - s)^{\alpha-1}}{\Gamma(\alpha)(1-\sum_{j=1}^{\infty} \eta_j \xi_j)} y(s)\, \mathrm{d}s \\ &= \int_0^1 \frac{(1-s)^{\alpha-2} P(s)}{\Gamma(\alpha) \Delta} y(s)\, \mathrm{d}s, \end{aligned}$$

where

$$P(s) = \alpha - 1 - \sum_{s \leqslant \xi_j} \eta_j \left(\frac{\xi_j - s}{1-s} \right)^{\alpha-1} (1-s) \quad \text{and} \quad \Delta = 1 - \sum_{j=1}^{\infty} \eta_j \xi_j.$$

Hence,

$$\begin{aligned} u(t) &= C_2 t - I_{0+}^{\alpha} y(t) \\ &= -\int_0^t \frac{\Delta (t-s)^{\alpha-1}}{\Gamma(\alpha) \Delta} y(s)\, \mathrm{d}s + \int_0^1 \frac{(1-s)^{\alpha-2} t P(s)}{\Gamma(\alpha) \Delta} y(s)\, \mathrm{d}s. \end{aligned}$$

Therefore,

$$G(t,s) = \frac{1}{\Delta \Gamma(\alpha)} \begin{cases} tP(s)(1-s)^{\alpha-2} - \Delta (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ tP(s)(1-s)^{\alpha-2}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

and

$$\frac{\partial G(t,s)}{\partial t} = \frac{1}{\Delta \Gamma(\alpha)} \begin{cases} P(s)(1-s)^{\alpha-2} - \Delta (\alpha-1)(t-s)^{\alpha-2}, & 0 \leqslant s \leqslant t \leqslant 1, \\ P(s)(1-s)^{\alpha-2}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{5}$$

It is easy to check that $G(t,s)$ and $\partial G(t,s)/\partial t$ are uniformly continuous on $[0,1] \times [0,1]$. □

Lemma 4. *Take $a, b \in (0,1)$ with $a < b$ such that $a > b^{\alpha-1}$, $(\alpha-1)b^{\alpha-2} < 1$, then we have*

$$0 \leqslant G(t,s) \leqslant (\alpha-1)g(s), \quad t,s \in [0,1];$$
$$G(t,s) \geqslant \rho g(s), \quad t \in [a,b],\ s \in [0,1];$$
$$0 \leqslant \frac{\partial G(t,s)}{\partial t} \leqslant (\alpha-1)g(s), \quad t,s \in [0,1];$$
$$\frac{\partial G(t,s)}{\partial t} \geqslant \rho g(s), \quad t \in [a,b],\ s \in [0,1],$$

where

$$g(s) = \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)\Delta},$$
$$0 < \rho_1 = \Delta\big(a - b^{\alpha-1}\big) \leqslant 1, \qquad 0 < \rho_2 = \Delta\big(1 - (\alpha-1)b^{\alpha-2}\big) \leqslant 1,$$

then

$$0 < \rho = \min\{\rho_1, \rho_2\} \leqslant 1.$$

Proof. By direct calculation, we get $P'(s) \geqslant 0$, $s \in [0,1]$, and so $P(s)$ is nondecreasing with respect to s. For $s \in [0,1]$, we get

$$\begin{aligned} P(s) &= \alpha - 1 - \sum_{s \leqslant \xi_j} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\alpha-1} (1-s) \\ &\geqslant P(0) = \alpha - 1 - \sum_{j=1}^{\infty} \eta_j \xi_j^{\alpha-1} \geqslant 1 - \sum_{j=1}^{\infty} \eta_j \xi_j = \Delta, \end{aligned}$$

and, obviously,

$$P(s) = \alpha - 1 - \sum_{s \leqslant \xi_j} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\alpha-1} (1-s) \leqslant \alpha - 1, \quad s \in [0,1].$$

Hence, for $t, s \in [0,1]$, we have

$$G(t,s) \leqslant \frac{tP(s)(1-s)^{\alpha-2}}{\Gamma(\alpha)\Delta} \leqslant \frac{P(s)(1-s)^{\alpha-2}}{\Gamma(\alpha)\Delta} \leqslant (\alpha-1)g(s),$$
$$\frac{\partial G(t,s)}{\partial t} \leqslant \frac{P(s)(1-s)^{\alpha-2}}{\Gamma(\alpha)\Delta} \leqslant (\alpha-1)g(s).$$

