ISSN 1392-5113 Nonlinear Analysis: Modelling and Control, 2016, Vol. 21, No. 4, 448–464
http://dx.doi.org/10.15388/NA.2016.4.2

An elegant operational matrix based on harmonic numbers: Effective solutions for linear and nonlinear fourth-order two point boundary value problems

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Received: September 22, 2014 / **Revised:** November 22, 2015 / **Published online:** March 7, 2016

Abstract. This paper analyzes the solution of fourth-order linear and nonlinear two point boundary value problems. The suggested method is quite innovative and it is completely different from all previous methods used for solving such kind of boundary value problems. The method is based on employing an elegant operational matrix of derivatives expressed in terms of the well-known harmonic numbers. Two algorithms are presented and implemented for obtaining new approximate solutions of linear and nonlinear fourth-order boundary value problems. The two algorithms rely on employing the new introduced operational matrix for reducing the differential equations with their boundary conditions to systems of linear or nonlinear algebraic equations, which can be efficiently solved by suitable solvers. For this purpose, the two spectral methods, namely, Petrov–Galerkin and collocation methods are applied. Some illustrative examples are considered aiming to ascertain the wide applicability, validity, and efficiency of the two proposed algorithms. The obtained numerical results are satisfactory and the approximate solutions are very close to the analytical solutions and they are more accurate than those obtained by some other existing techniques in literature.

Keywords: shifted Legendre polynomials, harmonic numbers, fourth-order boundary value problems, Petrov–Galerkin method, collocation method.

1 Introduction

Spectral methods on bounded domains typically employ grids consisting of zeros of Chebyshev polynomials, or zeros of Legendre polynomials, or some other points related to various orthogonal polynomials (see [34]). The aim of spectral methods is to approximate functions (solutions of differential equations) by means of truncated series of orthogonal polynomials. There are three well-known methods of spectral methods, namely, tau, collocation and Galerkin methods (see, for example, [5, 8, 13, 19, 29]). The choice of the suitable used spectral method suggested for solving the given equation depends certainly on the type of the differential equation and also on the type of the boundary conditions governed by it. The choice of test functions distinguishes between the three versions of spectral schemes.

In Galerkin method, the test functions are the same as the trial functions and they are chosen such that each member of them satisfies the boundary conditions governed by the given differential equation. Petrov–Galerkin method is widely used for solving ordinary and partial differential equations, see, for example, [1, 9, 12, 18, 30]. The Petrov–Galerkin methods have generally come to be known as "stabilized" formulations because they prevent the spatial oscillations and sometimes yield nodally exact solutions, where the classical Galerkin method would fail badly (see [35]). The main difference between Galerkin and Petrov–Galerkin methods is that the test functions in Petrov–Galerkin methods are not identical with the trial functions unlike Galerkin methods (see, for example, [9]).

High order boundary value problems play important parts in physics, engineering disciplines and applied mathematics. There is a great number of authors interested in solving these kinds of boundary value problems. In this respect, there is a huge number of articles handle both of high odd- and high even-order boundary value problems. For example, in the two papers [9, 12], the authors have employed Petrov–Galerkin methods together with the two orthogonal polynomials, namely, ultraspherical and generalized Jacobi polynomials for handling odd-order differential equations, while in the two papers [10, 11], the authors handled high even-order differential equations by applying the Galerkin method. For this purpose, they constructed suitable basis functions satisfying the underlying boundary conditions on the given differential equation. The suggested algorithms in these articles are suitable for handling one and two dimensional linear high even-order boundary value problems.

Fourth-order boundary value problems are of old and recent interests. For some old studies about these problems, see, for example, [4, 16]. Many other techniques were used for solving fourth-order boundary value problems, for example, variational iteration method is applied in [25], non-polynomial sextic spline method is applied in [20], quintic non-polynomial spline method in [22] and Jacobi Gelrkin methods in [14, 15]. Theorems which list the conditions for the existence and uniqueness of solution of such problems are thoroughly discussed in a book by Agarwal [2].

The employment of operational matrices for solving different kinds of differential equations is considered as a common technique. There are several studies in this respect. This approach has two main advantages, the first is its simplicity, and the second is the accuracy of the approximate solutions resulted from their uses. For example, in [13], the authors employed the operational matrices of derivatives of Chebyshev polynomials of the second kind to solve the singular Lane–Emden type equations. Some other studies in [26, 27] employ operational matrices of derivatives for solving the same type of equations. Other kinds of differential equations were handled by the same technique (see, for example, [23, 31, 36]).

In this paper, we aim to introduce a novel operational matrix of derivatives in terms of the well-known harmonic numbers, and then employ the introduced operational matrix to numerically solve both of linear and nonlinear fourth-order boundary value problems.

For this purpose, Petrov–Galerkin method is applied on linear equations and the typical collocation method is applied on nonlinear equations.

2 Preliminaries

In this section, we present some definitions and relations that will be used throughout this paper.

2.1 Shifted Legendre polynomials

The shifted Legendre polynomials $L_k^*(x)$ are defined on $[a,b]$ as

$$L_k^*(x) = L_k\left(\frac{2x-a-b}{b-a}\right), \quad k = 0,1,\dots,$$

where $L_k(x)$ are the Legendre polynomials. These polynomials may be generated by using the recurrence relation

$$(k+1)L_{k+1}^*(x) = (2k+1)\frac{2x-b-a}{b-a}L_k^*(x) - kL_{k-1}^*(x), \quad k = 1,2,\dots, \tag{1}$$

with $L_0^*(x) = 1$, $L_1^*(x) = (2x-b-a)/(b-a)$ or alternatively from Rodrigues' formula

$$L_k^*(x) = \frac{1}{(b-a)^k k!}D^k\left[\left(x^2-(b+a)x+ba\right)^k\right], \quad D \equiv \frac{\mathrm{d}}{\mathrm{d}x}.$$

The orthogonality relation for $L_k^*(x)$ on $[a,b]$ is given by

$$\int_a^b L_j^*(x)L_k^*(x)\,\mathrm{d}x = \begin{cases} (b-a)/(2k+1), & k = j,\\ 0, & k \neq j. \end{cases} \tag{2}$$

The following theorem is useful in the sequel.

