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Adaptive NN output-feedback control for stochastic time-delay nonlinear systems with unknown control coefficients and perturbations*

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Abstract. This paper addresses the problem of adaptive output-feedback control for more general class of stochastic time-varying delay nonlinear systems with unknown control coefficients and perturbations. By using Lyapunov–Krasovskii functional, backstepping and tuning function technique, a novel adaptive neural network (NN) output-feedback controller is constructed with fewer learning parameters. The designed controller guarantees that all the signals in the closed-loop system are 4-moment (or mean square) semi-globally uniformly ultimately bounded (SGUUB). Finally, a simulation example is shown to demonstrate the effectiveness of the proposed control scheme.

Keywords: stochastic nonlinear system, output-feedback control, time-varying delays, neural networks, unknown control coefficients.

1 Introduction

In order to obtain global stability, some restrictions such as matching conditions, extended matching conditions or growth conditions are often imposed on system nonlinearities. To handle the above restrictions, neural networks (NNs) are gradually into people's vision due to their ability to adaptively compensate for nonlinear functions. In the past two decades, based on the theoretical results of stochastic stability in [6, 7, 11] and NNs in [14, 16, 23], radial basis function neural network (RBF NN) approximation approach has been successfully used for various classes of stochastic nonlinear systems, see [1, 8, 12, 13, 15, 17, 27] and the references therein.

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It is well known that time-delays are frequently encountered in control systems. Recent years have witnessed some developments of stochastic nonlinear time-delay systems based on backstepping, NNs and other design technique, see [2, 3, 4, 5, 9, 10, 18, 20, 21, 22, 24, 25, 26] and [28]. In [3], the decentralized output-feedback control was considered by applying NNs. [18] and [28] proposed the adaptive NN control scheme for stochastic nonlinear systems with constant time-delays. However, in [28], the constant time-delays only appeared in the drift terms. In [4] and [5], the authors further investigated stochastic nonlinear systems with time-varying delays, but the diffusion terms still did not contain time-delays. Subsequently, the output-feedback control problems were presented in [2] and [10] for stochastic nonlinear systems with time-varying delays existing both in the drift and diffusion terms, but the considered time-delays only depended on measurable output. Later, for more general classes of stochastic nonlinear systems with time-varying delays depending on unmeasurable states, NN tracking problems were solved in [24, 26]. However, output-feedback control is still an open problem for this kind of systems.

The lasted reference [25] solved the adaptive NN output-feedback control problem for a class of stochastic time-varying delay nonlinear systems without considering perturbations. In addition, [25] required the delay-depended drift and diffusion terms to be bounded by two known functions. Then, one may ask the following interesting problems:

> *How to relax the conditions on time-varying delay-depended drift and diffusion terms by NN? Under the weaker conditions, how to design an adaptive output-feedback controller for a class of stochastic nonlinear time-delay systems with both unknown control directions and perturbations?*

This paper focuses on solving the above problems. The main contributions are listed as follows: (i) Compared with [25], by utilizing a novel RBF NN approximation approach and introducing a proper linear transformation, the restrictions on system nonlinearities are much weaker and the design procedure is simpler. In addition, the tuning function approach is used to overcome the problem of over-parameterization generated by the unknown perturbations. (ii) An adaptive NN output-feedback controller is constructed for more general class of stochastic time-varying delay nonlinear systems. It should be pointed out that the knowledge of NN nodes and weights is not necessary to be prior known and the proposed control scheme can ensure all the signals in the closed-loop system to be 4-moment (or mean square) semi-globally uniformly ultimately bounded (SGUUB).

The remainder of this paper is organized as follows. Section 2 begins with the mathematical preliminaries. Section 3 states the main problem. The design and analysis of the controller are presented in Sections 4 and 5, respectively. In Section 6, a simulation example is given. Section 7 concludes the paper. The necessary proof is provided in Appendix.