Furthermore, for $0 \leqslant s \leqslant t \leqslant 1$, we get

$$\begin{aligned} G(t,s) &= \frac{tP(s)(1-s)^{\alpha-2} - \Delta(t-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} \\ &= \frac{tP(s)(1-s)^{\alpha-2} - \Delta(t-s)^{\alpha-2}(t-s)}{\Delta\Gamma(\alpha)} \geqslant 0, \end{aligned}$$

and, obviously, for $0 \leqslant t \leqslant s \leqslant 1$, we get

$$G(t,s) \geqslant 0.$$

On the other hand, for $0 \leqslant s \leqslant t \leqslant 1$, we have

$$\begin{aligned}
\frac{\partial G(t,s)}{\partial t} &= \frac{P(s)(1-s)^{\alpha-2} - \Delta(\alpha-1)(t-s)^{\alpha-2}}{\Delta\Gamma(\alpha)} \\
&\geqslant \frac{P(0)(1-s)^{\alpha-2} - \Delta(\alpha-1)(t-s)^{\alpha-2}}{\Delta\Gamma(\alpha)} \\
&\geqslant \frac{(\alpha-1-\sum_{j=1}^{\infty}\eta_j\xi_j^{\alpha-1})(1-s)^{\alpha-2} - \Delta(\alpha-1)(t-s)^{\alpha-2}}{\Delta\Gamma(\alpha)} \\
&\geqslant \frac{((\alpha-1-\sum_{j=1}^{\infty}\eta_j\xi_j^{\alpha-1}) - \Delta(\alpha-1))(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha)} \\
&= \frac{((\alpha-1)\sum_{j=1}^{\infty}\eta_j\xi_j - \sum_{j=1}^{\infty}\eta_j\xi_j^{\alpha-1})(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha)} \geqslant 0,
\end{aligned}$$

and for $0 \leqslant t \leqslant s \leqslant 1$, $\partial G(t,s)/\partial t \geqslant 0$ obviously holds.

For $t \in [a,b]$, $s \in [0,1]$, we get

$$\begin{aligned}
G(t,s) &= \frac{tP(s)(1-s)^{\alpha-2} - \Delta(t-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} \\
&\geqslant \frac{aP(s)(1-s)^{\alpha-2} - \Delta(b-bs)^{\alpha-1}}{\Delta\Gamma(\alpha)} \\
&\geqslant \frac{(P(s)a - \Delta b^{\alpha-1})(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha)} \\
&\geqslant \frac{\Delta(a-b^{\alpha-1})(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha)} = \rho_1 g(s),
\end{aligned}$$

and for $t \in [a,b]$, $s \in [0,1]$, we have

$$\begin{aligned}
\frac{\partial G(t,s)}{\partial t} &= \frac{P(s)(1-s)^{\alpha-2} - \Delta(\alpha-1)(t-s)^{\alpha-2}}{\Delta\Gamma(\alpha)} \\
&\geqslant \frac{P(s)(1-s)^{\alpha-2} - \Delta(\alpha-1)(b-bs)^{\alpha-2}}{\Delta\Gamma(\alpha)} \\
&\geqslant \frac{(P(s) - \Delta(\alpha-1)b^{\alpha-2})(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha)} \\
&\geqslant \frac{\Delta(1-(\alpha-1)b^{\alpha-2})(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha)} = \rho_2 g(s).
\end{aligned}$$

Therefore, the proof of Lemma 4 is completed. □

Now we define a cone P on $C^1[0,1]$ and an operator $A: P \to C^1[0,1]$ as follows:

$$P = \Big\{u \in C^1[0,1],\ u(t) \geqslant 0,\ u'(t) \geqslant 0,\ t \in [0,1],\ \min_{t\in[a,b]} u^{(j)}(t) \geqslant \gamma\|u\|,\ j = 0,1\Big\},$$

where $0 < \gamma = \rho/(\alpha-1) < 1$ $(0 < \rho \leqslant 1 < \alpha - 1)$, a and b are the same as in Lemma 4, and

$$Au(t) = \int_0^1 G(t,s) f\big(s, u(s), u'(s)\big)\,\mathrm{d}s, \quad u \in P.$$

Problems (1) has a positive solution if and only if u is a fixed point of A in P.