Theorem 1. *If the q times repeated integration of $L_k^*(x)$ is denoted by*

$$I_k^{(q)}(x) = \overbrace{\int\int\cdots\int}^{q \text{ times}} L_k^*(x)\,\overbrace{\mathrm{d}x\,\mathrm{d}x\,\cdots\,\mathrm{d}x}^{q \text{ times}},$$

then

$$I_k^{(q)}(x) = \frac{(b-a)^q}{2^{2q}}\sum_{r=0}^{q}\frac{(-1)^r\binom{q}{r}(k+q-2r+1/2)\Gamma(k-r+1/2)}{\Gamma(k+q-r+3/2)}L_{k+q-2r}^*(x) + \pi_{q-1}(x), \tag{3}$$

and $\pi_{q-1}(x)$ is a polynomial of degree at most $(q-1)$.

Proof. See [7]. $\square$

2.2 Harmonic numbers

The nth harmonic number is defined as (see [28])

$$H_n = \sum_{i=1}^{n} \frac{1}{i}. \tag{4}$$

The recurrence relation satisfied by H_n is

$$H_n - H_{n-1} = \frac{1}{n}, \quad n = 1, 2, \dots.$$

The numbers H_n have the integral representation

$$H_n = \int_0^1 \frac{1 - x^n}{1 - x} \, \mathrm{d}x.$$

The following Lemma is useful in what follows.

Lemma 1. *The harmonic numbers H_k satisfy the following three-term recurrence relation*:

$$(2k-1)H_{k-1} - (k-1)H_{k-2} = kH_k, \quad k \geqslant 2. \tag{5}$$

Proof. The recurrence relation (5) can be easily proved with the aid of relation (4). □

3 Harmonic numbers operational matrix of derivatives

In this section, a novel operational matrix of derivatives in terms of the well-known harmonic numbers will be introduced. First, choose the following set of basis functions

$$\phi_k(x) = (x-a)^2(b-x)^2 L_k^*(x), \quad k = 0, 1, 2, \dots. \tag{6}$$

It is worthy to note here that the set of polynomials $\{\phi_k(x)\colon k = 0, 1, 2, \dots\}$ is a linearly independent set. Moreover, the polynomials $\{\phi_k(x)\colon k = 0, 1, \dots\}$ are orthogonal with respect to the weight function $w(x) = (x-a)^{-4}(b-x)^{-4}$, that is,

$$\int_a^b \frac{\phi_k(x)\phi_j(x)}{(x-a)^4(b-x)^4} \, \mathrm{d}x = \begin{cases} 0, & k \neq j, \\ (b-a)/(2k+1), & k = j. \end{cases}$$

Let us denote $H_w^r(I)$ ($r = 0, 1, 2, \dots$) as the weighted Sobolev spaces, whose inner products and norms are denoted by $(\cdot,\cdot)_{r,w}$ and $\|\cdot\|_{r,w}$, respectively (see [5]). To account for homogeneous boundary conditions, we define

$$H_{0,w}^2(I) = \bigl\{v \in H_w^2(I)\colon v(a) = v(b) = v'(a) = v'(b) = 0\bigr\},$$

where $I = (a, b)$. Now, define the following subspace of $H^2_{0,w}(I)$:

$$V_N = \operatorname{span}\{\phi_0(x), \phi_1(x), \dots, \phi_N(x)\}.$$

Any function $y(x) \in H^2_{0,w}(I)$ can be expanded in terms of the polynomials $\phi_k(x)$ as

$$y(x) = \sum_{k=0}^{\infty} c_k \phi_k(x), \tag{7}$$

where

$$c_k = \frac{2k+1}{b-a} \int_a^b \frac{y(x)\phi_k(x)}{(x-a)^4(b-x)^4}\,\mathrm{d}x. \tag{8}$$

In Eq. (7), $y(x)$ can be approximated by the first $(N+1)$ terms, that is,

$$y(x) \simeq y_N(x) = \sum_{k=0}^{N} c_k \phi_k(x) = \boldsymbol{C}^{\mathrm{T}} \boldsymbol{\Phi}(x), \tag{9}$$

where

$$\boldsymbol{C}^{\mathrm{T}} = [c_0, c_1, \dots, c_N], \qquad \boldsymbol{\Phi}(x) = [\phi_0(x), \phi_1(x), \dots, \phi_N(x)]^{\mathrm{T}}. \tag{10}$$

Now, the main theorem, from which a novel operational matrix of derivatives can be expressed in terms of harmonic numbers will be stated and proved.