2 Mathematical preliminaries

In this paper, the following notations are to be used. $\mathbb{R}^+$ denotes the set of all the nonnegative real numbers; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathcal{C}^i$ denotes the family of all the functions with continuous ith partial derivations; $\mathcal{C}^{2,1}(\mathbb{R}^n\times[-d,\infty),\mathbb{R}^+)$ denotes the family of all non-negative functions $V(x,t)$ on $\mathbb{R}^n\times[-d,\infty)$, which are $\mathcal{C}^2$ in x and $\mathcal{C}^1$ in t. $\mathcal{C}([-d,0];\mathbb{R}^n)$ denotes the space of continuous $\mathbb{R}^n$-valued functions on $[-d,0]$ endowed with the norm $\|\cdot\|$ defined by $\|f\|=\sup_{x\in[-d,0]}|f(x)|$ for $f\in\mathcal{C}([-d,0],\mathbb{R}^n)$; $\mathcal{C}^b_{\mathcal{F}_0}([-d,0],\mathbb{R}^n)$ denotes the family of all $\mathcal{F}_0$-measurable bounded $\mathcal{C}([-d,0];\mathbb{R}^n)$-valued random variables $\xi=\{\xi(\theta){:}\ -d\leqslant\theta\leqslant 0\}$. X^{T} denotes the transpose of a given vector or matrix X, $\mathrm{Tr}\{X\}$ denotes its trace when X is square. $\|X\|$ is the Euclidean norm of a vector X or its inducted matrix norm. $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and largest eigenvalues of a square matrix, respectively. To simply the procedure, we sometimes denote $X(t)$ by X for any variable $X(t)$.

Consider the following stochastic nonlinear time-delay system:

$$\mathrm{d}x(t)=f\bigl(t,x(t),x\bigl(t-d(t)\bigr)\bigr)\,\mathrm{d}t+g\bigl(t,x(t),x\bigl(t-d(t)\bigr)\bigr)\,\mathrm{d}\omega\quad\forall t\geqslant 0 \tag{1}$$

with initial data $\{x(\theta){:}\ -d\leqslant\theta\leqslant 0\}=\xi\in\mathcal{C}^b_{\mathcal{F}_0}([-d,0],\mathbb{R}^n)$, where $d(t):\mathbb{R}^+\to[0,d]$ is a Borel measurable function; ω is an r-dimensional standard wiener process defined on a probability space $\{\Omega,\mathcal{F},\mathbf{P}\}$ with Ω being a sample space, $\mathcal{F}$ being a σ-field and $\mathbf{P}$ being the probability measure. $f:\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $g:\mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times r}$ are locally Lipschitz with $f(0,0,t)\equiv 0$ and $g(0,0,t)\equiv 0$. For any given $V(x(t),t)\in C^{2,1}$, together with stochastic system (1), the differential operator $\mathcal{L}$ is defined as

$$\mathcal{L}V=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f+\frac{1}{2}\,\mathrm{Tr}\biggl\{g^{\mathrm{T}}\frac{\partial^2 V}{\partial x^2}g\biggr\}. \tag{2}$$

Definition 1. Let $p\geqslant 1$, consider stochastic nonlinear time-delay system (1), the solution $\{x(t),\ t\geqslant 0\}$ with initial condition $\xi\in S_0$ (S_0 is some compact set containing the origin) is said to be p-moment semi-globally uniformly ultimately bounded if there exists a constant d such that $\mathbf{E}\{\|x(t,\xi)\|^p\}\leqslant d$ for all $t\geqslant T$ holds for some $T\geqslant 0$.

In the sequel, radial basis function neural network (RBF NN) will be applied to estimate the unknown nonlinear functions. It has been proven in [23] that by choosing sufficiently large node number, for any unknown continuous function $f(x)$ over a compact set $S_x\subset\mathbb{R}^q$, there is a RBF NN $W^{*\mathrm{T}}S(x)$ such that for an expected level of accuracy ε ($0<\varepsilon<1$), it holds

$$f(x)=W^{*\mathrm{T}}S(x)+\delta(x),\quad \bigl|\delta(x)\bigr|\leqslant\varepsilon, \tag{3}$$

where $\delta(x)$ is the approximation error, and $S(x)=[s_1(x),\dots,s_N(x)]^{\mathrm{T}}$ is the known function vector with $N>1$ being the RBF NN node number. For $1\leqslant i\leqslant N$, the basis functions $s_i(x)$ are chosen as $s_i(x)=\exp[-(x-b_i)^{\mathrm{T}}(x-b_i)/\varsigma^2]$, where ς is the width of the function, $b_i=[b_{i1},\dots,b_{in}]^{\mathrm{T}}$ is the center of the receptive field. W^* is the ideal constant weight vector with the form $W^*=\arg\min_{W\in\mathbb{R}^N}\{\sup_{x\in S_x}|f(x)-W^{\mathrm{T}}S(x)|\}$, where $\arg\min$ is the value of variable W when the objective function $\sup_{x\in S_x}|f(x)-W^{\mathrm{T}}S(x)|$ is minimum with $W=[w_1,\dots,w_N]^{\mathrm{T}}$ being the weight vector.