Lemma 5. *The operator $A: P \to C^1[0,1]$ is continuous.*

Proof. First, for $u \in P$, by the continuity of $G(t,s)$, $s^\sigma f(s,u(s),u'(s))$, and the integrability of $s^{-\sigma}$,

$$Au(t) = \int_0^1 G(t,s) f\big(s, u(s), u'(s)\big)\,\mathrm{d}s, \quad u \in P,$$

is well defined on P. It thus follows from the uniform continuity of $G(t,s)$ in $[0,1]\times[0,1]$ and

$$\big|Au(t_2) - Au(t_1)\big| \leqslant \int_0^1 \big|G(t_2,s) - G(t_1,s)\big| s^{-\sigma} s^\sigma f\big(s, u(s), u'(s)\big)\,\mathrm{d}s$$

that $Au \in C[0,1]$, $u \in P$. Furthermore, by the uniform continuity of $\partial G(t,s)/\partial t$ for $t, s \in [0,1]$, we get

$$(Au)'(t) = \int_0^1 \frac{\partial G(t,s)}{\partial t} f\big(s, u(s), u'(s)\big)\,\mathrm{d}s \in C[0,1].$$

Let $u_n, u \in P$, $u_n \to u$ in $C^1[0,1]$. Since $\partial^j G(t,s)/\partial t^j$ $(j = 0,1)$ is uniformly continuous, there exists $M > 0$ such that

$$\left|\frac{\partial^j G(t,s)}{\partial t^j}\right| \leqslant M, \quad t, s \in [0,1],\ j = 0,1.$$

On the other hand, since $u_n \to u$ in $C^1[0,1]$, there exists $A > 0$ such that $\|u_n\| \leqslant A$ $(n = 1, 2, \dots)$, and then $\|u\| \leqslant A$. Furthermore, $s^\sigma f(s, x_0, x_1)$ is continuous on $[0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$, so $s^\sigma f(s, x_0, x_1)$ is uniformly continuous on $[0,1] \times [0,A] \times [0,A]$. Hence, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $s_1, s_2 \in [0,1]$, $x_0^1, x_0^2, x_1^1, x_1^2 \in [0,A]$, $|s_1 - s_2| < \delta$, $|x_0^1 - x_0^2| < \delta$, $|x_1^1 - x_1^2| < \delta$, we have

$$\big|s_1^\sigma f\big(s_1, x_0^1, x_1^1\big) - s_2^\sigma f\big(s_2, x_0^2, x_1^2\big)\big| < \varepsilon. \tag{6}$$

By $\|u_n - u\| \to 0$, for the above $\delta > 0$, there exists N such that, for all $n > N$, we have

$$\big|u_n(t) - u(t)\big|, \big|u_n'(t) - u'(t)\big| \leqslant \|u_n - u\| < \delta \quad \forall t \in [0,1].$$

Hence, for any $t \in [0,1]$, $n > N$, by (6), we have

$$\big|t^\sigma f\big(t, u_n(t), u_n'(t)\big) - t^\sigma f\big(t, u(t), u'(t)\big)\big| < \varepsilon. \tag{7}$$