Theorem 2. *If the polynomials $\phi_k(x)$ are selected as in* (6)*, then for all $k \geqslant 1$, one has*

$$D\phi_k(x) = \frac{2}{b-a} \sum_{\substack{j=0 \\ (k+j)\ \text{odd}}}^{k-1} (2j+1)(1 + 4H_k - 4H_j)\phi_j(x) + \xi_k(x), \tag{11}$$

where $\xi_k(x)$ is given by

$$\xi_k(x) = 2(x-a) \begin{cases} (b-x)(a+b-2x), & k \text{ even}, \\ (a-b)(b-x), & k \text{ odd}. \end{cases} \tag{12}$$

Proof. Our strategy is to prove Theorem 2 on $[-1, 1]$, and hence the proof on the general interval $[a, b]$ can be easily transported. Now, we intend to prove the relation

$$D\psi_k(x) = \sum_{\substack{j=0 \\ (k+j)\ \text{odd}}}^{k-1} (2j+1)(1 + 4H_k - 4H_j)\psi_j(x) + \gamma_k(x), \tag{13}$$

where

$$\psi_k(x) = \left(1 - x^2\right)^2 L_k(x) \quad \text{and} \quad \gamma_k(x) = 4\left(x^2 - 1\right) \begin{cases} x, & k \text{ even}, \\ 1, & k \text{ odd}. \end{cases}$$

We proceed by induction on k. For $k = 1$, it is clear that each of the two sides of (13) is equal to $(1-x^2)(1-5x^2)$. Now, assume that relation (13) is valid for $(k-2)$ and $(k-1)$, and we will show its validity for k. The polynomials $\psi_k(x)$ satisfy the same recurrence relation of Legendre polynomials, that is,

$$\psi_k(x) = \frac{2k-1}{k} x\psi_{k-1}(x) - \frac{(k-1)}{k}\psi_{k-2}(x), \quad k \geqslant 2, \tag{14}$$

which gives immediately after differentiation

$$D\psi_k(x) = \frac{2k-1}{k} xD\psi_{k-1}(x) + \frac{2k-1}{k}\psi_{k-1}(x) - \frac{(k-1)}{k}D\psi_{k-2}(x). \tag{15}$$

The application of the induction hypothesis twice on $D\psi_{k-1}(x)$ and $D\psi_{k-2}(x)$ in (15) yields

$$\begin{aligned} D\psi_k(x) = {}& \frac{(2k-1)x}{k} \sum_{\substack{j=0\\(k+j)\ \text{even}}}^{k-2} (2j+1)(1-4H_j+4H_{k-1})\psi_j(x) \\ & - \frac{(k-1)}{k} \sum_{\substack{j=0\\(k+j)\ \text{odd}}}^{k-3} (2j+1)(1-4H_j+4H_{k-2})\psi_j(x) \\ & + \frac{2k-1}{k}\psi_{k-1}(x) + \frac{2k-1}{k}x\gamma_{k-1}(x) - \frac{k-1}{k}\gamma_{k-2}(x). \end{aligned} \tag{16}$$

Substituting the recurrence relation (14) in the form

$$x\psi_j(x) = \frac{j+1}{2j+1}\psi_{j+1}(x) + \frac{j}{2j+1}\psi_{j-1}(x)$$

into relation (16), then after performing some rather lengthy manipulations, we get

$$\begin{aligned} D\psi_k(x) = {}& \sum_{\substack{j=1\\(k+j)\ \text{odd}}}^{k-3} g_{kj}\psi_j(x) \\ & + \left[2k-1+\frac{4(k-1)(2k-1)(H_{k-1}-H_{k-2})}{k}\right]\psi_{k-1}(x) \\ & + \theta_k\left[\frac{4(2k-1)H_{k-1}}{k} - \frac{4(k-1)H_{k-2}}{k} - \frac{(7k-4)}{k}\right]\psi_0(x) \\ & + \frac{2k-1}{k}x\gamma_{k-1}(x) - \frac{k-1}{k}\gamma_{k-2}(x), \end{aligned} \tag{17}$$

where

$$\begin{aligned} g_{kj} = {}& 2j+1 - \frac{4j(2k-1)H_{j-1}}{k} + \frac{4(2j+1)(k-1)H_j}{k} - \frac{4(j+1)(2k-1)H_{j+1}}{k} \\ & - \frac{4(2j+1)(k-1)H_{k-2}}{k} + \frac{4(2j+1)(2k-1)H_{k-1}}{k}, \end{aligned} \tag{18}$$

and

$$\theta_k = \begin{cases} 1, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

Now, the elements g_{kj} in (18) after making use of the recurrence relation in Lemma 1, can be simplified to take the formula

$$g_{kj} = (2j+1)(1+4H_k-4H_j). \tag{19}$$

Repeated use of Lemma 1 in (17), and after performing some rather manipulation, lead to

$$\begin{aligned} D\psi_k(x) = {} & \sum_{\substack{j=0 \\ (k+j)\,\text{odd}}}^{k-1} (2j+1)(4H_k-4H_j+1)\psi_j(x) - \frac{4(2k-1)}{k}\theta_k\psi_0(x) \\ & + \left(\frac{2k-1}{k}\right)x\gamma_{k-1}(x) - \left(\frac{k-1}{k}\right)\gamma_{k-2}(x), \end{aligned} \tag{20}$$

and by noting that

$$-\frac{4(2k-1)}{k}\theta_k\psi_0(x) + \left(\frac{2k-1}{k}\right)x\gamma_{k-1}(x) - \left(\frac{k-1}{k}\right)\gamma_{k-2}(x) = \gamma_k(x),$$

then the proof of formula (13) is completed.

