ISSN 1392-5113 Nonlinear Analysis: Modelling and Control, 2016, Vol. 21, No. 5, 716–729
http://dx.doi.org/10.15388/NA.2016.5.10

An existence result for a class of nonlinear integral equations of fractional orders

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Received: November 25, 2015 / **Revised:** May 4, 2016 / **Published online:** June 28, 2016

Abstract. Using a measure of non-compactness argument, we study in this paper the existence of solutions for a class of functional equations involving a fractional integral with respect to another function. Some examples are presented to illustrate the obtained results.

Keywords: integral equation, fractional, measure of non-compactness, Darbo's theorem.

1 Introduction and preliminaries

In this paper, we are concerned with the existence of solutions to the nonlinear integral equation

$$y(t) = f\big(t, y\big(\mu(t)\big)\big) + g\big(t, y\big(\nu(t)\big)\big) \int_a^t \frac{h'(\tau)u(t,\tau,y(c_1(\tau)),y(c_2(\tau)),\dots,y(c_n(\tau)))}{(h(t)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau, \tag{1}$$

where $\alpha \in (0,1)$, $0 \leqslant a < T$, $f,g : [a,T]\times\mathbb{R} \to \mathbb{R}$, $\mu,\nu,c_i : [a,T] \to [a,T]$, $i = 1,\dots,n$, $u : [a,T]\times[a,T]\times\mathbb{R}^n \to \mathbb{R}$, and $h : [a,T] \to \mathbb{R}$. Equation (1) can be written in the form

$$y(t) = f\big(t, y\big(\mu(t)\big)\big) + \Gamma(\alpha) g\big(t, y\big(\nu(t)\big)\big) I^{\alpha}_{a+,h}\big(u\big(t,\cdot,y\big(c_1(\cdot)\big),\dots,y\big(c_n(\cdot)v\big)\big)\big)(t), \quad t \in [a,T],$$

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where $I^{\alpha}_{a^+,h}$ is the fractional integral of order α with respect to the function h defined by (see [16])

$$I^{\alpha}_{a^+,h}\psi(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{h'(\tau)}{(h(t)-h(\tau))^{1-\alpha}}\psi(\tau)\,\mathrm{d}\tau, \quad t \in [a,T].$$

In the case $h(\tau)=\tau$, Eq. (1) models some problems related to queuing theory and biology (see [12]).

Using a measure of non-compactness argument, we provide sufficient conditions for the existence of at least one solution to Eq. (1). Such technique was used by many authors to establish existence results for various classes of integral equations. For more details, we refer to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15] and the references therein. To the best of our knowledge, integral equations of type (1) were not considered before.

At first, let us recall some definitions and preliminaries about the concept of measure of non-compactness.

Let $\mathbb{E}$ be a Banach space with respect to a given norm $\|\cdot\|$. Let $\mathcal{P}(\mathbb{E})$ be the set of all nonempty bounded subsets of $\mathbb{E}$. We say that $\sigma : \mathcal{P}(\mathbb{E}) \to [0,\infty)$ is a measure of non-compactness (see [1]) if the following properties hold:

(P1) For all $M \in \mathcal{P}(\mathbb{E})$, we have
$$\sigma(M)=0 \quad \Longrightarrow \quad M \text{ is precompact;}$$

(P2) For every pair $(M_1,M_2) \in \mathcal{P}(\mathbb{E}) \times \mathcal{P}(\mathbb{E})$, we have
$$M_1 \subseteq M_2 \quad \Longrightarrow \quad \sigma(M_1) \leqslant \sigma(M_2);$$

(P3) For every $M \in \mathcal{P}(\mathbb{E})$,
$$\sigma(\overline{M}) = \sigma(M) = \sigma(\overline{\mathrm{co}}\,M),$$
where $\overline{\mathrm{co}}\,M$ denotes the closed convex hull of M;

(P4) For every pair $(M_1,M_2) \in \mathcal{P}(\mathbb{E}) \times \mathcal{P}(\mathbb{E})$ and $\eta \in (0,1)$, we have
$$\sigma\big(\eta M_1 + (1-\eta)M_2\big) \leqslant \eta\sigma(M_1) + (1-\eta)\sigma(M_2);$$

(P5) If $\{M_n\}$ is a sequence of closed and decreasing (w.r.t $\subseteq$) sets in $\mathcal{P}(\mathbb{E})$ such that $\sigma(M_n) \to 0$ as $n \to \infty$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty and compact.

Our main tool in this paper is the Darbo's theorem (see [1]).

Lemma 1. *Let H be a nonempty, bounded, closed and convex subset of the Banach space $\mathbb{E}$. Let $T : H \to H$ be a continuous mapping such that*

$$\sigma(T\mathcal{W}) \leqslant K\sigma(\mathcal{W}), \quad \mathcal{W} \subseteq H,$$

where $K \in (0,1)$ is a constant. Then T has at least one fixed point.