Lemma 1 [Young's inequality]. *For all $(x,y)\in\mathbb{R}^2$, $xy\leqslant\varepsilon^p p^{-1}|x|^p+(q\varepsilon^q)^{-1}|y|^q$ holds, where $\varepsilon>0$, $p,q>1$, and $(p-1)(q-1)=1$.*

Lemma 2. (*See* [19].) *For any smooth function $f(x)$, $x \in \mathbb{R}^n$, there exists a smooth function $\bar{f}(x)$ such that $f(x) - f(0) = (\int_0^1 \partial f(\lambda)/\partial\lambda|_{\lambda=\alpha x}\,\mathrm{d}\alpha)x = x\bar{f}(x)$.*

3 Problem description

In this paper, we consider a class of stochastic nonlinear systems in the following form:

$$\begin{aligned}
\mathrm{d}\eta_i &= h_i\eta_{i+1}\,\mathrm{d}t + \phi_i\bigl(t,\eta(t),\eta\bigl(t-d(t)\bigr)\bigr)\,\mathrm{d}t + \theta_i^{\mathrm{T}}\psi_i(y)\,\mathrm{d}t \\
&\quad + \varphi_i\bigl(t,\eta(t),\eta\bigl(t-d(t)\bigr)\bigr)\,\mathrm{d}\omega, \quad i = 1,\dots,n-1, \\
\mathrm{d}\eta_n &= h_n u\,\mathrm{d}t + \phi_n\bigl(t,\eta(t),\eta\bigl(t-d(t)\bigr)\bigr)\,\mathrm{d}t + \theta_n^{\mathrm{T}}\psi_n(y)\,\mathrm{d}t \\
&\quad + \varphi_n\bigl(t,\eta(t),\eta\bigl(t-d(t)\bigr)\bigr)\,\mathrm{d}\omega, \\
y &= \eta_1,
\end{aligned} \tag{4}$$

where $\eta = [\eta_1,\dots,\eta_n]^{\mathrm{T}} \in \mathbb{R}^n$ is the system state vector and $\eta(t-d(t)) = [\eta_1(t-d(t)), \dots, \eta_n(t-d(t))]^{\mathrm{T}}$ is the time-delay state variable, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are control input and system output, respectively, $\eta_2,\dots,\eta_n$ are unmeasurable state variables, control coefficients $h_1,\dots,h_n$ are unknown constants. $d(t): \mathbb{R}^+ \to [0,d]$ is time-varying delay. ω is an r-dimensional standard wiener process. For $i = 1,\dots,n$, $\phi_i: \mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ and $\varphi_i: \mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^r$ are unknown smooth functions with $\phi_i(t,0,0) = 0$ and $\varphi_i(t,0,0) = 0$. For $i = 1,2,\dots,n$, $\theta_i \in \mathbb{R}^{m_i}$ are unknown constant system parameters and $\psi_i: \mathbb{R} \to \mathbb{R}^{m_i}$ are known smooth vector-valued functions.

The control objective of the paper is to construct an adaptive NN output-feedback controller for system (4) such that all the signals in the closed-loop system are 4-moment (or mean-square) SGUUB. To realise this objective, we need the following assumptions.

Assumption 1. The nonzero control coefficients $h_1,\dots,h_n$ are of known signs and satisfy $\underline{h} \leqslant |h_i| \leqslant \bar{h}$ $(i = 1,\dots,n)$, where $\underline{h}$ and $\bar{h}$ are known positive constants denoting the low and upper bounds of $h_1,\dots,h_n$, respectively.

Defining $h = h_1\cdots h_n/\underline{h}^n$ and using Assumption 1, it is easy to find a constant $h_M \geqslant 1$ such that

$$1 \leqslant |h| \leqslant h_M, \quad \underline{h}^n = \underbrace{\underline{h}\times\cdots\times\underline{h}}_{n}. \tag{5}$$

Assumption 2. The time-varying delay $d(t)$ in system (4) satisfies $\dot{d}(t) \leqslant \gamma < 1$ for a given constant γ.