Thus, for $n > N$, $t \in [0,1]$, by (7), we have

$$\begin{aligned}
&\big|(Au_n)(t) - (Au)(t)\big| \\
&\quad = \left|\int_0^1 G(t,s) f\big(s, u_n(s), u_n'(s)\big)\,\mathrm{d}s - \int_0^1 G(t,s) f\big(s, u(s), u'(s)\big)\,\mathrm{d}s\right| \\
&\quad = \left|\int_0^1 G(t,s) s^{-\sigma}\big(s^\sigma f\big(s, u_n(s), u_n'(s)\big) - s^\sigma f\big(s, u(s), u'(s)\big)\big)\,\mathrm{d}s\right| \\
&\quad \leqslant M \int_0^1 s^{-\sigma}\big(s^\sigma f\big(s, u_n(s), u_n'(s)\big) - s^\sigma f\big(s, u(s), u'(s)\big)\big)\,\mathrm{d}s \\
&\quad \leqslant M\varepsilon \int_0^1 s^{-\sigma}\,\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
&\big|(Au_n)'(t) - (Au)'(t)\big| \\
&\quad = \left|\int_0^1 \frac{\partial G(t,s)}{\partial t} f\big(s, u_n(s), u_n'(s)\big)\,\mathrm{d}s - \int_0^1 \frac{\partial G(t,s)}{\partial t} f\big(s, u(s), u'(s)\big)\,\mathrm{d}s\right| \\
&\quad = \left|\int_0^1 \frac{\partial G(t,s)}{\partial t} s^{-\sigma}\big(s^\sigma f\big(s, u_n(s), u_n'(s)\big) - s^\sigma f\big(s, u(s), u'(s)\big)\big)\,\mathrm{d}s\right| \\
&\quad \leqslant M \int_0^1 s^{-\sigma}\big(s^\sigma f\big(s, u_n(s), u_n'(s)\big) - s^\sigma f\big(s, u(s), u'(s)\big)\big)\,\mathrm{d}s \\
&\quad \leqslant M\varepsilon \int_0^1 s^{-\sigma}\,\mathrm{d}s,
\end{aligned}$$

and hence, we get $\|Au_n - Au\|_0 \to 0$, $\|(Au_n)' - (Au)'\|_0 \to 0$ $(n \to \infty)$. That is, $\|Au_n - Au\| \to 0 (n \to \infty)$, namely, A is continuous in the space $C^1[0,1]$. □

Lemma 6. *$A : P \to P$ is completely continuous.*

Proof. From Lemma 4 we have $(Au)^{(j)}(t) \geqslant 0$ $(j = 0, 1)$, $t \in [0, 1]$ and

$$\begin{aligned}\max_{t\in[0,1]} (Au)^{(j)}(t) &= \max_{t\in[0,1]} \int_0^1 \frac{\partial^j G(t,s)}{\partial t^j} f\bigl(s, u(s), u'(s)\bigr)\,\mathrm{d}s \\ &\leqslant \int_0^1 \max_{t\in[0,1]} \frac{\partial^j G(t,s)}{\partial t^j} f\bigl(s, u(s), u'(s)\bigr)\,\mathrm{d}s \\ &\leqslant \int_0^1 (\alpha-1) g(s) f\bigl(s, u(s), u'(s)\bigr)\,\mathrm{d}s, \quad j = 0, 1,\end{aligned}$$

so $\|Au\|_0$, $\|(Au)'\|_0 \leqslant \int_0^1 (\alpha-1) g(s) f(s, u(s), u'(s))\,\mathrm{d}s$. Consequently,

$$\|Au\| = \max\bigl\{\|Au\|_0, \bigl\|(Au)'\bigr\|_0\bigr\} \leqslant \int_0^1 (\alpha-1) g(s) f\bigl(s, u(s), u'(s)\bigr)\,\mathrm{d}s.$$

On the other hand, for all $u \in P$, $t \in [a, b]$, by Lemma 4, we have

$$\begin{aligned}(Au)^{(j)}(t) &= \int_0^1 \frac{\partial^j G(t,s)}{\partial t^j} f\bigl(s, u(s), u'(s)\bigr)\,\mathrm{d}s \\ &\geqslant \int_0^1 \rho_{j+1} g(s) f\bigl(s, u(s), u'(s)\bigr)\,\mathrm{d}s \\ &\geqslant \rho_{j+1} \int_0^1 \frac{\partial^{(j)} G(t,s)}{\partial t^j} f\bigl(s, u(s), u'(s)\bigr)\,\mathrm{d}s \\ &\geqslant \frac{\rho_{j+1}}{\alpha-1} \int_0^1 (\alpha-1) g(s) f\bigl(s, u(s), u'(s)\bigr)\,\mathrm{d}s \\ &\geqslant \frac{\rho_{j+1}}{\alpha-1}\|Au\| \geqslant \frac{\rho}{\alpha-1}\|Au\| = \gamma\|Au\|, \quad j = 0, 1.\end{aligned}$$

Thus, $A(P) \subset P$.