Let us fix some notations that will be used throw this paper. We denote by $C([a,T];\mathbb{R})$ the set of real continuous functions defined in $[a,T]$. Such set is a Banach space with respect to the norm

$$\|z\| = \max\{|z(t)|:\ t \in [a,T]\}, \quad z \in C([a,T];\mathbb{R}).$$

We denote by $\mathcal{P}(C([a,T];\mathbb{R}))$ the set of all nonempty bounded subsets of $C([a,T];\mathbb{R})$. Let $\mathcal{W} \in \mathcal{P}(C([a,T];\mathbb{R}))$. For $w \in \mathcal{W}$ and $\rho \geqslant 0$, set

$$\omega(w,\rho) = \sup\{|w(t) - w(s)|:\ t,s \in [a,T],\ |t-s| \leqslant \rho\}.$$

We define the mapping $\Omega : \mathcal{P}(C([a,T];\mathbb{R})) \times [0,\infty) \to [0,\infty)$ by

$$\Omega(\mathcal{W},\rho) = \sup\{\omega(w,\rho):\ w \in \mathcal{W}\}, \quad (\mathcal{W},\rho) \in \mathcal{P}(C([a,T];\mathbb{R})) \times [0,\infty).$$

It was proved in [5] that the mapping $\sigma : \mathcal{P}(C([a,T];\mathbb{R})) \to [0,\infty)$ defined by

$$\sigma(\mathcal{W}) = \lim_{\rho \to 0^+} \Omega(\mathcal{W},\rho), \quad \mathcal{W} \in \mathcal{P}(C([a,T];\mathbb{R})),$$

is a measure of non-compactness in the Banach space $C([a,T];\mathbb{R})$.

2 Main result

We consider the following assumptions:

(H1) The functions

$$\mu, \nu, c_i : [a,T] \to [a,T], \quad i = 1,\dots,n,$$

are continuous;

(H2) There exist nonnegative constants L and p such that

$$|\mu(t) - \mu(s)| \leqslant L|t-s|^p, \quad (t,s) \in [a,T] \times [a,T];$$

(H3) There exist nonnegative constants D and q such that

$$|\nu(t) - \nu(s)| \leqslant D|t-s|^q, \quad (t,s) \in [a,T] \times [a,T];$$

(H4) The function $f : [a,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|f(t,u) - f(t,v)| \leqslant \lambda|u-v|$$

for all $(t,u,v) \in [a,T] \times \mathbb{R}^2$, where λ is a nonnegative constant;

(H5) The function $g : [a,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|g(t,u) - g(t,v)| \leqslant \theta|u-v|$$

for all $(t,u,v) \in [a,T] \times \mathbb{R}^2$, where θ is a nonnegative constant;

(H6) The function $u : [a,T] \times [a,T] \times \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies

$$\big|u(t,\tau,x_1,x_2,\dots,x_n)\big| \leqslant \varphi\Big(\max_{i=1,\dots,n} |x_i|\Big)$$

for all $(t,\tau,x_1,x_2,\dots,x_n) \in [a,T] \times [a,T] \times \mathbb{R}^n$, where $\varphi : [0,\infty) \to [0,\infty)$ is nondecreasing;

(H7) The function $h : [a,T] \to \mathbb{R}$ is C^1 and nondecreasing;

(H8) There exists $r_0 > 0$ such that

$$\lambda r_0 + M + (\theta r_0 + N)\frac{\varphi(r_0)}{\alpha}\big(h(T) - h(a)\big)^{\alpha} < r_0,$$

where

$$M = \max\big\{\big|f(t,0)\big|\colon\ t \in [a,T]\big\} \quad \text{and} \quad N = \max\big\{\big|g(t,0)\big|\colon\ t \in [a,T]\big\}.$$

Our main result is the following.

Theorem 1. *Under assumptions* (H1)–(H8)*, Eq.* (1) *has at least one solution* $y^* \in C([a,T];\mathbb{R})$*. Moreover, such solution satisfies*

$$\|y^*\| \leqslant r_0.$$

Proof. For any $y \in C([a,T];\mathbb{R})$, let

$$(Ty)(t) = f\big(t, y\big(\mu(t)\big)\big) + g\big(t, y\big(\nu(t)\big)\big) \int_a^t \frac{h'(\tau)u(t,\tau,y(c_1(\tau)),y(c_2(\tau)),\dots,y(c_n(\tau)))}{(h(t)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau$$

for all $t \in [a,T]$. We claim that

$$TC\big([a,T];\mathbb{R}\big) \subseteq C\big([a,T];\mathbb{R}\big). \tag{2}$$

To prove our claim, we have just to justify that the function

$$\gamma : t \in [a,T] \mapsto \gamma(t) = \int_a^t \frac{h'(\tau)u(t,\tau,y(c_1(\tau)),y(c_2(\tau)),\dots,y(c_n(\tau)))}{(h(t)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau$$

