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On the effect of α-admissibility and θ-contractivity to the existence of fixed points of multivalued mappings

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Abstract. In this study, we give some fixed point results for multivalued mappings using α-admissibility and θ-contractivity of multivalued mappings on complete metric spaces. Our results are proper generalizations of some fixed point results related to multivalued contraction. We also provide an example showing this fact. Finally, we obtain some ordered fixed point results for multivalued mappings as corollary.

Keywords: fixed point, multivalued mapping, multivalued θ-contraction, α-admissible mapping.

1 Introduction and preliminaries

Fixed point theory is one of the most rapidly growing research areas in nonlinear functional analysis and a very powerful tool of the current mathematical applications. Except for a great number of extensions of Banach's contraction mapping principle for single valued mappings, it was also naturally extended to multivalued mappings by Nadler [25] in 1969, which is also sometimes referred to as Nadler's multivalued contraction principle. Since then, there has been continuous and intense research activity in multivalued mapping fixed point theory and by now there are a number of results that extend this result in many different directions (see [8, 9, 11, 18, 21, 29, 30]).

Recently, Samet et al. [31] introduced the interesting notion called α-admissible self-mappings of metric spaces. Using this concept, they give a class of mappings known as α-ψ-contractive mapping including Banach contractions. Also, Jleli and Samet [16] introduced a new type of contraction called θ-contraction, and it was extended to multivalued mapping case by Hançer et al. [12]. In this work, considering the concepts of multivalued θ-contraction and multivalued α-admissible mappings, we present some multivalued fixed point results on complete metric spaces. Moreover, two examples and some corollaries are given to illustrate the usability of the obtained results.

In the following lines, we present some notational and terminological conventions, which will be used throughout this paper.

Let (X,d) be a metric space. We denote by $P(X)$ the family of all nonempty subsets of X, by $CB(X)$ the family of all nonempty closed and bounded subsets of X and by $K(X)$ the family of all nonempty compact subsets of X. We define the Pompeiu–Hausdorff distance with respect to d by

$$H(A,B)=\max\Big\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\Big\}$$

for every $A,B\in CB(X)$, where $d(x,B)=\inf\{d(x,y)\colon\ y\in B\}$. It is well known that H is a metric on $CB(X)$. We can find detailed information about the Pompeiu–Hausdorff metric in [2, 6, 15]. An element $x\in X$ is called a fixed point of a multivalued mapping $T:X\to P(X)$ if $x\in Tx$. Let $T:X\to CB(X)$. Then T is said to be a multivalued contraction if there exists $L\in[0,1)$ such that $H(Tx,Ty)\leqslant Ld(x,y)$ for all $x,y\in X$ (see [25]). In 1969, Nadler [25] proved a fundamental fixed point theorem for multivalued mappings: Every multivalued contraction mappings on complete metric spaces has a fixed point.

Let (X,d) be a metric space, $T:X\to P(X)$ and $\alpha:X\times X\to[0,\infty)$ be a function. Then we say that:

1. T is an α-admissible mapping whenever $\alpha(x,y)\geqslant 1$ for each $x\in X$ and $y\in Tx$ implies $\alpha(y,z)\geqslant 1$ for all $z\in Ty$.
2. T is an α_*-admissible mapping whenever $\alpha(x,y)\geqslant 1$ for each $x\in X$ and $y\in Tx$ implies $\alpha_*(Tx,Ty)\geqslant 1$, where $\alpha_*(Tx,Ty)=\inf\{\alpha(a,b)\colon\ a\in Tx,\ b\in Ty\}$.
3. α has property (B) whenever $\{x_n\}$ is a sequence in X such that if $\alpha(x_n,x_{n+1})\geqslant 1$ for all $n\in\mathbb{N}$ and $x_n\to x$, then $\alpha(x_n,x)\geqslant 1$ for all $n\in\mathbb{N}$.

It is easy to see that α_*-admissible mapping are also α-admissible mapping, but the converse may not be true as shown in Example 15 of [22].

Let Ψ be the family of nondecreasing functions $\psi:[0,\infty)\to[0,\infty)$ such that $\sum_{n=1}^{\infty}\psi^n(t)<\infty$ for all $t>0$, where ψ^n is the nth iterate of ψ. It is easily proved that if $\psi\in\Psi$, then $\psi(0)=0$ and $\psi(t)<t$ for all $t>0$.