Assumption 3. For $1 \leqslant i \leqslant n$, the nonlinear functions ϕ_i and φ_i satisfy the following inequalities:

$$\begin{aligned}
\bigl\|\phi_i\bigl(t,\eta(t),\eta\bigl(t-d(t)\bigr)\bigr)\bigr\| &\leqslant \phi_{i1}\bigl(\bigl\|\eta(t)\bigr\|\bigr) + \phi_{i2}\bigl(\bigl\|\eta\bigl(t-d(t)\bigr)\bigr\|\bigr), \\
\bigl\|\varphi_i\bigl(t,\eta(t),\eta\bigl(t-d(t)\bigr)\bigr)\bigr\| &\leqslant \varphi_{i1}\bigl(\bigl\|\eta(t)\bigr\|\bigr) + \varphi_{i2}\bigl(\bigl\|\eta\bigl(t-d(t)\bigr)\bigr\|\bigr),
\end{aligned} \tag{6}$$

where $\phi_{i1}(\cdot)$ and $\varphi_{i1}(\cdot)$ are unknown nonnegative smooth functions, $\phi_{i2}(\cdot)$ and $\varphi_{i2}(\cdot)$ are unknown class-$\mathcal{K}_\infty$ functions.

Remark 1. For system (4) without perturbations $\theta_i^{\mathrm{T}}\psi_i(y)$, in the similar references [2, 10], the functions ϕ_i and φ_i only appear in the forms of $\phi_i(t, y(t), y(t-d(t)))$ and $\varphi_i(t, y(t), y(t-d(t)))$. In [25], the functions f_i and h_i may contain unmeasured state variables. However, the bounds of the time-delay functions should only contain $y(t)$ and $y(t-d(t))$. [24] and [26] relaxed the bound functions to contain state variables and solved the tracking problems. For system (4) with perturbations $\theta_i^{\mathrm{T}}\psi_i(y)$, this paper handles the output-feedback control problems by allowing the bound functions $\phi_{ij}(\cdot)$ and $\varphi_{ij}(\cdot)$ $(i = 1, \dots, n, j = 1, 2)$ to be both unknown and contain state variables.

4 Output-feedback controller design

The design procedure of output-feedback controller is divided into two parts. Firstly, by introducing an equivalent linear state transformation to lump the control coefficients into one, RBF NN is used to estimate the unknown nonlinear functions and the tuning function method is utilized to avoid over-parameterization. Then, an adaptive output-feedback controller is designed to guarantee all the signals in the closed-loop system to be 4-moment (or mean-square) SGUUB.

4.1 Full-order observer design

To make system (4) more feasible, we introduce a linear state transformation

$$x_i = \frac{\underline{h}^n}{h_i \cdots h_n}\eta_i, \quad i = 1, \dots, n. \tag{7}$$

Then, system (4) is changed into the following equivalent system:

$$\begin{aligned} \mathrm{d}x_i &= x_{i+1}\,\mathrm{d}t + f_i\big(t, x(t), x\big(t-d(t)\big)\big)\,\mathrm{d}t + \Theta_i^{\mathrm{T}}\psi_i(y)\,\mathrm{d}t \\ &\quad + g_i\big(t, x(t), x\big(t-d(t)\big)\big)\,\mathrm{d}\omega, \quad i = 1, \dots, n-1, \\ \mathrm{d}x_n &= \underline{h}^n u dt + f_n\big(t, x(t), x\big(t-d(t)\big)\big)\,\mathrm{d}t + \Theta_n^{\mathrm{T}}\psi_n(y)\,\mathrm{d}t \\ &\quad + g_n\big(t, x(t), x\big(t-d(t)\big)\big)\,\mathrm{d}\omega, \\ y &= hx_1, \end{aligned} \tag{8}$$