is continuous in $[a,T]$. Let $\{t_n\}$ be a sequence in $[a,T]$ such that $\{t_n\}$ converges to a certain $t \in [a,T]$. Without restriction of the generality, we may assume that $t_n \geqslant t$ for n large enough. We have

$$\big|\gamma(t_n) - \gamma(t)\big| = \left|\int_a^{t_n} \frac{h'(\tau)U(t_n,\tau)}{(h(t_n)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau - \int_a^{t} \frac{h'(\tau)U(t,\tau)}{(h(t)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau\right|,$$

where

$$U(t_n,\tau)=u\big(t_n,\tau,y\big(c_1(\tau)\big),y\big(c_2(\tau)\big),\dots,y\big(c_n(\tau)\big)\big),$$
$$U(t,\tau)=u\big(t,\tau,y\big(c_1(\tau)\big),y\big(c_2(\tau)\big),\dots,y\big(c_n(\tau)\big)\big).$$

Therefore,

$$\begin{aligned}\big|\gamma(t_n)-\gamma(t)\big| &\leqslant \left|\int_a^t\left(\frac{h'(\tau)U(t_n,\tau)}{(h(t_n)-h(\tau))^{1-\alpha}}-\frac{h'(\tau)U(t,\tau)}{(h(t)-h(\tau))^{1-\alpha}}\right)\mathrm{d}\tau\right| \\ &\quad+\left|\int_t^{t_n}\frac{h'(\tau)U(t_n,\tau)}{(h(t_n)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau\right| \\ &\leqslant \left|\int_a^t\frac{h'(\tau)}{(h(t)-h(\tau))^{1-\alpha}}\big(U(t_n,\tau)-U(t,\tau)\big)\,\mathrm{d}\tau\right| \\ &\quad+\left|\int_a^t\left(\frac{h'(\tau)U(t_n,\tau)}{(h(t_n)-h(\tau))^{1-\alpha}}-\frac{h'(\tau)U(t_n,\tau)}{(h(t)-h(\tau))^{1-\alpha}}\right)\mathrm{d}\tau\right| \\ &\quad+\int_t^{t_n}\frac{h'(\tau)|U(t_n,\tau)|}{(h(t_n)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau \\ &:=A_n+B_n+C_n.\end{aligned}$$

A simple application of the Dominated Convergence Theorem, yields

$$\lim_{n\to\infty}A_n=0.$$

On the other hand, we have

$$\begin{aligned}B_n &\leqslant \varphi\big(\|y\|\big)\int_a^t\left(\frac{h'(\tau)}{(h(t)-h(\tau))^{1-\alpha}}-\frac{h'(\tau)}{(h(t_n)-h(\tau))^{1-\alpha}}\right)\mathrm{d}\tau \\ &=\frac{\varphi(\|y\|)}{\alpha}\Big(\big(h(t)-h(a)\big)^\alpha+\big(h(t_n)-h(t)\big)^\alpha-\big(h(t_n)-h(a)\big)^\alpha\Big).\end{aligned}$$

Passing to the limit as $n\to\infty$, we get

$$\lim_{n\to\infty}B_n=0.$$

Finally,

$$C_n\leqslant\varphi\big(\|y\|\big)\int_t^{t_n}\frac{h'(\tau)}{(h(t_n)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau=\frac{\varphi(\|y\|)}{\alpha}\big(h(t_n)-h(t)\big)^\alpha.$$

Passing to the limit as $n\to\infty$, we obtain

$$\lim_{n\to\infty}C_n=0.$$

As consequence, we have

$$\lim_{n\to\infty} \bigl|\gamma(t_n) - \gamma(t)\bigr| = 0,$$

which proves (2). Then

$$T : C\bigl([a,T];\mathbb{R}\bigr) \to C\bigl([a,T];\mathbb{R}\bigr)$$

is well-defined.

For $r > 0$, set

$$B(0,r) = \bigl\{y \in C\bigl([a,T];\mathbb{R}\bigr)\colon \|y\| \leqslant r\bigr\}.$$