A multivalued mapping $T:X\to CB(X)$ is called multivalued α-ψ-contractive whenever, for all $x,y\in X$,

$$\alpha(x,y)H(Tx,Ty)\leqslant\psi\big(d(x,y)\big),$$

and multivalued α_*-ψ-contractive whenever, for all $x,y\in X$,

$$\alpha_*(Tx,Ty)H(Tx,Ty)\leqslant\psi\big(d(x,y)\big).$$

The fixed point results for these type mappings are given as follows (also see [3, 13, 14, 17, 19, 23]):

Theorem 1. (*See* [24].) *Let (X,d) be a complete metric space, $T: X \to CB(X)$ be a multivalued mapping, $\alpha: X \times X \to [0,\infty)$ be a function, and $\psi \in \Psi$ be a strictly increasing mapping. Assume that the following conditions hold*:

(i) *T is α-admissible and multivalued α-ψ-contractive on X*;
(ii) *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geqslant 1$*;
(iii) *T is continuous or α has property* (B).

Then T has a fixed point in X.

Theorem 2. (*See* [4].) *Let (X,d) be a complete metric space, $T: X \to CB(X)$ be a multivalued mapping, $\alpha: X \times X \to [0,\infty)$ be a function, and $\psi \in \Psi$ be a strictly increasing mapping. Assume that the following conditions hold*:

(i) *T is α_*-admissible and multivalued α_*-ψ-contractive on X*;
(ii) *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geqslant 1$*;
(iii) *T is continuous or α has property* (B).

Then T has a fixed point in X.

On the other hand, Jleli and Samet [16] introduced a new type of contractive mappings. We called it as θ-contraction and proved a fixed point theorem for mappings of this type for which the Banach contraction principle and some other known contractive conditions in the literature can be obtained as special cases. We denote the family of all functions $\theta: (0,\infty) \to (1,\infty)$ satisfying the following properties by Θ:

(Θ_1) θ is nondecreasing;
(Θ_2) For each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty}\theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$;
(Θ_3) There exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0^+}(\theta(t)-1)/t^r = l$.

By considering conditions (Θ_1)–(Θ_3), Jleli and Samet [16] introduced the concept of θ-contraction as follows:

Let (X,d) be a metric space. A mapping $T: X \to X$ is said to be an θ-contraction if there exists $\theta \in \Theta$ and $k \in (0,1)$ such that

$$\theta\bigl(d(Tx,Ty)\bigr) \leqslant \bigl[\theta\bigl(d(x,y)\bigr)\bigr]^k \tag{1}$$

for all $x, y \in X$ with $d(Tx,Ty) > 0$. If we consider the different type of mapping θ, then we obtain some classes of contractions known in the literature. For example, let $\theta: (0,\infty) \to (1,\infty)$ be given by $\theta(t) = \mathrm{e}^{\sqrt{t}}$. It is clear that $\theta \in \Theta$. Then, for each $T: X \to X$ mapping satisfying (1), we have

$$d(Tx,Ty) \leqslant k^2 d(x,y) \tag{2}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$. It is clear that for $x, y \in X$ such that $Tx = Ty$, the above inequality also holds. Therefore, T is an ordinary contraction. Similarly, let $\theta : (0, \infty) \to (1, \infty)$ be given by $\theta(t) = \mathrm{e}^{\sqrt{t\mathrm{e}^t}}$. It is clear that $\theta \in \Theta$. Then, for each mapping $T : X \to X$ satisfying (1), we have

$$\frac{d(Tx, Ty)}{d(x, y)} \mathrm{e}^{d(Tx,Ty)-d(x,y)} \leqslant k^2 \tag{3}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$.

In addition, we have concluded that every θ-contraction T is a contractive mapping, i.e., $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Thus, every θ-contraction mapping is continuous. On the other side, example in [16] shows that the mapping $T : X \to X$ is not ordinary contraction, but it is a θ-contraction with $\theta(t) = \mathrm{e}^{\sqrt{t\mathrm{e}^t}}$. Thus, the following theorem, which was given as a corollary by Jleli and Samet is a proper generalization of Banach's contraction principle.

Theorem 3. (*See* [16, *Cor.* 2.1].) *Let (X, d) be a complete metric space and $T : X \to X$ be a θ-contraction. Then T has a unique fixed point in X.*

Recently, Hançer et al. [12] focused on θ-contraction for multivalued mappings and introduced the concept of multivalued θ-contractions. See also recent paper [32]. Let (X, d) be a complete metric space and $T : X \to CB(X)$. Then T is said to be a multivalued θ-contraction if there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that

$$\theta\bigl(H(Tx, Ty)\bigr) \leqslant \bigl[\theta\bigl(d(x, y)\bigr)\bigr]^k$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$. It is clear that every multivalued contraction mappings is also multivalued θ-contraction with $\theta(t) = \mathrm{e}^{\sqrt{t}}$.