Now, if x in (13) is replaced by $(2x-a-b)/(b-a)$, then after some manipulations, it can be shown that

$$D\phi_k(x) = \frac{2}{b-a}\sum_{\substack{j=0 \\ (k+j)\,\text{odd}}}^{k-1} (2j+1)(1+4H_k-4H_j)\phi_j(x) + \xi_k(x)$$

and

$$\xi_k(x) = 2(x-a)\begin{cases} (b-x)(a+b-2x), & k \text{ even}, \\ (a-b)(b-x), & k \text{ odd}. \end{cases}$$

This completes the proof of Theorem 2. □

Now, and with the aid of Theorem 2, the first derivative of the vector $\boldsymbol{\Phi}(x)$ defined in (10) can be expressed in the matrix form:

$$\frac{\mathrm{d}\boldsymbol{\Phi}(x)}{\mathrm{d}x} = M\boldsymbol{\Phi}(x) + \boldsymbol{\xi}(x), \tag{21}$$

where $\boldsymbol{\xi}(x) = (\xi_0(x), \xi_1(x), \dots, \xi_N(x))^{\mathrm{T}}$ and $M = (m_{kj})_{0\leqslant k,j\leqslant N}$ is an $(N+1)\times(N+1)$ matrix whose nonzero elements can be given explicitly from relation (11) as

$$m_{kj} = \begin{cases} 2/(b-a)(2j+1)(1+4H_k-4H_j), & k>j,\ (k+j) \text{ odd}, \\ 0, & \text{otherwise}. \end{cases}$$

For example, for $N = 5$, the operational matrix M is the following (6×6) matrix:

$$M = \frac{2}{b-a} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 \\ 25/3 & 0 & 35/3 & 0 & 0 & 0 \\ 0 & 16 & 0 & 14 & 0 & 0 \\ 152/15 & 0 & 62/3 & 0 & 81/5 & 0 \end{pmatrix}.$$

Corollary 1. *The second-, third- and fourth-order derivatives of the vector $\boldsymbol{\Phi}(x)$ are given respectively by*

$$\frac{\mathrm{d}^2\boldsymbol{\Phi}(x)}{\mathrm{d}x^2} = M^2\boldsymbol{\Phi}(x) + M\boldsymbol{\xi}(x) + \boldsymbol{\xi}^{(1)}(x), \tag{22}$$

$$\frac{\mathrm{d}^3\boldsymbol{\Phi}(x)}{\mathrm{d}x^3} = M^3\boldsymbol{\Phi}(x) + M^2\boldsymbol{\xi}(x) + M\boldsymbol{\xi}^{(1)}(x) + \boldsymbol{\xi}^{(2)}(x), \tag{23}$$

$$\frac{\mathrm{d}^4\boldsymbol{\Phi}(x)}{\mathrm{d}x^4} = M^4\boldsymbol{\Phi}(x) + M^3\boldsymbol{\xi}(x) + M^2\boldsymbol{\xi}^{(1)}(x) + M\boldsymbol{\xi}^{(2)}(x) + \boldsymbol{\xi}^{(3)}(x). \tag{24}$$

4 Solution of fourth-order two point BVPs

In this section, we are interested in developing two numerical algorithms for solving both of linear and nonlinear fourth-order two point BVPs. The operational matrix of derivatives that introduced in Section 3 is employed for this purpose. The linear equations are handled by the application of a Petrov–Galerkin method, while the nonlinear equations are handled by the application of the typical collocation method.

4.1 Linear fourth-order BVPs

Consider the linear fourth-order boundary value problem

$$\begin{aligned} &y^{(4)}(x) + f_3(x)y^{(3)}(x) + f_2(x)y^{(2)}(x) \\ &\quad + f_1(x)y^{(1)}(x) + f_0(x)y(x) = g(x), \quad x \in (a,b), \end{aligned} \tag{25}$$

subject to the homogenous boundary conditions

$$y(a) = y(b) = y'(a) = y'(b) = 0. \tag{26}$$

If $y(x)$ is approximated as

$$y(x) \simeq y_N(x) = \sum_{k=0}^{N} c_k\phi_k(x) = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}(x), \tag{27}$$

then making use of relations (21)–(24), the following approximations for $y^{(\ell)}(x)$, $1 \leqslant \ell \leqslant 4$, are obtained:

$$y^{(1)}(x) \simeq \boldsymbol{C}^{\mathrm{T}}\big(M\boldsymbol{\Phi}(x) + \boldsymbol{\xi}\big), \qquad y^{(2)}(x) \simeq \boldsymbol{C}^{\mathrm{T}}\big(M^2\boldsymbol{\Phi}(x) + \boldsymbol{\eta_2}(x)\big), \tag{28}$$

$$y^{(3)}(x) \simeq \boldsymbol{C}^{\mathrm{T}}\big(M^3\boldsymbol{\Phi}(x) + \boldsymbol{\eta_3}(x)\big), \qquad y^{(4)}(x) \simeq \boldsymbol{C}^{\mathrm{T}}\big(M^4\boldsymbol{\Phi}(x) + \boldsymbol{\eta_4}(x)\big), \tag{29}$$

where

$$\begin{aligned}
\boldsymbol{\eta_2}(x) &= M\boldsymbol{\xi}(x) + \boldsymbol{\xi}^{(1)}(x),\\
\boldsymbol{\eta_3}(x) &= M^2\boldsymbol{\xi}(x) + M\boldsymbol{\xi}^{(1)}(x) + \boldsymbol{\xi}^{(2)}(x),\\
\boldsymbol{\eta_4}(x) &= M^3\boldsymbol{\xi}(x) + M^2\boldsymbol{\xi}^{(1)}(x) + M\boldsymbol{\xi}^{(2)}(x) + \boldsymbol{\xi}^{(3)}(x).
\end{aligned}$$

If we substitute by relations (27)–(29) into Eq. (25), then one can write the residual $R(x)$, of this equation as

$$\begin{aligned}
R(x) = {}& \boldsymbol{C}^{\mathrm{T}}\big(M^4\boldsymbol{\Phi}(x) + \boldsymbol{\eta_4}(x)\big) + f_3(x)\boldsymbol{C}^{\mathrm{T}}\big(M^3\boldsymbol{\Phi}(x) + \boldsymbol{\eta_3}(x)\big)\\
&+ f_2(x)\big(M^2\boldsymbol{\Phi}(x) + \boldsymbol{\eta_2}(x)\big) + f_1(x)\boldsymbol{C}^{\mathrm{T}}\big(M\boldsymbol{\Phi}(x) + \boldsymbol{\xi}(x)\big)\\
&+ f_0(x)\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}(x) - g(x).
\end{aligned} \tag{30}$$