Let $y \in B(0,r)$ for some $r > 0$. For all $t \in [a,T]$, we have

$$\begin{aligned}
\bigl|(Ty)(t)\bigr| &\leqslant \bigl|f\bigl(t,y(\mu(t))\bigr) - f(t,0)\bigr| + \bigl|f(t,0)\bigr| \\
&\quad + \bigl(\bigl|g\bigl(t,y\bigl(\nu(t)\bigr)\bigr) - g(t,0)\bigr| + \bigl|g(t,0)\bigr|\bigr) \\
&\quad \times \int_a^t \frac{h'(\tau)|u(t,\tau,y(c_1(\tau)),y(c_2(\tau)),\dots,y(c_n(\tau)))|}{(h(t)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau \\
&\leqslant \lambda\bigl|y\bigl(\mu(t)\bigr)\bigr| + \bigl|f(t,0)\bigr| \\
&\quad + \bigl(\theta\bigl|y\bigl(\nu(t)\bigr)\bigr| + \bigl|g(t,0)\bigr|\bigr)\int_a^t \frac{h'(\tau)\varphi(\max_{i=1,\dots,n}|y(c_i(\tau))|)}{(h(t)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau \\
&\leqslant \lambda\|y\| + M + \bigl(\theta\|y\| + N\bigr)\frac{\varphi(\|y\|)}{\alpha}\bigl(h(t)-h(a)\bigr)^{\alpha} \\
&\leqslant \lambda r + M + (\theta r + N)\frac{\varphi(r)}{\alpha}\bigl(h(T)-h(a)\bigr)^{\alpha}.
\end{aligned}$$

Taking $r = r_0$, from (H8), we obtain $\|Ty\| \leqslant r_0$. As consequence, we get

$$T\bigl(B(0,r_0)\bigr) \subseteq B(0,r_0)$$

and $T : B(0,r_0) \to B(0,r_0)$ is well-defined.

Now, we claim that T is a continuous operator in $B(0,r_0)$. In order to prove our claim, take $y,z \in B(0,r_0)$ and $\varepsilon > 0$ such that

$$\|y - z\| \leqslant \varepsilon.$$

For all $t \in [a,T]$, we have

$$\begin{aligned}
&\bigl|(Ty)(t) - (Tz)(t)\bigr| \\
&\quad\leqslant \bigl|f\bigl(t,y\bigl(\mu(t)\bigr)\bigr) - f\bigl(t,z\bigl(\mu(t)\bigr)\bigr)\bigr| \\
&\qquad + \bigl|g\bigl(t,y\bigl(\nu(t)\bigr)\bigr) - g\bigl(t,z\bigl(\nu(t)\bigr)\bigr)\bigr|\int_a^t \frac{h'(\tau)|u(t,\tau,y(c_1(\tau)),\dots,y(c_n(\tau)))|}{(h(t)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau \\
&\qquad + \bigl(\bigl|g\bigl(t,z\bigl(\nu(t)\bigr)\bigr) - g(t,0)\bigr| + \bigl|g(t,0)\bigr|\bigr)\int_a^t \frac{h'(\tau)(|V(t,\tau) - W(t,\tau)|)}{(h(t)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau,
\end{aligned}$$

where

$$V(t,\tau) = u\big(t,\tau, y\big(c_1(\tau)\big),\dots,y\big(c_n(\tau)\big)\big),$$
$$W(t,\tau) = u\big(t,\tau, z\big(c_1(\tau)\big),\dots,z\big(c_n(\tau)\big)\big).$$

Using the considered assumptions, for all $t \in [a,T]$, we obtain

$$\begin{aligned}
&\big|(Ty)(t) - (Tz)(t)\big| \\
&\quad\leqslant \lambda\big|y\big(\mu(t)\big) - z\big(\mu(t)\big)\big| \\
&\qquad + \theta\big|y\big(\nu(t)\big) - z\big(\nu(t)\big)\big|\varphi\Big(\max_{i=1,\dots,n}\big|y\big(c_i(\tau)\big)\big|\Big)\int_a^t \frac{h'(\tau)}{(h(t)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau \\
&\qquad + \big(\theta\big|z\big(\nu(t)\big)\big| + N\big)\omega_\varepsilon \int_a^t \frac{h'(\tau)}{(h(t)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau \\
&\quad\leqslant \lambda\|y-z\| + \frac{\theta\|y-z\|\varphi(\|y\|)}{\alpha}\big(h(t)-h(a)\big)^\alpha + \frac{(\theta\|z\|+N)\omega_\varepsilon}{\alpha}\big(h(t)-h(a)\big)^\alpha \\
&\quad\leqslant \lambda\varepsilon + \big(h(T)-h(a)\big)^\alpha\left(\frac{\theta\varepsilon\varphi(r_0) + (\theta r_0+N)\omega_\varepsilon}{\alpha}\right),
\end{aligned}$$

where

$$\begin{aligned}
\omega_\varepsilon = \sup\big\{&\big|u(t,\tau,u_1,\dots,u_n) - u(t,\tau,v_1,\dots,v_n)\big|\colon t,\tau\in[a,T], \\
&u_i, v_i \in [-r_0, r_0],\ |u_i - v_i| \leqslant \varepsilon,\ i=1,\dots,n\big\}.
\end{aligned}$$

Note that from the uniform continuity of the function u in $[a,T]\times[a,T]\times[-r_0,r_0]^n$ we observe easily that $\lim_{\varepsilon\to 0^+}\omega_\varepsilon = 0$. Then

$$\|Ty - Tz\| \leqslant \lambda\varepsilon + \big(h(T)-h(a)\big)^\alpha\left(\frac{\theta\varepsilon\varphi(r_0) + (\theta r_0+N)\omega_\varepsilon}{\alpha}\right).$$

Passing to the limit as $\varepsilon \to 0^+$, we deduce the continuity of the operator T in $B(0,r_0)$.