Theorem 4. (*See* [12].) *Let (X, d) be a complete metric space, and $T : X \to K(X)$ be a multivalued θ-contraction. Then T has a fixed point in X.*

As shown in [12, Ex. 1], we cannot take $CB(X)$ instead of $K(X)$ in Theorem 4. Nevertheless, it can be seen $CB(X)$ instead of $K(X)$ by adding the following condition on θ (see [12]) as follows:

(Θ_4) $\theta(\inf A) = \inf \theta(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

Note that if θ satisfies (Θ_1), then it satisfies (Θ_4) if and only if it is right continuous. Now, let Θ_* denote the family of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying (Θ_1)–(Θ_4). The function $\theta : (0, \infty) \to (1, \infty)$ defined by $\theta(t) = \mathrm{e}^{\sqrt{t}}$ for $t \leqslant 1$, and $\theta(t) = 3$ for $t > 1$, is belonging to $\Theta \setminus \Theta_*$.

Therefore, considering the class Θ_*, we can provide the following theorem, which is a real generalization of Nadler's result.

Theorem 5. (*See* [12].) *Let (X, d) be a complete metric space, and $T : X \to CB(X)$ be a multivalued θ-contraction with $\theta \in \Theta_*$. Then T has a fixed point in X.*

2 Main results

Before we give our main results, we recall some definitions and facts for multivalued mappings. Let X and Y be two topological spaces. Then a multivalued mapping $T: X \to P(Y)$ is said to be upper semicontinuous (lower semicontinuous) if the inverse image of closed sets (open sets) is closed (open). A multivalued mapping is continuous if it is upper as well as lower semicontinuous.

Lemma 1. (*See* [15].) *Let (X, d) be a metric space, and let $T : X \to P(X)$ be an upper semicontinuous mapping such that Tx is closed for all $x \in X$. If $x_n \to x_0$, $y_n \to y_0$ and $y_n \in Tx_n$, then $y_0 \in Tx_0$.*

Now, we give concept of multivalued (α, θ)-contraction. Let (X, d) be a metric space, $T : X \to CB(X)$ and $\alpha : X \times X \to [0, \infty)$ be a function. We define the set $S_{T,\alpha} \subseteq X \times X$ by

$$S_{T,\alpha} = \bigl\{(x, y)\colon\ \alpha(x, y) \geqslant 1 \text{ and } H(Tx, Ty) > 0\bigr\}.$$

For simplicitly, we denote it further just by S. Note that $(x, x) \notin S$. Then we say that T is a multivalued (α, θ)-contraction if there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that

$$\theta\bigl(H(Tx, Ty)\bigr) \leqslant \bigl[\theta\bigl(d(x, y)\bigr)\bigr]^k \tag{4}$$

for all $(x, y) \in S$.

Now, we give some fixed point results for mappings of this type on a complete metric space.

Theorem 6. *Let (X, d) be a complete metric space, $T : X \to K(X)$ be a multivalued mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Assume that the following conditions hold*:

(i) *T is an α-admissible and multivalued (α, θ)-contraction with $\theta \in \Theta$*;
(ii) *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geqslant 1$*;
(iii) *T is upper semicontinuous or α has property* (B).

Then T has a fixed point in X.

Proof. Suppose that T has no fixed point. Then $d(x, Tx) > 0$ for all $x \in X$. Let x_0 and x_1 be as mentioned in the hypotheses. Then we have $H(Tx_0, Tx_1) > 0$. Therefore, $(x_0, x_1) \in S$. So, from (4) and considering (Θ_1), we obtain that

$$\theta\bigl(d(x_1, Tx_1)\bigr) \leqslant \theta\bigl(H(Tx_0, Tx_1)\bigr) \leqslant \bigl[\theta\bigl(d(x_0, x_1)\bigr)\bigr]^k, \tag{5}$$

where $k \in (0, 1)$. Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) = d(x_1, Tx_1).$$

Then from (5) we get

$$\theta\bigl(d(x_1, x_2)\bigr) \leqslant \theta\bigl(H(Tx_0, Tx_1)\bigr) \leqslant \bigl[\theta\bigl(d(x_0, x_1)\bigr)\bigr]^k.$$