The application of Petrov–Galerkin method (see [5]) yields the following $(N+1)$ linear equations in the unknown expansion coefficients c_i:

$$\int_a^b R(x)L_i^*(x)\,\mathrm{d}x = 0, \quad i = 0, 1, \ldots, N. \tag{31}$$

Thus Eq. (31) generates a set of $(N+1)$ linear equations, which can be solved for the unknown components of the vector $\boldsymbol{C}$, and hence the approximate spectral solution $y_N(x)$ given in (27) can be obtained.

Remark 1. It is worthy to note here that, with the aid of a suitable transformation, problem (25), governed by the nonhomogeneous boundary conditions

$$y(a) = \alpha_1, \quad y(b) = \alpha_2, \qquad y'(a) = \beta_1, \quad y'(b) = \beta_2, \tag{32}$$

can be easily transformed to a problem similar to (25)–(26) (see [10]).

4.2 Solution of nonlinear fourth-order two point BVPs

Consider the following nonlinear fourth-order boundary value problem:

$$y^{(4)}(x) = F\big(x, y(x), y^{(1)}(x), y^{(2)}(x), y^{(3)}(x)\big), \tag{33}$$

governed by the homogenous boundary conditions

$$y(a) = y(b) = y'(a) = y'(b) = 0. \tag{34}$$

If $y^{(\ell)}(x)$, $0 \leqslant \ell \leqslant 4$, are approximated as in (27)–(29), then the following nonlinear equations in the unknown vector **C** can be obtained:

$$\begin{aligned}
\boldsymbol{C}^{\mathrm{T}}\big(M^4\boldsymbol{\Phi}(x)+\boldsymbol{\eta_4}(x)\big) \approx F\big(&x,\, \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Phi}(x),\, \boldsymbol{C}^{\mathrm{T}}\big(M\boldsymbol{\Phi}(x)+\boldsymbol{\xi}(x)\big),\\
&\boldsymbol{C}^{\mathrm{T}}\big(M^2\boldsymbol{\Phi}(x)+\boldsymbol{\eta}_2(x)\big),\, \boldsymbol{C}^{\mathrm{T}}\big(M^3\boldsymbol{\Phi}(x)+\boldsymbol{\eta}_3(x)\big)\big).
\end{aligned} \tag{35}$$

To find a numerical solution $y_N(x)$, Eq. (35) is collocated at $(N+1)$ points. There are several choices for these points. For example, they may taken to be the zeros of the polynomials $L^*_{N+1}(x)$, or $T^*_{N+1}(x)$, or $U^*_{N+1}(x)$, where $L^*_i(x)$, $T^*_i(x)$, $U^*_i(x)$ are respectively the shifted Legendre and shifted Chebyshev polynomials of the first and second kinds. It should be noted here that, for every choice of the collocation points, a set of $(N+1)$ nonlinear equations is generated in the expansion coefficients c_i. With the aid of the well-known Newton's iterative method, this nonlinear system can be solved, and hence the corresponding approximate solution $y_N(x)$ can be obtained.

5 Convergence and error analysis

In this section, the convergence and error analysis of the suggested approximate solution will be investigated. We will state and prove a theorem in which the expansion in (7) of a function $y(x) = (x-a)^2(b-x)^2Q(x) \in H^2_{0,w}(I)$, where $Q(x)$ is of bounded second derivative, converges uniformly to $y(x)$. Moreover, an upper bound for the global error (in L^2_w norm) is given.

Theorem 3. *A function $y(x) = (x-a)^2(b-x)^2Q(x) \in H^2_{0,w}(I)$, $w = (1/(x-a)^4 \times (b-x)^4)$ with $|Q''(x)| \leqslant M$, can be expanded as an infinite sum of the basis given in* (7). *This series converges uniformly to $y(x)$, and the coefficients in* (7) *satisfy the inequality*

$$|c_k| < \frac{\sqrt{\pi/2}M(b-a)^2}{(k-3/2)^{3/2}} \quad \forall k \geqslant 2. \tag{36}$$

Proof. From Eq. (8) and with the aid of (6), one has

$$c_k = \frac{2k+1}{b-a} \int_a^b Q(x) L^*_k(x)\,\mathrm{d}x. \tag{37}$$

If the last relation is integrated by parts twice with the aid of Theorem 1 (for $q=2$), then for all $k \geqslant 2$, we have

$$c_k = \frac{1}{4}(b-a)(2k+1) \int_a^b \left[\frac{L^*_{k-2}(x)}{(2k-1)(2k+1)} - \frac{2L^*_k(x)}{(2k-1)(2k+3)} + \frac{L^*_{k+2}(x)}{(2k+1)(2k+3)}\right] \times Q''(x)\mathrm{d}x. \tag{38}$$

Making use of the substitution $(2x-a-b)/(b-a) = \cos\theta$ enables one to put the coefficients c_k in the form

$$c_k = \frac{1}{8}(b-a)^2(2k+1) \int_0^\pi \left[\frac{L_{k-2}(\cos\theta)}{(2k-1)(2k+1)} - \frac{2L_k(\cos\theta)}{(2k-1)(2k+3)} + \frac{L_{k+2}(\cos\theta)}{(2k+1)(2k+3)}\right] \times Q''\left(\frac{1}{2}(a+b+(b-a)\cos\theta)\right)\sin\theta\,\mathrm{d}\theta, \quad k \geqslant 2. \tag{39}$$