Let $\mathcal{W}$ be a nonempty subset of $B(0,r_0)$. Let $\rho > 0$ be fixed, $z \in \mathcal{W}$ and $t_1, t_2 \in [a,T]$ be such that $|t_1 - t_2| \leqslant \rho$. Without restriction of the generality, we may assume $t_1 \geqslant t_2$. We have

$$\begin{aligned}
&\big|(Tz)(t_1) - (Tz)(t_2)\big| \\
&\quad\leqslant \big|f\big(t_1, z\big(\mu(t_1)\big)\big) - f\big(t_1, z\big(\mu(t_2)\big)\big)\big| + \big|f\big(t_1, z\big(\mu(t_2)\big)\big) - f\big(t_2, z\big(\mu(t_2)\big)\big)\big| \\
&\qquad + \big(\big|g\big(t_1, z\big(\nu(t_1)\big)\big) - g\big(t_1, z\big(\nu(t_2)\big)\big)\big| + \big|g\big(t_1, z\big(\nu(t_2)\big)\big) - g\big(t_2, z\big(\nu(t_2)\big)\big)\big|\big) \\
&\qquad \times \int_a^{t_1} \frac{h'(\tau)|u(t_1,\tau,z(c_1(\tau)),\dots,z(c_n(\tau)))|}{(h(t_1)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau \\
&\qquad + \big(\big|g\big(t_2, z\big(\nu(t_2)\big)\big) - g(t_2,0)\big| + \big|g(t_2,0)\big|\big)
\end{aligned}$$

$$\times\left(\int_a^{t_1}\frac{h'(\tau)u(t_1,\tau,z(c_1(\tau)),\ldots,z(c_n(\tau)))}{(h(t_1)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau - \int_a^{t_2}\frac{h'(\tau)u(t_2,\tau,z(c_1(\tau)),\ldots,z(c_n(\tau)))}{(h(t_2)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau\right).$$

Using the considered assumptions, we obtain

$$\begin{aligned}
&\big|(Tz)(t_1)-(Tz)(t_2)\big| \\
&\quad\leqslant \lambda\big|z\big(\mu(t_1)\big)-z\big(\mu(t_2)\big)\big| + \omega_f(\rho) + \big(\theta\big|z\big(\nu(t_1)\big)-z\big(\nu(t_2)\big)\big| + \omega_g(\rho)\big) \\
&\qquad\times \frac{\varphi(\max_{i=1,\ldots,n}|z(c_i(\tau))|)}{\alpha}\big(h(T)-h(a)\big)^\alpha + \big(\theta\big|z\big(\nu(t_2)\big)\big| + N\big) \\
&\qquad\times\left(\int_a^{t_1}\frac{h'(\tau)u(t_1,\tau,z(c_1(\tau)),\ldots,z(c_n(\tau)))}{(h(t_1)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau\right. \\
&\qquad - \int_a^{t_2}\frac{h'(\tau)u(t_1,\tau,z(c_1(\tau)),\ldots,z(c_n(\tau)))}{(h(t_1)-h(\tau))^{1-\alpha}}\,\mathrm{d}\tau \\
&\qquad + \int_a^{t_2}\left|\frac{h'(\tau)u(t_1,\tau,z(c_1(\tau)),\ldots,z(c_n(\tau)))}{(h(t_1)-h(\tau))^{1-\alpha}}\right. \\
&\qquad \left. - \frac{h'(\tau)u(t_1,\tau,z(c_1(\tau)),\ldots,z(c_n(\tau)))}{(h(t_2)-h(\tau))^{1-\alpha}}\right|\mathrm{d}\tau \\
&\qquad + \int_a^{t_2}\frac{h'(\tau)}{(h(t_2)-h(\tau))^{1-\alpha}}\big|u\big(t_1,\tau,z\big(c_1(\tau)\big),\ldots,z\big(c_n(\tau)\big)\big) \\
&\qquad \left. - u\big(t_2,\tau,z\big(c_1(\tau)\big),\ldots,z\big(c_n(\tau)\big)\big)\big|\,\mathrm{d}\tau\right) \\
&\quad\leqslant \lambda\omega_1(\rho) + \omega_f(\rho) + \big(\theta\omega_2(\rho)+\omega_g(\rho)\big)\frac{\varphi(r_0)}{\alpha}\big(h(T)-h(a)\big)^\alpha \\
&\qquad + (\theta r_0 + N)\left(\frac{\varphi(r_0)}{\alpha}\big(h(t_1)-h(t_2)\big)^\alpha + \frac{\varphi(r_0)}{\alpha}\big(\big(h(t_2)-h(a)\big)^\alpha\right. \\
&\qquad \left. + \big(h(t_1)-h(t_2)\big)^\alpha - \big(h(t_1)-h(a)\big)^\alpha\big) + \frac{\omega_3(\rho)}{\alpha}\big(h(t_2)-h(a)\big)^\alpha\right) \\
&\quad\leqslant \lambda\omega_1(\rho) + \omega_f(\rho) + \big(\theta\omega_2(\rho)+\omega_g(\rho)\big)\frac{\varphi(r_0)}{\alpha}\big(h(T)-h(a)\big)^\alpha \\
&\qquad + (\theta r_0 + N)\left(\frac{2\varphi(r_0)}{\alpha}\omega(h,\rho) + \frac{\omega_3(\rho)}{\alpha}\big(h(T)-h(a)\big)^\alpha\right),
\end{aligned}$$