Also, since T is α-admissible, we get $\alpha(x_1, x_2) \geqslant 1$ for $x_2 \in Tx_1$. Again, since $x_2 \in Tx_1$, $(x_1, x_2) \in S$. So, from (4) we obtain that

$$\theta\bigl(d(x_2, Tx_2)\bigr) \leqslant \theta\bigl(H(Tx_1, Tx_2)\bigr) \leqslant \bigl[\theta\bigl(d(x_1, x_2)\bigr)\bigr]^k. \tag{6}$$

Since Tx_2 is compact, there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) = d(x_2, Tx_2).$$

Then from (6) we have

$$\theta\bigl(d(x_2, x_3)\bigr) \leqslant \theta\bigl(H(Tx_1, Tx_2)\bigr) \leqslant \bigl[\theta\bigl(d(x_1, x_2)\bigr)\bigr]^k.$$

In this way, we can construct a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, $(x_n, x_{n+1}) \in S$ and

$$\theta\bigl(d(x_n, x_{n+1})\bigr) \leqslant \bigl[\theta\bigl(d(x_n, x_{n-1})\bigr)\bigr]^k \tag{7}$$

for all $n \in \mathbb{N}$. Denote $d_n = d(x_n, x_{n+1})$ for $n \in \mathbb{N} \cup \{0\}$. Then $d_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$, and using (7), we have

$$\theta(d_n) \leqslant \bigl[\theta(d_{n-1})\bigr]^k \leqslant \bigl[\theta(d_{n-2})\bigr]^{k^2} \leqslant \cdots \leqslant \bigl[\theta(d_0)\bigr]^{k^n}.$$

Thus, we obtain

$$1 < \theta(d_n) \leqslant \bigl[\theta(d_0)\bigr]^{k^n} \tag{8}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (8), we obtain

$$\lim_{n\to\infty} \theta(d_n) = 1. \tag{9}$$

From (Θ_2), $\lim_{n\to\infty} d_n = 0^+$, and so, from (Θ_3) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d_n) - 1}{(d_n)^r} = l.$$

Suppose that $l < \infty$. In this case, let $D = l/2 > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$,

$$\left|\frac{\theta(d_n) - 1}{(d_n)^r} - l\right| \leqslant D.$$

This implies that, for all $n \geqslant n_0$,

$$\frac{\theta(d_n) - 1}{(d_n)^r} \geqslant l - D = D.$$

Then, for all $n \geqslant n_0$,

$$n(d_n)^r \leqslant An\bigl[\theta(d_n) - 1\bigr],$$

where $A = 1/D$.

Suppose now that $l = \infty$. Let $D > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$,

$$\frac{\theta(d_n) - 1}{(d_n)^r} \geqslant D.$$

This implies that, for all $n \geqslant n_0$,

$$n(d_n)^r \leqslant An\big[\theta(d_n) - 1\big],$$

where $A = 1/D$.

Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$,

$$n(d_n)^r \leqslant An\big[\theta(d_n) - 1\big].$$

Using (8), we obtain, for all $n \geqslant n_0$,

$$n(d_n)^r \leqslant An\big[\big[\theta(d_0)\big]^{k^n} - 1\big].$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n(d_n)^r = 0.$$

Thus, there exits $n_1 \in \mathbb{N}$ such that $n(d_n)^r \leqslant 1$ for all $n \geqslant n_1$. So, for all $n \geqslant n_1$, we have

$$d_n \leqslant \frac{1}{n^{1/r}}. \tag{10}$$

In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geqslant n_1$. Using the triangular inequality for the metric and from (10), we have

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= d_n + d_{n+1} + \cdots + d_{m-1} \\ &= \sum_{i=n}^{m-1} d_i \leqslant \sum_{i=n}^{\infty} d_i \leqslant \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} i^{-1/r}$ we get $d(x_n, x_m) \to 0$ as $n \to \infty$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$.

If T is upper semicontinuous, then by Lemma 1 we have $z \in Tz$. This contradicts with the assumption that T has no fixed point.

Now, assume that α has property (B). Then $\alpha(x_n, z) \geqslant 1$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} x_n = z$ and $d(z, Tz) > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_{n+1}, Tz) > 0$ for all $n \geqslant n_0$. Therefore, $H(Tx_n, Tz) > 0$ for all $n \geqslant n_0$. Thus, $(x_n, z) \in S$ for all $n \geqslant n_0$. Now, from (4) and (Θ_1) we have

$$\theta\big(d(x_{n+1}, Tz)\big) \leqslant \theta\big(H(Tx_n, Tz)\big) \leqslant \big[\theta\big(d(x_n, z)\big)\big]^k,$$

and so,

$$d(x_{n+1}, Tz) \leqslant d(x_n, z) \tag{11}$$

for all $n \geqslant n_0$. Letting $n \to \infty$ in (11), we obtain $d(z, Tz) = 0$. That is, we get $z \in Tz$. This contradicts with the assumption that T has no fixed point.