Next, we will prove that AV is relatively compact in $C^1[0, 1]$ for bounded $V \subset P$. Since V is bounded, there exists $D > 0$ such that, for any $u \in V$, $\|u\| \leqslant D$, and by the continuity of $t^\sigma f(t, x_0, x_1)$ on $[0, 1] \times [0, D] \times [0, D]$, there exists $C > 0$ such that

$|s^\sigma f(s,u(s),u'(s))| \leqslant C$ for $s \in [0,1]$, $u \in V$. Hence, for $t \in [0,1],\ u \in V$, we have

$$\begin{aligned}\bigl|Au(t)\bigr| &= \int_0^1 G(t,s) f\bigl(s,u(s),u'(s)\bigr)\,\mathrm{d}s = \int_0^1 G(t,s)s^{-\sigma}s^{\sigma} f\bigl(s,u(s),u'(s)\bigr)\,\mathrm{d}s\\ &\leqslant C\int_0^1 (\alpha-1)g(s)s^{-\sigma}\,\mathrm{d}s = \frac{CB_1}{\Gamma(\alpha-1)\Delta},\end{aligned}$$

where $B_1 = \int_0^1 (1-s)^{\alpha-2}s^{-\sigma}\,\mathrm{d}s$. Similarly, we can derive

$$\bigl|(Au)'(t)\bigr| \leqslant \frac{CB_1}{\Gamma(\alpha-1)\Delta}, \quad t\in[0,1],\ u\in V,$$

which shows that AV is bounded in $C^1[0,1]$. Next, we will verify that $(AV)'$ is equicontinuous. Let $t_1, t_2 \in [0,1], t_1 < t_2, u \in V$, we get

$$\begin{aligned}\bigl|(Au)'(t_2) - (Au)'(t_1)\bigr| &= \Biggl|\int_0^1 \frac{P(s)(1-s)^{\alpha-2}}{\Gamma(\alpha)\Delta} f\bigl(s,u(s),u'(s)\bigr)\,\mathrm{d}s.\\ &\quad - \int_0^{t_2} \frac{(\alpha-1)(t_2-s)^{\alpha-2}}{\Gamma(\alpha)} f\bigl(s,u(s),u'(s)\bigr)\,\mathrm{d}s\\ &\quad - \int_0^1 \frac{P(s)(1-s)^{\alpha-2}}{\Gamma(\alpha)\Delta} f\bigl(s,u(s),u'(s)\bigr)\,\mathrm{d}s\\ &\quad + \int_0^{t_1} \frac{(\alpha-1)(t_1-s)^{\alpha-2}}{\Gamma(\alpha)} f\bigl(s,u(s),u'(s)\bigr)\,\mathrm{d}s\Biggr|\\ &= \Biggl|\frac{1}{\Gamma(\alpha-1)\Delta}\int_0^{t_2} (t_2-s)^{\alpha-2}s^{-\sigma}s^{\sigma} f\bigl(s,u(s),u'(s)\bigr)\,\mathrm{d}s\\ &\quad - \frac{1}{\Gamma(\alpha-1)\Delta}\int_0^{t_1} (t_1-s)^{\alpha-2}s^{-\sigma}s^{\sigma} f\bigl(s,u(s),u'(s)\bigr)\,\mathrm{d}s\Biggr|\\ &\leqslant \frac{C}{\Gamma(\alpha-1)\Delta}\Biggl[\int_0^{t_2} (t_2-s)^{\alpha-2}s^{-\sigma}\,\mathrm{d}s - \int_0^{t_1} (t_1-s)^{\alpha-2}s^{-\sigma}\,\mathrm{d}s\Biggr].\end{aligned}$$

Furthermore,

$$\int_0^t (t-s)^{\alpha-2}s^{-\sigma}\,\mathrm{d}s = t^{\alpha-\sigma-1}\int_0^1 (1-s)^{\alpha-2}s^{-\sigma}\,\mathrm{d}s.$$

Thus, we obtain

$$\left|(Au)'(t_2) - (Au)'(t_1)\right| \leqslant \frac{CB_1}{\Gamma(\alpha)\Delta}\left(t_2^{\alpha-1-\sigma} - t_1^{\alpha-1-\sigma}\right)$$

for all $u \in V$. From above and the uniform continuity of $t^{\alpha-1-\sigma}$ on $[0,1]$ and together with Lemma 2, we can derive that AV is relatively compact in $C^1[0,1]$, and so we get that $A : P \to P$ is completely continuous. □

Definition 4. The map α is said to be a non-negative continuous concave functional on P, provided $\alpha : P \to R^+$ is continuous and

$$\alpha\left(tx + (1-t)y\right) \geqslant \alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$, $t \in [0,1]$.