where

$$\omega_1(\rho) = \sup\{|z(\mu(t)) - z(\mu(s))|\colon t,s \in [a,T],\ |t-s| \leqslant \rho\},$$
$$\omega_2(\rho) = \sup\{|z(\nu(t)) - z(\nu(s))|\colon t,s \in [a,T],\ |t-s| \leqslant \rho\},$$
$$\omega_f(\rho) = \sup\{|f(t,u) - f(s,u)|\colon t,s \in [a,T],\ |t-s| \leqslant \rho,\ u \in [-r_0,r_0]\},$$
$$\omega_g(\rho) = \sup\{|g(t,u) - g(s,u)|\colon t,s \in [a,T],\ |t-s| \leqslant \rho,\ u \in [-r_0,r_0]\},$$
$$\omega_3(\rho) = \sup\{|u(t_1,s,u_1,\dots,u_n) - u(t_2,s,u_1,\dots,u_n)|\colon t_1,t_2,s \in [0,T],\ |t_1-t_2| \leqslant \rho,\ u_i \in [-r_0,r_0],\ i=1,\dots,n\}.$$

Observe that

$$\omega_1(\rho) \leqslant \sup\{|z(t) - z(s)|\colon t,s \in [a,T],\ |t-s| \leqslant L\rho^p\} = \omega(z, L\rho^p).$$

Similarly,

$$\omega_2(\rho) \leqslant \sup\{|z(t) - z(s)|\colon t,s \in [a,T],\ |t-s| \leqslant D\rho^q\} = \omega(z, D\rho^q).$$

Note also that

$$\lim_{\rho\to 0^+} \omega_f(\rho) = \lim_{\rho\to 0^+} \omega_g(\rho) = \lim_{\rho\to 0^+} \omega_3(\rho) = 0.$$

Therefore,

$$\begin{aligned}\Omega(T\mathcal{W},\rho) \leqslant{}& \lambda\Omega(\mathcal{W}, L\rho^p) + \omega_f(\rho)\\ &+ \bigl(\theta\Omega(\mathcal{W}, D\rho^q) + \omega_g(\rho)\bigr)\frac{\varphi(r_0)}{\alpha}\bigl(h(T)-h(a)\bigr)^\alpha\\ &+ (\theta r_0 + N)\left(\frac{2\varphi(r_0)}{\alpha}\omega(h,\rho) + \frac{\omega_3(\rho)}{\alpha}\bigl(h(T)-h(a)\bigr)^\alpha\right).\end{aligned}$$

Passing to the limit as $\rho \to 0^+$, we get

$$\sigma(T\mathcal{W}) \leqslant \left(\lambda + \theta\frac{\varphi(r_0)}{\alpha}\bigl(h(T)-h(a)\bigr)^\alpha\right)\sigma(\mathcal{W}).$$

Then we proved that for any nonempty subset $\mathcal{W}$ of $B(0,r_0)$, we have

$$\sigma(T\mathcal{W}) \leqslant K\sigma(\mathcal{W}),$$

where

$$K = \lambda + \theta\frac{\varphi(r_0)}{\alpha}\bigl(h(T)-h(a)\bigr)^\alpha.$$

Note that from (H8) we have $K < 1$. Applying Darbo's theorem (see Lemma 1), we deduce that the operator T has at least one fixed point $y^* \in B(0,r_0)$, which is a solution to Eq. (1). □