As a result, this proves that T has a fixed point in X. □

Remark 1. As we mentioned earlier, we cannot extend the range of T, i.e., $K(X)$ cannot be replaced by $CB(X)$ in Theorem 6. Example 1 in [12] shows this fact by taking $\alpha(x, y) = 1$. However, we can take $CB(X)$ instead of $K(X)$ by adding condition (Θ_4) on θ.

Theorem 7. *Let (X, d) be a complete metric space, $T : X \to CB(X)$ be a multivalued mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Assume that the following conditions hold*:

(i) *T is an α-admissible and multivalued (α, θ)-contraction with $\theta \in \Theta_*$*;
(ii) *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geqslant 1$*;
(iii) *T is upper semicontinuous or α has property* (B). *Then T has a fixed point in X.*

Proof. We begin proceeding as in the proof of Theorem 6. Considering condition (Θ_4), we can write

$$\theta\bigl(d(x_1, Tx_1)\bigr) = \inf_{y \in Tx_1} \theta\bigl(d(x_1, y)\bigr).$$

So, from

$$\theta\bigl(d(x_1, Tx_1)\bigr) \leqslant \theta\bigl(H(Tx_0, Tx_1)\bigr) \leqslant \bigl[\theta\bigl(d(x_0, x_1)\bigr)\bigr]^k$$

we have

$$\inf_{y \in Tx_1} \theta\bigl(d(x_1, y)\bigr) \leqslant \bigl[\theta\bigl(d(x_0, x_1)\bigr)\bigr]^k < \bigl[\theta\bigl(d(x_0, x_1)\bigr)\bigr]^{(k+1)/2}. \tag{12}$$

Then from (12) there exists $x_2 \in Tx_1$ such that

$$\theta\bigl(d(x_1, x_2)\bigr) \leqslant \bigl[\theta\bigl(d(x_0, x_1)\bigr)\bigr]^{(k+1)/2}.$$

The rest of the proof is analogous to Theorem 6. □

Remark 2. If we take $\alpha(x, y) = 1$ in Theorem 7, we obtain Theorem 5. In addition, if we take $\theta(t) = \mathrm{e}^{\sqrt{t}}$, we obtain Nadler's result.

Now, we give an example showing that T is a multivalued (α, θ)-contraction but not multivalued θ-contraction. Therefore, Theorem 7 (resp. Theorem 6) can be applied to this example, but Theorems 1, 2, 4 and 5 cannot.

Example 1. Consider the complete metric space (X, d), where $X = \{0, 2, 4, \dots\}$ and $d : X \times X \to [0, \infty)$ is given by

$$d(x, y) = \begin{cases} 0, & x = y, \\ x + y, & x \neq y. \end{cases}$$

Define $T: X \to CB(X)$ by

$$Tx = \begin{cases} \{x\}, & x \in \{0, 2\}, \\ \{0, 2, \ldots, x-2\}, & x \geqslant 4, \end{cases}$$

and $\alpha: X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 0, & (x, y) \in \{(0, 2), (2, 0)\}, \\ 2, & \text{otherwise.} \end{cases}$$

Then it is clear that T is an α-admissible mapping.

Now, we claim that T is a multivalued (α, θ)-contraction with $k = \mathrm{e}^{-1}$ and $\theta(t) = \mathrm{e}^{\sqrt{t\mathrm{e}^t}}$. To see this, we have to show that

$$\theta\bigl(H(Tx, Ty)\bigr) \leqslant \bigl[\theta\bigl(d(x, y)\bigr)\bigr]^k$$

for all $(x, y) \in S$ or, equivalently,

$$\frac{H(Tx, Ty)}{d(x, y)} \mathrm{e}^{H(Tx, Ty) - d(x, y)} \leqslant \mathrm{e}^{-2} \tag{13}$$

for all $(x, y) \in S$. Note that

$$\begin{aligned} S &= \bigl\{(x, y) \in X \times X\colon\ \alpha(x, y) \geqslant 1 \text{ and } H(Tx, Ty) > 0\bigr\} \\ &= \bigl\{(x, y) \in X \times X\colon\ (x, y) \notin \bigl\{(0, 2), (2, 0)\bigr\} \text{ and } x \neq y\bigr\}. \end{aligned}$$