where $x = [x_1, \dots, x_n]^{\mathrm{T}}$, $x(t-d(t)) = [x_1(t-d(t)), \dots, x_n(t-d(t))]^{\mathrm{T}}$, $f_j(\cdot) = \underline{h}^n(h_j \cdots h_n)^{-1}\phi_j(\cdot)$, $g_j(\cdot) = \underline{h}^n(h_j \cdots h_n)^{-1}\varphi_j(\cdot)$ and $\Theta_j = \underline{h}^n(h_j \cdots h_n)^{-1}\theta_j$ $(j = 1, \dots, n)$. According to Assumption 3 and (7), for $i = 1, \dots, n$, there must exist unknown nonnegative smooth functions f_{i1}, g_{i1} and class-$\mathcal{K}_\infty$ unknown functions f_{i2}, g_{i2} such that

$$\begin{aligned} \big\|f_i\big(t, x(t), x\big(t-d(t)\big)\big)\big\| &\leqslant f_{i1}\big(\|x(t)\|\big) + f_{i2}\big(\big\|x\big(t-d(t)\big)\big\|\big), \\ \big\|g_i\big(t, x(t), x\big(t-d(t)\big)\big)\big\| &\leqslant g_{i1}\big(\|x(t)\|\big) + g_{i2}\big(\big\|x\big(t-d(t)\big)\big\|\big). \end{aligned} \tag{9}$$

We now turn to design an adaptive output-feedback controller for (8). Since $\eta_2, \dots, \eta_n$ are unmeasured and the unknown control coefficients exist in (7), all the states $x_1, \dots, x_n$

are unavailable, the following full-order observer is needed

$$\begin{aligned} \dot{\hat{x}}_i &= \hat{x}_{i+1} + b_i(x_1 - \hat{x}_1), \quad i = 1, \dots, n-1, \\ \dot{\hat{x}}_n &= \underline{h}^n u + b_n(x_1 - \hat{x}_1), \end{aligned} \tag{10}$$

where $b_1, \dots, b_n$ are constants satisfying

$$A^{\mathrm{T}} P + PA \leqslant -I, \quad A = \begin{pmatrix} -b_1 & 1 & 0 & \dots & 0 \\ -b_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -b_{n-1} & 0 & \dots & 0 & 1 \\ -b_n & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{11}$$

and P is a certain symmetric and positive definite matrix. Define the observer error $\tilde{x} = x - \hat{x}$ with $\tilde{x} = [\tilde{x}_1, \dots, \tilde{x}_n]^{\mathrm{T}}$. By (8), (10) and (11), one has

$$\begin{aligned} \mathrm{d}\tilde{x} &= A\tilde{x}\,\mathrm{d}t + f\bigl(t, x(t), x\bigl(t - d(t)\bigr)\bigr)\,\mathrm{d}t \\ &\quad + \Theta^{\mathrm{T}} \psi(y)\,\mathrm{d}t + g\bigl(t, x(t), x\bigl(t - d(t)\bigr)\bigr)\,\mathrm{d}\omega, \end{aligned} \tag{12}$$

where $f(\cdot) = [f_1(\cdot), \dots, f_n(\cdot)]^{\mathrm{T}}$, $\Theta = \mathrm{diag}[\Theta_1, \dots, \Theta_n]$, $\psi(\cdot) = [\psi_1, \dots, \psi_n]^{\mathrm{T}}$ and $g(\cdot) = [g_1(\cdot), \dots, g_n(\cdot)]^{\mathrm{T}}$.

Consider $V_0 = (1/2)(\tilde{x}^{\mathrm{T}} P \tilde{x})^2$, in view of (2), (11), (12) and Lemmas 1, 2, one can get

$$\begin{aligned} \mathcal{L}V_0 &\leqslant -\lambda_{\min}(P)\|\tilde{x}\|^4 + 2\tilde{x}^{\mathrm{T}} P \tilde{x}\tilde{x}^{\mathrm{T}} P f + \frac{3}{2}\|\tilde{x}\|^4 + \frac{1}{2}\|P\|^8 r_\Theta^4 \|\bar{\psi}\|^4 y^4 \\ &\quad + \mathrm{Tr}\bigl\{g^{\mathrm{T}}\bigl(2P\tilde{x}\tilde{x}^{\mathrm{T}} P + \tilde{x}^{\mathrm{T}} P \tilde{x} P\bigr) g\bigr\}, \end{aligned} \tag{13}$$