Now, taking into consideration that $|Q''(x)| \leqslant M$ and making use of the inequality (see [6]):

$$\sqrt{\sin\theta}\big|L_k(\cos\theta)\big| < \sqrt{\frac{2}{\pi(k+1/2)}}, \quad 0 \leqslant \theta \leqslant \pi,$$

then from (39) it can be shown that

$$|c_k| < \frac{\sqrt{\pi}M(b-a)^2(2k+1)}{\sqrt{2k-3}(2k-1)(2k+3)},$$

and accordingly the following inequality holds:

$$|c_k| < \frac{\sqrt{\pi/2}M(b-a)^2}{(k-3/2)^{3/2}} \quad \forall k \geqslant 2.$$

This prove (36), and hence completes the proof of Theorem 3. □

Theorem 4. *If $y(x)$ satisfies the assumptions of Theorem 3, and if we consider the expansion $y_N(x) = \sum_{k=0}^{N} c_k\phi_k(x)$, then the following error estimate (in the sense of L_w^2-norm, $w = (x-a)^{-4}(b-x)^{-4}$) holds:*

$$\|y - y_N\|_w < \frac{\sqrt{2\pi/3}M(b-a)^{5/2}}{N^{3/2}}. \tag{40}$$

Proof. The orthogonality property of the polynomials $\{\phi_i(x)\}$ together with Eq. (8) enables one to get

$$\|y - y_N\|_w^2 = \sum_{k=N+1}^{\infty} \frac{(b-a)}{(2k+1)} c_k^2.$$

In view of Theorem 3, we can write

$$\|y - y_N\|_w^2 < 2\pi M^2(b-a)^5 \sum_{k=N+1}^{\infty} \frac{1}{k^4}.$$

Applying the integral test for series (see [33]) yields

$$\|y - y_N\|_w^2 < 2\pi M^2(b-a)^5 \int\limits_N^{\infty} x^{-4}\mathrm{d}x = \frac{2\pi M^2(b-a)^5}{3N^3},$$

and accordingly

$$\|y - y_N\|_w < \frac{\sqrt{2\pi/3}M(b-a)^{5/2}}{N^{3/2}}.$$

Theorem 4 is now proved. □

6 Numerical results and discussions

In this section, the two presented algorithms, namely, PGOMM and COMM, which presented in Section 4, are applied to solve linear and nonlinear fourth-order two point boundary value problems. As expected, the accuracy increases as the number of terms of the basis expansion increases.

Example 1. Consider the fourth-order linear equation (see [17, 21, 24])

$$\begin{aligned} &y^{(4)}(x) - (c+1)y''(x) + cy(x) = \frac{1}{2}cx^2 - 1, \quad 0 \leqslant x \leqslant 1, \\ &y(0) = y'(0) = 1, \qquad y(1) = \sinh(1) + \frac{3}{2}, \quad y'(1) = 1 + \cosh(1). \end{aligned} \tag{41}$$

The exact solution of (41) is

$$y(x) = 1 + \frac{x^2}{2} + \sinh(x).$$

In Table 1, the maximum absolute errors E, which resulted from the application of PGOMM, are listed for various values of N and c, while in Table 2, a comparison between the best absolute errors obtained by the application of PGOMM (at $N = 10$) with the best errors obtained by the methods namely, ADM, HPM, DTM developed in [24], and the method namely RKHSM developed in [17]. Moreover, in Table 3, we display a comparison between the relative errors obtained by the application of PGOMM ($N = 11$) with the relative errors resulted from the application of 20th-order homotopy analysis method in [21]. The relative errors are calculated by the formula

$$\delta(x) = \left| \frac{y_{\text{exact}}(x) - y_{\text{approximate}}(x)}{y_{\text{exact}}(x)} \right|.$$

Remark 2. From Table 2 it is clear that the approximate solution obtained by using PGOMM is in good agreement with the exact solution for all values of c, even if c is very large, unlike the approximate solutions obtained by applying ADM and HPM in [24].

Table 1. Maximum pointwise error $|y_{\text{exact}} - y_N|$ using PGOMM for Example 1.

N	$c = 10$	$c = 10^6$	$c = 10^8$	$c = 10^{10}$	$c = 10^{20}$
4	$8.34845 \cdot 10^{-11}$	$7.2599 \cdot 10^{-11}$	$7.26161 \cdot 10^{-11}$	$7.26161 \cdot 10^{-11}$	$7.26162 \cdot 10^{-11}$
6	$4.57783 \cdot 10^{-14}$	$3.66014 \cdot 10^{-14}$	$3.64329 \cdot 10^{-14}$	$3.63589 \cdot 10^{-14}$	$3.63373 \cdot 10^{-14}$
8	$6.18429 \cdot 10^{-16}$	$6.04551 \cdot 10^{-16}$	$6.24934 \cdot 10^{-16}$	$5.30175 \cdot 10^{-16}$	$6.17345 \cdot 10^{-16}$

Table 2. Comparison between the maximum absolute errors for Example 1 using different methods.