Definition 5. The map β is said to be a non-negative continuous convex functional on P, provided $\beta : P \to R^+$ is continuous and

$$\beta\left(tx + (1-t)y\right) \leqslant \beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$, $t \in [0,1]$.

3 Main result

Let φ, θ be non-negative continuous convex functionals on P, ϕ be a non-negative continuous concave functional on P, and ψ be a non-negative continuous functional on P. Then, for non-negative numbers e, c, d, h, we define the following convex sets:

$$\begin{aligned}
P(\varphi, h) &= \left\{x \in P \mid \varphi(x) < h\right\},\\
P(\varphi, \phi, c, h) &= \left\{x \in P \mid \phi(x) \geqslant c,\ \varphi(x) \leqslant h\right\},\\
P(\varphi, \theta, \phi, c, d, h) &= \left\{x \in P \mid c \leqslant \phi(x),\ \theta(x) \leqslant d,\ \varphi(x) \leqslant h\right\},\\
R(\varphi, \psi, e, h) &= \left\{x \in P \mid e \leqslant \psi(x),\ \varphi(x) \leqslant h\right\}.
\end{aligned}$$

We will apply the following fixed point theorem of Avery and Peterson to solve problem (1).

Lemma 7. (*See* [1, 13].) *Let P be a cone of E, φ and θ be non-negative continuous convex functionals on P, ϕ be a non-negative continuous concave functional on P, and ψ be a non-negative continuous functional on P, $\psi(\mu x) \leqslant \mu\psi(x)$ for $0 \leqslant \mu \leqslant 1$ such that, for some positive numbers L and h,*

$$\phi(x) \leqslant \psi(x) \quad \textit{and} \quad \|x\| \leqslant L\varphi(x)$$

for all $x \in \overline{P(\varphi, h)}$. Let

$$A : \overline{P(\varphi, h)} \to \overline{P(\varphi, h)}$$

is completely continuous, and there exist positive numbers e, c, d with $e < c$ such that the following conditions are satisfied:

(S1) $\{x \in P(\varphi, \theta, \phi, c, d, h)\colon \phi(x) > c\} \neq \phi$ *and* $\phi(Ax) > c$ *for* $x \in P(\varphi, \theta, \phi, c, d, h)$;
(S2) $\phi(Ax) > c$ *for* $x \in P(\varphi, \phi, c, h)$ *and* $\theta(Ax) > d$;
(S3) $0 \notin R(\varphi, \psi, e, h)$ *and* $\psi(Ax) < e$ *for* $x \in R(\varphi, \psi, e, h)$ *with* $\psi(x) = e$.

Then A *has at least three fixed points* x_1, x_2, x_3 *such that*

$$\varphi(x_i) \leqslant h, \quad i = 1, 2, 3,$$

and

$$c < \phi(x_1), \qquad e < \psi(x_2), \qquad \phi(x_2) < c, \qquad \psi(x_3) < e.$$

Let the convex functions $\psi(u) = \theta(u) = \varphi(u) = \|u\|$ on P, and define a concave function $\phi(u) = \min\{\min_{t\in[a,b]} |u(t)|, \min_{t\in[a,b]} |u'(t)|\}$, where a, b are the same as in Lemma 4.

Theorem 1. *Assume that there exist positive numbers* e, c, d, h *with* $c > e$, $d > \max\{1/\rho, e^{1-a/2}\}c$, $h > rc/(\rho Q)$ *and* $h \geqslant d$ *such that*:

(H3) $t^\sigma f(t, x, y)) < h/r$ *for* $(t, x, y) \in [0, 1] \times [0, h]^2$;
(H4) $f(t, x, y) \geqslant c/(\rho Q)$ *for* $(t, x, y) \in [a, b] \times [c, d]^2$;
(H5) $t^\sigma f(t, x, y) < e/r$ *for* $(t, x, y) \in [0, 1] \times [0, e]^2$, *where* $r = (\alpha-1)\int_0^1 g(s)s^{-\sigma}\,\mathrm{d}s$, $Q = \int_a^b g(s)\,\mathrm{d}s$.