Thus, without loss of generality, we may assume $x > y$ for all $(x, y) \in S$ in the following cases:

Case 1. Let $y = 0$ and $x \geqslant 4$. Then $H(Tx, Ty) = x - 2$ and $d(x, y) = x$, and so, we have

$$\frac{H(Tx, Ty)}{d(x, y)} \mathrm{e}^{H(Tx, Ty) - d(x, y)} \leqslant \frac{x-2}{x} \mathrm{e}^{-2} \leqslant \mathrm{e}^{-2}.$$

Case 2. Let $y = 2$ and $x = 4$. Then $H(Tx, Ty) = 2$ and $d(x, y) = 6$, and so, we have

$$\frac{H(Tx, Ty)}{d(x, y)} \mathrm{e}^{H(Tx, Ty) - d(x, y)} \leqslant \frac{1}{3} \mathrm{e}^{-4} \leqslant \mathrm{e}^{-2}.$$

Case 3. Let $y = 2$ and $x > 4$. Then $H(Tx, Ty) = x$ and $d(x, y) = x + 2$, and so, we have

$$\frac{H(Tx, Ty)}{d(x, y)} \mathrm{e}^{H(Tx, Ty) - d(x, y)} \leqslant \frac{x}{x+2} \mathrm{e}^{-2} \leqslant \mathrm{e}^{-2}.$$

Case 4. Let $x > y \geqslant 4$. Then $H(Tx, Ty) = x - 2$ and $d(x, y) = x + y$, and so, we have

$$\frac{H(Tx, Ty)}{d(x, y)} \mathrm{e}^{H(Tx,Ty)-d(x,y)} = \frac{x-2}{x+y} \mathrm{e}^{-2-y} \leqslant \mathrm{e}^{-2}.$$

This shows that T is a multivalued (α, θ)-contraction.

For $x_0 = 2$ and $x_1 \in Tx_0 = \{2\}$, we have $\alpha(x_0, x_1) = \alpha(2, 2) = 2 \geqslant 1$.

Finally, since τ_d is discrete topology, T is upper semicontinuous. Therefore, all conditions of Theorem 7 (resp. Theorem 6) are satisfied. Then T has a fixed point in X.

Note that α has no property (B). Indeed, considering the sequence $\{x_n\} = \{2, 4, 6, 0, 0, 0, \dots\}$ in X, then $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in N$ and $x_n \to 0$, but $\alpha(x_1, z) = \alpha(2, 0) = 0 \ngeqslant 1$. Also, since $\alpha(2, 4) \geqslant 1$ but $\alpha_*(T2, T4) = 0$, then T is not an α_*-admissible.

On the other hand, since $H(T0, T2) = 2 = d(0, 2)$, then for all $\theta \in \Theta$ and $k \in (0, 1)$, we have

$$\theta\big(H(Tx, Ty)\big) = \theta(2) > \big[\theta(2)\big]^k = \big[\theta\big(d(x, y)\big)\big]^k.$$

That is, T is not multivalued θ-contraction. Therefore, Theorems 4 and 5 cannot be applied to this example. Also, T is not multivalued contraction.

Finally, since $H(T0, T4) = 2$, $d(0, 4) = 4$ and $\alpha(0, 4) = 2$, then for all $\psi \in \Psi$, we have

$$4 = \alpha(0, 4)H(T0, T4) \nleqslant \psi\big(d(0, 4)\big) < d(0, 4) = 4.$$

Thus, T is not multivalued α-ψ-contractive mapping. Therefore, Theorems 1 and 2 cannot be applied, too.

Corollary 1. *Let (X, d) be a complete metric space, $T : X \to CB(X)$ (resp. $K(X)$) be a multivalued mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Assume that the following conditions hold*:

(i) *T be an α_*-admissible*;

(ii) *There exist $\theta \in \Theta_*$ (resp. $\theta \in \Theta$) and $k \in (0, 1)$ such that*

$$\theta\big(H(Tx, Ty)\big) \leqslant \big[\theta\big(d(x, y)\big)\big]^k \tag{14}$$

for all $(x, y) \in S_*$, where

$$S_* = \big\{(x, y)\colon\ \alpha_*(Tx, Ty) \geqslant 1 \text{ and } H(Tx, Ty) > 0\big\} \subseteq X \times X;$$

(iii) *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geqslant 1$*;

(iv) *T is upper semicontinuous or α has property* (B).