Method	$c = 10$	$c = 10^6$
PGOMM ($N = 10$)	$6.1 \cdot 10^{-16}$	$7.9 \cdot 10^{-16}$
ADM in [24]	$7.6 \cdot 10^{-3}$	$2.6 \cdot 10^{+6}$
HPM in [24]	$7.6 \cdot 10^{-3}$	$2.6 \cdot 10^{+6}$
DTM in [24]	$1.6 \cdot 10^{-8}$	$4.2 \cdot 10^{-3}$
RKHSM(u_1^{101}) in [17]	$1.7 \cdot 10^{-9}$	$4.1 \cdot 10^{-10}$

Table 3. Comparison between the relative errors $\delta(x)$ obtained by different methods for Example 1.

x	$c = 100$		$c = 10^8$	
	PGOMM $(N = 11)$	20th-order [21]	PGOMM $(N = 11)$	20th-order [21]
0.1	0.0	$8.6 \cdot 10^{-9}$	0.0	$8.9 \cdot 10^{-5}$
0.2	$5.68434 \cdot 10^{-14}$	$4.6 \cdot 10^{-9}$	$5.68434 \cdot 10^{-14}$	$4.7 \cdot 10^{-5}$
0.3	$5.68434 \cdot 10^{-14}$	$4.4 \cdot 10^{-9}$	$5.68434 \cdot 10^{-14}$	$3.5 \cdot 10^{-5}$
0.4	$7.10543 \cdot 10^{-14}$	$4.3 \cdot 10^{-9}$	$8.52651 \cdot 10^{-14}$	$3.4 \cdot 10^{-5}$
0.5	$1.13687 \cdot 10^{-13}$	$3.9 \cdot 10^{-9}$	$1.13687 \cdot 10^{-13}$	$3.0 \cdot 10^{-5}$
0.6	$8.52651 \cdot 10^{-14}$	$3.5 \cdot 10^{-9}$	$8.52651 \cdot 10^{-14}$	$2.6 \cdot 10^{-5}$
0.7	$4.19220 \cdot 10^{-13}$	$2.9 \cdot 10^{-9}$	$4.19220 \cdot 10^{-13}$	$2.0 \cdot 10^{-5}$
0.8	$1.49214 \cdot 10^{-13}$	$2.5 \cdot 10^{-9}$	$1.49214 \cdot 10^{-13}$	$2.6 \cdot 10^{-5}$
0.9	$3.90799 \cdot 10^{-13}$	$3.8 \cdot 10^{-9}$	$3.76588 \cdot 10^{-13}$	$4.0 \cdot 10^{-5}$

Table 4. Comparison between the absolute errors for Example 2.

x	COMM $(N = 4)$	COMM $(N = 6)$	COMM $(N = 8)$	RKHSM (u_1^{101}) [17]
0.0	0.0	0.0	0.0	0.0
0.1	$7.69876 \cdot 10^{-11}$	$7.43711 \cdot 10^{-14}$	$2.77556 \cdot 10^{-17}$	$2.78 \cdot 10^{-8}$
0.2	$5.87005 \cdot 10^{-10}$	$1.74749 \cdot 10^{-13}$	0.0	$8.09 \cdot 10^{-8}$
0.3	$1.15724 \cdot 10^{-9}$	$1.96093 \cdot 10^{-13}$	$5.55112 \cdot 10^{-17}$	$1.20 \cdot 10^{-7}$
0.4	$9.58059 \cdot 10^{-10}$	$4.42368 \cdot 10^{-13}$	$2.498 \cdot 10^{-16}$	$1.25 \cdot 10^{-7}$
0.5	$8.1097 \cdot 10^{-11}$	$8.16014 \cdot 10^{-15}$	$5.55112 \cdot 10^{-17}$	$9.56 \cdot 10^{-8}$
0.6	$1.10346 \cdot 10^{-9}$	$4.51764 \cdot 10^{-13}$	$3.46945 \cdot 10^{-16}$	$4.82 \cdot 10^{-8}$
0.7	$1.25497 \cdot 10^{-9}$	$1.88627 \cdot 10^{-13}$	$4.30211 \cdot 10^{-16}$	$7.38 \cdot 10^{-9}$
0.8	$6.25015 \cdot 10^{-10}$	$1.89848 \cdot 10^{-13}$	$5.41234 \cdot 10^{-16}$	$1.07 \cdot 10^{-8}$
0.9	$7.9979 \cdot 10^{-11}$	$7.90253 \cdot 10^{-14}$	$3.1572 \cdot 10^{-16}$	$7.08 \cdot 10^{-9}$
1.0	$7.63278 \cdot 10^{-17}$	$3.18772 \cdot 10^{-17}$	$5.74221 \cdot 10^{-16}$	0.0

Example 2. Consider the following nonlinear fourth-order boundary value problem (see [17]):

$$\begin{aligned} &y^{(4)}(x) - \mathrm{e}^x y''(x) + y(x) + \sin\bigl(y(x)\bigr) = f(x), \quad 0 \leqslant x \leqslant 1, \\ &y(0) = y'(0) = 1, \qquad y(1) = 1 + \sinh(1), \quad y'(1) = \cosh(1) \end{aligned} \tag{42}$$

with $f(x) = 1 - (-2 + \mathrm{e}^x)\sinh(x) + \sin(\sinh(x) + 1)$ and the exact solution

$$y(x) = \sinh(x) + 1.$$

In Table 4, the absolute errors are listed for various values of N. In order to compare the absolute errors obtained by applying COMM with those obtained by applying RHKSM in [17], we list the absolute errors obtained by the application of RHKSM in the last column of this table. This table shows that the approximate solution of problem (42) obtained by using COMM is of high efficiency and more accurate than the approximate solution obtained by RHKSM [17].

Example 3. Consider the following fourth-order nonlinear boundary value problem (see [3]):

$$\begin{aligned} &y^{(4)}(x) + \bigl(y''(x)\bigr)^2 = \sin x + \sin^2(x), \quad 0 \leqslant x \leqslant 1, \\ &y(0) = 0, \quad y'(0) = 1, \qquad y(1) = \sin 1, \quad y'(1) = \cos 1. \end{aligned} \tag{43}$$

Table 5. Maximum absolute errors of $|y - y_N|$ for Example 3 for $N = 2, 4, 6, 8, 10$.