Then problem (1) *has at least three fixed points* u_1, u_2, u_3 *satisfying*

$$\|u_i\| \leqslant h, \quad i = 1, 2, 3,$$

and

$$c < \min\Bigl\{\min_{t\in[a,b]} \bigl|u_1(t)\bigr|, \min_{t\in[a,b]} \bigl|u_1'(t)\bigr|\Bigr\}, \qquad e < \|u_2\|,$$
$$\min\Bigl\{\min_{t\in[a,b]} \bigl|u_2(t)\bigr|, \min_{t\in[a,b]} \bigl|u_2'(t)\bigr|\Bigr\} < c, \qquad \|u_3\| < e.$$

Proof. Let $u \in \overline{P(\varphi, h)}$. By condition (H3), we get

$$\|Au\|_0 = \max_{t\in[0,1]} \bigl|Au(t)\bigr| \leqslant (\alpha - 1)\int_0^1 g(s)s^{-\sigma}s^\sigma f\bigl(s, u(s), u'(s)\bigr)\,\mathrm{d}s \leqslant h,$$

$$\|(Au)'\|_0 = \max_{t\in[0,1]} \left|\frac{\partial(Au)(t)}{\partial t}\right| \leqslant (\alpha - 1)\int_0^1 g(s)s^{-\sigma}s^\sigma f\bigl(s, u(s), u'(s)\bigr)\,\mathrm{d}s \leqslant h.$$

Consequently, we obtain $\varphi(Au) = \|Au\| \leqslant h$. This, together with Lemmas 5 and 6, means that $A : \overline{P(\varphi, h)} \to \overline{P(\varphi, h)}$ is completely continuous.

Take $u(t) = ce^{t-0.5a}$, $t \in [0,1]$. By simple calculation, we have that $u \in P$, $\|u\| < d$, and $\phi(u) > c$ and so

$$\bigl\{u \in P(\varphi, \theta, \phi, c, d, h)\colon\ c < \phi(u)\bigr\} \neq \emptyset.$$

For $u \in P(\varphi, \theta, \phi, c, d, h)$, by (H4), we get

$$\begin{aligned}\phi(Au) &= \min\Bigl\{\min_{t\in[a,b]} \bigl|Au(t)\bigr|, \min_{t\in[a,b]} \bigl|(Au)'(t)\bigr|\Bigr\} \\ &\geqslant \rho \int_0^1 g(s) f\bigl(s, u(s), u'(s)\bigr)\,\mathrm{d}s > \int_a^b \rho g(s) \frac{c}{\rho Q}\,\mathrm{d}s = b,\end{aligned}$$

which shows that condition (S1) is satisfied.

Take $u \in P(\varphi, \phi, c, h)$ and $\|Au\| > d$. Since $Au \in P$, we obtain

$$\phi(Au) = \min\Bigl\{\min_{t\in[a,b]} \bigl|Au(t)\bigr|, \min_{t\in[a,b]} \bigl|(Au)'(t)\bigr|\Bigr\} \geqslant \rho\|Au\| \geqslant \rho d > c,$$

which implies that condition (S2) holds.

Next, we will verify that condition (S3) holds. For $\psi(0) = 0$, we have $0 \in R(\varphi, \psi, e, h)$. Let $u \in R(\varphi, \psi, e, h)$ and $\psi(u) = \|u\| = e$, by (H5), we get

$$\begin{aligned}\|Au\|_0 = \max_{t\in[0,1]} \bigl|Au(t)\bigr| &\leqslant \int_0^1 (\alpha - 1) g(s) s^{-\sigma} s^{\sigma} f\bigl(s, u(s), u'(s)\bigr)\,\mathrm{d}s \\ &< \frac{e(\alpha-1)}{r} \int_0^1 g(s) s^{-\sigma}\,\mathrm{d}s \leqslant e,\end{aligned}$$

and

$$\begin{aligned}\bigl\|(Au)'\bigr\|_0 = \max_{t\in[0,1]} \left|\frac{\partial(Au)(t)}{\partial t}\right| &\leqslant \int_0^1 (\alpha - 1) g(s) s^{-\sigma} s^{\sigma} f\bigl(s, u(s), u'(s)\bigr)\,\mathrm{d}s \\ &< \frac{e(\alpha-1)}{r} \int_0^1 g(s) s^{-\sigma}\,\mathrm{d}s \leqslant e.\end{aligned}$$

Consequently, we have $\psi(Au) = \|Au\| < e$. Thus, condition (S3) holds.