Then T has a fixed point in X.

Proof. Let $(x, y) \in S$. Then, since T is an α_*-admissible mapping, we obtain

$$\alpha_*(Tx, Ty) \geqslant 1,$$

and so, $(x, y) \in S_*$, that is, $S \subseteq S_*$. Therefore, from (14) we have

$$\theta\big(H(Tx, Ty)\big) \leqslant \big[\theta\big(d(x, y)\big)\big]^k$$

for all $(x, y) \in S$. That is, T is a multivalued (α, θ)-contraction. Also, since T is an α_*-admissible mapping, then it is α-admissible. Therefore, all conditions of Theorem 7 (resp. Theorem 6) are satisfied. Thus, T has a fixed point in X. □

3 Application in partially ordered metric spaces

Recently, there have been so many interesting developments in fixed point theory in metric spaces endowed with a partial order. The first result in this direction was given by Ran and Reurings [28] where they extended the Banach contraction principle in partially ordered sets with some application to a matrix equation, and followed by Nieto et al. [27] without continuity of a mapping $T : X \to X$. Later, many important and valuable results appeared in this direction (see [1,5,7,20,26]). In 2004, Feng and Liu [10] defined relations between two sets. Motivated by these works, in this section, we will construct various fixed point results on a metric space endowed with a partial order for multivalued mappings. Let X be a nonempty set. If (X, d) is a metric space and $(X, \preccurlyeq)$ is partially ordered, then $(X, d, \preccurlyeq)$ is called a partially ordered metric space. Moreover, a partially ordered metric space $(X, d, \preccurlyeq)$ is regular if for every sequence $\{x_n\}$ such that $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$, then $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$. Let A and B be two nonempty subsets of a partially ordered set $(X, \preccurlyeq)$. Then the relations between A and B are defined as follows:

1. $A \prec_1 B$: if for every $a \in A$, there exists $b \in B$ such that $a \preccurlyeq b$;
2. $A \prec_2 B$: if for every $b \in B$, there exists $a \in A$ such that $a \preccurlyeq b$;
3. $A \prec B$: if $A \prec_1 B$ and $A \prec_2 B$.

$\prec_1$ and $\prec_2$ are different relations between A and B. For example, let $X = \mathbb{R}$, $A = [1/2, 1]$, $B = [0, 1]$, $\preccurlyeq$ be usual order on X, then $A \prec_1 B$ but $A \not\prec_2 B$; if $A = [0, 1]$, $B = [0, 1/2]$, then $A \prec_2 B$ while $A \not\prec_1 B$. $\prec_1$, $\prec_2$ and $\prec$ are reflexive and transitive, but are not antisymmetric. For instance, let $X = \mathbb{R}$, $A = [0, 3]$, $B = [0, 1] \cup [2, 3]$, $\preccurlyeq$ be usual order on X, then $A \prec B$ and $B \prec A$, but $A \neq B$. Hence, they are not partial orders (see [10]).

Corollary 2. *Let $(X, d, \preccurlyeq)$ be a partially ordered complete metric space, $T : X \to CB(X)$ (resp. $K(X)$) be a multivalued mapping, and $\alpha : X \times X \to [0, \infty)$ be a function. Assume that the following conditions hold*:

(i) *There exist $x_0 \in X$ such that $\{x_0\} \prec_1 Tx_0$;*
(ii) *There exist $k \in (0, 1)$ and $\theta \in \Theta_*$ (resp. $\theta \in \Theta$) such that*

$$\theta\big(H(Tx, Ty)\big) \leqslant \big[\theta\big(d(x, y)\big)\big]^k$$

for all $(x, y) \in S_\preccurlyeq$, where

$$S_\preccurlyeq = \big\{(x, y)\colon x \preccurlyeq y \text{ and } H(Tx, Ty) > 0\big\} \subseteq X \times X;$$

(iii) *T is upper semicontinuous or X is regular*;
(iv) *For each $x \in X$ and $y \in Tx$ with $x \preccurlyeq y$, we have $y \preccurlyeq z$ for all $z \in Ty$.*

Then T has a fixed point in X.