3 Particular cases and examples

3.1 A functional equation involving the Riemann–Liouville fractional integral

Take

$$h(t) = t, \quad t \in [a, T],$$

in Eq. (1), we obtain the functional equation

$$y(t) = f\bigl(t, y\bigl(\mu(t)\bigr)\bigr) + \Gamma(\alpha) g\bigl(t, y\bigl(\nu(t)\bigr)\bigr) I_{a^+}^{\alpha}\bigl(u\bigl(t, \cdot, y\bigl(c_1(\cdot)\bigr), \dots, y\bigl(c_n(\cdot)\bigr)\bigr)\bigr)(t), \quad (3)$$

where $I_{a^+}^{\alpha}$ is the Riemann–Liouville fractional integral defined by (see [16])

$$I_{a^+}^{\alpha}\psi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\psi(\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau, \quad t \in [a, T].$$

We can rewrite Eq. (3) in the form

$$y(t) = f\bigl(t, y\bigl(\mu(t)\bigr)\bigr) + g\bigl(t, y\bigl(\nu(t)\bigr)\bigr) \int_a^t \frac{u(t, \tau, y(c_1(\tau)), \dots, y(c_n(\tau)))}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau.$$

Then from Theorem 1 we deduce the following existence result.

Corollary 1. *Suppose that assumptions* (H1)–(H6) *are satisfied. Suppose also that there is some* $r_0 > 0$ *such that*

$$\lambda r_0 + M + (\theta r_0 + N)\frac{\varphi(r_0)}{\alpha}(T-a)^{\alpha} < r_0.$$

Then Eq. (3) *has at least one solution* $y^* \in C([a, T]; \mathbb{R})$. *Moreover, such solution satisfies*

$$\|y^*\| \leqslant r_0.$$

We present the following example to illustrate the above result.

Example 1. We consider the integral equation

$$y(t) = \frac{2y(t^2)}{5} + \frac{1+t}{8} + \frac{y(\cos t) + t^2}{36} \int_0^t \frac{\ln(1 + |y(\tau)|)}{(1+t+\tau)\sqrt{t-\tau}}\,\mathrm{d}\tau. \quad (4)$$

Setting

$$f(t, x) = \frac{1+t}{8} + \frac{2u}{5}, \quad g(t, x) = \frac{u + t^2}{36}, \quad (t, x) \in [0, 1] \times \mathbb{R},$$

$$u(t, s, x) = \frac{\ln(1 + |x|)}{1+t+s}, \quad (t, s, x) \in [0, 1] \times [0, 1] \times \mathbb{R},$$

$$\mu(t) = t^2, \quad \nu(t) = \cos t, \quad c_1(t) = t, \quad t \in [0, 1], \qquad \alpha = \frac{1}{2},$$

we can rewrite Eq. (4) in the form

$$y(t) = f\bigl(t, y\bigl(\mu(t)\bigr)\bigr) + g\bigl(t, y\bigl(\nu(t)\bigr)\bigr) \int\limits_0^t \frac{u(t, \tau, y(c_1(\tau)))}{(t-\tau)^{1-\alpha}}\, \mathrm{d}\tau, \quad t \in [0,1].$$

For all $t, s \in [0,1]$, we have

$$\bigl|\mu(t) - \mu(s)\bigr| = \bigl|t^2 - s^2\bigr| = |t+s||t-s| \leqslant 2|t-s|.$$

Then assumption (H2) is satisfied with

$$L = 2 \quad \text{and} \quad p = 1.$$

For all $t, s \in [0,1]$, we have

$$\bigl|\nu(t) - \nu(s)\bigr| = |\cos t - \cos s| \leqslant |t-s|.$$

Then assumption (H3) is satisfied with

$$D = q = 1.$$

For all $(t, u, v) \in [0,1] \times \mathbb{R}^2$, we have

$$\bigl|f(t,u) - f(t,v)\bigr| \leqslant \frac{2}{5}|u-v|.$$

Then assumption (H4) is satisfied with

$$\lambda = \frac{2}{5}.$$

For all $(t, u, v) \in [0,1] \times \mathbb{R}^2$, we have

$$\bigl|g(t,u) - g(t,v)\bigr| \leqslant \frac{1}{36}|u-v|.$$

Then assumption (H5) is satisfied with

$$\theta = \frac{1}{36}.$$

For all $(t, s, x) \in [0,1] \times [0,1] \times \mathbb{R}$, we have

$$\bigl|u(t,s,x)\bigr| \leqslant \ln\bigl(1 + |x|\bigr) = \varphi\bigl(|x|\bigr),$$

where

$$\varphi(r) = \ln(1+r), \quad r \geqslant 0.$$

Then assumption (H6) is also satisfied. Note that in this case, we have

$$M = \frac{1}{4} \quad \text{and} \quad N = \frac{1}{36}.$$

Let $r_0 = 1$. We have

$$\lambda r_0 + M + (\theta r_0 + N)\frac{\varphi(r_0)}{\alpha}(T-a)^\alpha = \frac{2}{5} + \frac{1}{4} + \frac{1}{9}\ln 2 \approx 0.727 < r_0 = 1.$$