N	E_1	E_2	E_3
2	$8.63 \cdot 10^{-7}$	$1.31 \cdot 10^{-6}$	$5.66 \cdot 10^{-7}$
4	$9.88 \cdot 10^{-10}$	$2.12 \cdot 10^{-9}$	$4.29 \cdot 10^{-10}$
6	$3.53 \cdot 10^{-13}$	$7.98 \cdot 10^{-13}$	$1.71 \cdot 10^{-13}$
8	$1.666 \cdot 10^{-16}$	$2.78 \cdot 10^{-16}$	$1.11 \cdot 10^{-16}$
10	$1.11 \cdot 10^{-16}$	$1.11 \cdot 10^{-16}$	$1.11 \cdot 10^{-16}$

Table 6. Comparison between the absolute errors of $|y - y_N|$ for Example 3.

x	The method in [3] $(N = 2)$	E_1	E_2	E_3
0.0	0.0	0.0	0.0	0.0
0.1	$3.0026 \cdot 10^{-6}$	$2.339 \cdot 10^{-7}$	$3.905 \cdot 10^{-7}$	$1.293 \cdot 10^{-7}$
0.2	$1.0310 \cdot 10^{-5}$	$6.611 \cdot 10^{-7}$	$1.037 \cdot 10^{-6}$	$4.096 \cdot 10^{-7}$
0.3	$1.9108 \cdot 10^{-5}$	$8.634 \cdot 10^{-7}$	$1.308 \cdot 10^{-6}$	$5.657 \cdot 10^{-7}$
0.4	$2.9885 \cdot 10^{-5}$	$6.481 \cdot 10^{-7}$	$9.671 \cdot 10^{-7}$	$4.347 \cdot 10^{-7}$
0.5	$2.9885 \cdot 10^{-5}$	$1.199 \cdot 10^{-7}$	$1.841 \cdot 10^{-7}$	$7.693 \cdot 10^{-7}$
0.7	$2.1185 \cdot 10^{-5}$	$6.877 \cdot 10^{-7}$	$1.035 \cdot 10^{-6}$	$4.554 \cdot 10^{-7}$
0.9	$3.7203 \cdot 10^{-6}$	$2.0172 \cdot 10^{-7}$	$3.362 \cdot 10^{-7}$	$3.469 \cdot 10^{-7}$
1.0	$5.5144 \cdot 10^{-7}$	0.0	0.0	0.0

The exact solution of the above problem is

$$y(x) = \sin x.$$

In Table 5, we list the maximum absolute errors $|y - y_N|$ using COMM for various values of N. Let E_1, E_2 and E_3 denote the maximum absolute errors if the selected collocation points are respectively, the zeros of the shifted Legnedre polynomials $L^*_{N+1}(x)$, and the shifted Chebyshev polynomials of the first and second kinds $T^*_{N+1}(x)$ and $U^*_{N+1}(x)$. The numerical results show that the best errors are achieved when the selected collocation points are the zeros of Chebyshev polynomials of the second kind. Table 6 displays a comparison between the errors obtained by the application of COMM for the previous three choices of the collocation points with the method developed in [3] for the case $N = 2$. The comparison ascertains that our results are more accurate than those obtained in [3].

Example 4. Consider the following fourth-order nonlinear boundary value problem (see [32]):

$$\begin{aligned} &y^{(4)}(x) = y^2(x) + g(x), \quad 0 < x < 1, \\ &y(0) = y'(0) = 0, \qquad y(1) = y'(1) = 1, \end{aligned} \tag{44}$$

where $g(x) = -x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48$. The exact solution of the above problem is

$$y(x) = x^5 - 2x^4 + 2x^2.$$

Table 7 displays a comparison between the absolute errors obtained by the application of COMM for $N = 4$ with the two methods developed in [32]. The comparison ascertains that our method is more accurate than the two methods obtained in [32].

Table 7. Comparison between the absolute errors for Example 4.

x	Method in [32] $(\lvert\phi_2 - y\rvert)$	Method in [32] $(\lvert\psi_2 - y\rvert)$	COMM $(N = 2)$
0.0	0.0	0.0	0.0
0.2	$3.5906 \cdot 10^{-5}$	$8.1093 \cdot 10^{-10}$	$9.73223 \cdot 10^{-20}$
0.4	$1.0188 \cdot 10^{-4}$	$2.0542 \cdot 10^{-9}$	$5.29677 \cdot 10^{-19}$
0.6	$1.3579 \cdot 10^{-4}$	$2.2272 \cdot 10^{-9}$	$6.52792 \cdot 10^{-18}$
0.8	$8.5908 \cdot 10^{-5}$	$1.0115 \cdot 10^{-9}$	$2.91002 \cdot 10^{-17}$
1.0	$5.5799 \cdot 10^{-13}$	0.0	$8.79947 \cdot 10^{-17}$

7 Conclusions

In this article, a novel operational matrix of derivatives is introduced. This operational matrix is given in terms of the well-known harmonic numbers. Two algorithms based on the application of the Petrov–Galerkin and collocation spectral methods are presented and implemented for the sake of obtaining new approximate solutions of linear and nonlinear fourth-order boundary value problems. The derivation of these algorithms rely on selecting a set of basis functions satisfying the boundary conditions of the given boundary value problem in terms of shifted Legendre polynomials. The main advantages of the introduced algorithms are their simplicity in application and their high accuracy, since high accurate approximate solutions can be achieved by using a few number of terms of the suggested expansion. The numerical results are convincing and the resulted approximate solutions are very close to the exact ones.

Acknowledgment. The author would like to thank the referee for his carefully reading the paper and also for his constructive comments, which have improved the paper.

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