By Lemma 7, we get that (1) has at least three positive solutions u_1, u_2, u_3 satisfying

$$\|u_i\| \leqslant h, \quad i = 1, 2, 3,$$

and

$$\begin{aligned}&c < \min\Bigl\{\min_{t\in[a,b]} \bigl|u_1(t)\bigr|, \min_{t\in[a,b]} \bigl|u_1'(t)\bigr|\Bigr\}, \qquad e < \|u_2\|, \\ &\min\Bigl\{\min_{t\in[a,b]} \bigl|u_2(t)\bigr|, \min_{t\in[a,b]} \bigl|u_2'(t)\bigr|\Bigr\} < c, \qquad \|u_3\| < e.\end{aligned}$$

□

4 An example

Consider the following infinite-point boundary value problem:

$$
\begin{aligned}
&{}^{c}D_{0+}^{5/2}u(t) + f\big(t, u(t), u'(t)\big) = 0, \quad 0 < t < 1, \\
&u(0) = u''(0) = 0, \qquad u''(1) = \sum_{j=1}^{\infty} \frac{1}{2j^2} u\left(\frac{1}{j^2}\right),
\end{aligned}
\tag{8}
$$

where

$$
f(t,x,y) = \begin{cases} (x^2+y^2)/(2\sqrt{\pi t}), & (t,x,y) \in (0,1] \times [0,1/2] \times [0,1/2], \\ 499(\sqrt[6]{x} + \sqrt[6]{y})/(2\sqrt{\pi t}), & (t,x,y) \in (0,1] \times [1,20] \times [1,20], \\ 499/(2\sqrt{\pi t}), & (t,x,y) \in (0,1] \times [100,\infty) \times [100,\infty). \end{cases}
$$

Obviously, $\sqrt{t}f(t,x,y)$ is continuous in $[0,1]\times\mathbb{R}^+\times\mathbb{R}^+$, and $\sqrt{t}f(t,x,y) \leqslant 499\cdot 3/\sqrt{\pi}$ for $(t,x,y) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$.

For Theorem 1, we take $\alpha = 2.5$, $\sigma = 0.5$, $a = 0.3$, $b = 0.4$, $\eta_j = 1/(2j^2)$, $\xi_j = 1/j^2$, $\Delta = 1 - \Sigma_{j=1}^{\infty}\eta_j\xi_j \approx 0.4589$, $g(s) = (1-s)^{\alpha-2}/(\Gamma(\alpha)\Delta) = 2(1-s)^{1/2}/\sqrt{\pi}$, $r = \int_0^1 g(s)s^{-\sigma}\,\mathrm{d}s = \sqrt{\pi}$, $L = \int_a^b g(s)\,\mathrm{d}s = 2(0.7^{3/2} - 0.6^{3/2})/3$,

$$
\begin{aligned}
\rho_1 &= \Delta\big(a - b^{\alpha-1}\big) \approx 0.0216, \\
\rho_2 &= \Delta\big(1 - (\alpha-1)b^{\alpha-2}\big) \approx 0.0235,
\end{aligned}
$$

and as $\rho_1 < \rho_2$, $\rho = \rho_1$.

Let $e = 0.5$, $f = 1$, $g = 20$, $d = 500$. By direct calculation, we get that the conditions of Theorem 1 are satisfied. So, the BVP (8) has at least three positive solutions u_1, u_2, u_3 satisfying

$$
u_i \leqslant 500, \quad i = 1,2,3,
$$

and $1 < \phi(u_1)$, $0.5 < \|u_2\|$, $\phi(u_2) < 1$, $\|u_3\| < 0.5$.

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