Applying Corollary 1, we obtain the existence of at least one solution $y^* \in C([0,1];\mathbb{R})$ to Eq. (4) such that

$$\|y^*\| \leqslant 1.$$

3.2 A functional equation involving the Hadamard fractional integral

Taking

$$h(t) = \ln t, \quad t \in [a,T],\ 0 < a < T,$$

in Eq. (1), we obtain the functional equation

$$\begin{aligned} y(t) &= f\bigl(t, y\bigl(\mu(t)\bigr)\bigr) \\ &\quad + \Gamma(\alpha) g\bigl(t, y\bigl(\nu(t)\bigr)\bigr) J_{a^+}^\alpha \bigl(u\bigl(t,\cdot, y\bigl(c_1(\cdot)\bigr), \dots, y\bigl(c_n(\cdot)\bigr)\bigr)\bigr)(t), \end{aligned} \tag{5}$$

where $J_{a^+}^\alpha$ is the Hadamard fractional integral defined by (see [16])

$$J_{a^+}^\alpha \psi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{\psi(\tau)}{\tau}\, \mathrm{d}\tau.$$

We can rewrite Eq. (5) in the form

$$y(t) = f\bigl(t, y\bigl(\mu(t)\bigr)\bigr) + g\bigl(t, y\bigl(\nu(t)\bigr)\bigr) \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{u(t,\tau, y(c_1(\tau)), \dots, y(c_n(\tau)))}{\tau}\, \mathrm{d}\tau.$$

From Theorem 1 we deduce the following result.

Corollary 2. *Suppose that assumptions* (H1)–(H6) *are satisfied. Suppose also that there exists some* $r_0 > 0$ *such that*

$$\lambda r_0 + M + (\theta r_0 + N)\frac{\varphi(r_0)}{\alpha}\left(\ln \frac{T}{a}\right)^\alpha < r_0.$$

Then Eq. (5) *has at least one solution* $y^* \in C([a,T];\mathbb{R})$*. Moreover, we have*

$$\|y^*\| \leqslant r_0.$$

We present the following example to illustrate the above result.

Example 2. Let us consider the integral equation

$$y(t)=\frac{t}{32}+\frac{y(t)}{8}+\left(\frac{t^2}{64}+\frac{y(t)}{16}\right)\int_1^t\left(\ln\frac{t}{\tau}\right)^{-1/2}\frac{y(\tau)}{\tau}\,\mathrm{d}\tau \tag{6}$$

for all $t\in[1,2]$. Setting

$$f(t,x)=\frac{t}{32}+\frac{x}{8},\quad g(t,x)=\frac{t^2}{64}+\frac{x}{16},\quad (t,x)\in[1,2]\times\mathbb{R},$$
$$u(t,s,x)=x,\quad (t,s,x)\in[1,2]\times[1,2]\times\mathbb{R},$$
$$\mu(t)=\nu(t)=c_1(t)=t,\quad t\in[1,2],\qquad \alpha=\frac{1}{2},$$

we can rewrite Eq. (6) in the form

$$y(t)=f\bigl(t,y\bigl(\mu(t)\bigr)\bigr)+g\bigl(t,y\bigl(\nu(t)\bigr)\bigr)\int_1^t\left(\ln\frac{t}{\tau}\right)^{-1/2}\frac{u(t,\tau,y(\tau))}{\tau}\,\mathrm{d}\tau.$$

For all $(t,u,v)\in[1,2]\times\mathbb{R}^2$, we have

$$\bigl|f(t,u)-f(t,v)\bigr|\leqslant\frac{1}{8}|u-v|.$$

Then condition (H4) is satisfied with

$$\lambda=\frac{1}{8}.$$

For all $(t,u,v)\in[1,2]\times\mathbb{R}^2$, we have

$$\bigl|g(t,u)-g(t,v)\bigr|\leqslant\frac{1}{16}|u-v|.$$

Then condition (H5) is satisfied with

$$\theta=\frac{1}{16}.$$

It is clear that assumption (H6) is satisfied with

$$\varphi(r)=r,\quad r>0.$$

Note that in this case, we have

$$M=N=\frac{1}{16}.$$

On the other hand, for $r_0=1$, we have

$$\begin{aligned}\lambda r_0+M+(\theta r_0+N)\frac{\varphi(r_0)}{\alpha}\left(\ln\frac{T}{a}\right)^{\alpha}&=\frac{1}{8}+\frac{1}{16}+\frac{1}{4}\sqrt{\ln 2}\\&\approx 0.3956<1=r_0.\end{aligned}$$

By Corollary 2, Eq. (6) admits at least one solution $y^*\in C([1,2];\mathbb{R})$ such that

$$\|y^*\|\leqslant 1.$$

Acknowledgment. The second author would like to extend his sincere appreciation to the Deanship of Scientific Research at King Saud University for its funding of this research through the International Research Group Project No. IRG14-04.

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