Proof. Define a mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & x \preccurlyeq y, \\ 0, & \text{otherwise}. \end{cases}$$

It is clear that $S_{\preccurlyeq} = S$. That is, T is multivalued (α,θ)-contraction. Also, since $\{x_0\} \prec_1 Tx_0$, then there exists $x_1 \in Tx_0$ such that $x_0 \preccurlyeq x_1$, and so, $\alpha(x_0,x_1) \geqslant 1$. Now, let $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geqslant 1$, then $x \preccurlyeq y$, and so, by hypotheses (iv) we have $y \preccurlyeq z$ for all $z \in Ty$. Therefore, $\alpha(y,z) \geqslant 1$ for all $z \in Ty$. This shows that T is α-admissible mapping. Finally, if T is upper semicontinuous or X is regular, then T is upper semicontinuous or α has property (B). Therefore, from Theorem 7 (resp. Theorem 6), T has a fixed point in X. □

Now, we give an example to illustrate our above result.

Example 2. Consider the complete metric space (X,d), where $X = \{1/2^{n-1}\colon n \in \mathbb{N}\} \cup \{0\}$ and $d(x,y) = |x-y|$. Define a relation $\preccurlyeq$ on X as

$$x \preccurlyeq y \quad \Longleftrightarrow \quad \frac{y}{x} \in \mathbb{N} \quad \text{or} \quad x = y.$$

It is easy to see that this relation is a partial order. Note that 0 is not comparable to other elements and $1/2^{m-1} \preccurlyeq 1/2^{n-1}$ for all $m,n \in \mathbb{N}$ with $m \geqslant n$. Define a mapping $T : X \to CB(X)$ as

$$Tx = \begin{cases} \{\frac{1}{2^n}, 1\}, & x = \frac{1}{2^{n-1}},\ n > 1, \\ \{x\}, & x = 0, 1. \end{cases}$$

Since $H(T0,T1) = 1 = d(0,1)$, then for all $\theta \in \Theta$ and $k \in (0,1)$, we have

$$\theta\bigl(H(Tx,Ty)\bigr) = \theta(1) > \bigl[\theta(1)\bigr]^k = \bigl[\theta\bigl(d(x,y)\bigr)\bigr]^k.$$

Then T is not multivalued θ-contraction. Therefore, Theorems 4 and 5 cannot be applied to this example. Also, T is not multivalued contraction.

Now, we claim that T satisfies all conditions of Corollary 2 by $k = 1/\sqrt{2}$ and $\theta(t) = \mathrm{e}^{\sqrt{t}}$. Note that if $(x,y) \in S_{\preccurlyeq}$, then $x = 1/2^{m-1}$ and $y = 1/2^{n-1}$ for some $m,n \in \mathbb{N}$ with $m > n$. Thus, we get

$$\begin{aligned} H\biggl(T\frac{1}{2^{m-1}}, T\frac{1}{2^{n-1}}\biggr) &= H\biggl(\biggl\{\frac{1}{2^m},1\biggr\}, \biggl\{\frac{1}{2^n},1\biggr\}\biggr) \\ &= \biggl|\frac{1}{2^m} - \frac{1}{2^n}\biggr| = \frac{1}{2}\biggl|\frac{1}{2^{m-1}} - \frac{1}{2^{n-1}}\biggr| \\ &= \frac{1}{2}d\biggl(\frac{1}{2^{m-1}}, \frac{1}{2^{n-1}}\biggr) = k^2 d\biggl(\frac{1}{2^{m-1}}, \frac{1}{2^{n-1}}\biggr), \end{aligned}$$

that is, condition (iii) of Corollary 2 is satisfied.

Also, for $x_0 = 1$, we have $\{x_0\} = \{1\} \prec_1 \{1\} = Tx_0$. Therefore, condition (i) of Corollary 2 is satisfied.

If $x \in X \setminus \{0\}$, we take $y = 1 \in Tx$ with $x \preccurlyeq y$, we have $y \preccurlyeq z$ for all $z \in Ty = \{1\}$, and if $x = 0$, we take $y = 0 \in Tx = \{0\}$ with $x \preccurlyeq y$, we have $y \preccurlyeq z$ for all $z \in Ty = \{0\}$. Thus, condition (iv) of Corollary 2 is satisfied.

Finally, X is regular. Indeed, for every sequence $\{x_n\}$ such that $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$, there exists an $n_0 \in \mathbb{N}$ such that $\{x_n\} = \{x_1, x_2, x_3, \dots, x_{n_0-1}, x, x, x, \dots\}$ for all $n \geqslant n_0$. So, $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$.

Hence, all conditions of Corollary 2 are satisfied, and so, T has a fixed point in X.

Remark 3. We can give a similar result using $\prec_2$ instead of $\prec_1$ in Corollary 2.

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