where $r_\Theta > 0$ is an unknown number satisfying $\|\Theta\| \leqslant r_\Theta$. For (13), by Lemma 1, (9), $(a_1 + \dots + a_n)^2 \leqslant n\sum_{i=1}^n a_i^2$ and $(a+b)^4 \leqslant 8(a^4 + b^4)$, one has

$$\begin{aligned} &2\tilde{x}^{\mathrm{T}} P \tilde{x}\tilde{x}^{\mathrm{T}} P f \\ &\quad\leqslant \frac{3}{2}\|P\|^{8/3}\|\tilde{x}\|^4 + \frac{1}{2}\|f\|^4 \\ &\quad\leqslant \frac{3}{2}\|P\|^{8/3}\|\tilde{x}\|^4 + 4n\sum_{i=1}^n \bigl(f_{i1}^4\bigl(\|x\|\bigr) + f_{i2}^4\bigl(\bigl\|x\bigl(t - d(t)\bigr)\bigr\|\bigr)\bigr), \end{aligned} \tag{14}$$

$$\begin{aligned} &\mathrm{Tr}\bigl\{g^{\mathrm{T}}\bigl(2P\tilde{x}\tilde{x}^{\mathrm{T}} P + \tilde{x}^{\mathrm{T}} P \tilde{x} P\bigr) g\bigr\} \\ &\quad\leqslant 3n\sqrt{n}\|P\|^2\|\tilde{x}\|^2\|g\|^2 \\ &\quad\leqslant \frac{9n^3}{2}\|P\|^4\|\tilde{x}\|^4 + 4n\sum_{i=1}^n \bigl(g_{i1}^4\bigl(\|x\|\bigr) + g_{i2}^4\bigl(\bigl\|x\bigl(t - d(t)\bigr)\bigr\|\bigr)\bigr). \end{aligned} \tag{15}$$

Substituting (14) and (15) into (13) leads to

$$\mathcal{L}V_0 \leqslant -c_0\|\tilde{x}\|^4 + \pi\|\bar{\psi}\|^4 y^4 + 4n\sum_{i=1}^n \bigl(f_{i1}^4 + g_{i1}^4\bigr) + \varDelta_{0d}\bigl(\bigl\|x\bigl(t - d(t)\bigr)\bigr\|\bigr), \tag{16}$$

where $c_0 = \lambda_{\min}(P) - (3/2)\|P\|^{8/3} - (9n^3/2)\|P\|^4$, $\Delta_{0d}(\cdot) = 4n\sum_{i=1}^{n}(f_{i2}^4(\cdot) + g_{i2}^4(\cdot))$,

$$\pi = \frac{1}{2}\max\left\{\|P\|^8 r_\Theta^4, \frac{3h_M^{4/3}}{2\varrho^{4/3}}\right\}, \tag{17}$$

and $\varrho > 0$ is an arbitrary design constant.

4.2 Backstepping controller design

The entire system can be rewritten as

$$\begin{aligned} \mathrm{d}\tilde{x} &= A\tilde{x}\,\mathrm{d}t + f(\cdot)\,\mathrm{d}t + \Theta^{\mathrm{T}}\psi(y)\,\mathrm{d}t + g(\cdot)\,\mathrm{d}\omega, \\ \mathrm{d}y &= h(\hat{x}_2 + \tilde{x}_2)\,\mathrm{d}t + hf_1(\cdot)\,\mathrm{d}t + \theta_1^{\mathrm{T}}\psi_1(y)\,\mathrm{d}t + hg_1(\cdot)\,\mathrm{d}\omega, \\ \mathrm{d}\hat{x}_i &= (\hat{x}_{i+1} + b_i\tilde{x}_1)\,\mathrm{d}t, \quad i = 1, \dots, n-1, \\ \mathrm{d}\hat{x}_n &= (\underline{h}^n u + b_n\tilde{x}_1)\,\mathrm{d}t. \end{aligned} \tag{18}$$

Introduce the coordinate transformation

$$z_1 = y, \qquad z_i = \hat{x}_i - \alpha_i(\bar{z}_i, \hat{x}_2, \hat{\theta}, \hat{\vartheta}), \quad i = 2, \dots, n, \tag{19}$$

and define new states $X_1 = (x, z_1)^{\mathrm{T}}$ and $X_i = (\tilde{x}_1, x, \bar{z}_i)^{\mathrm{T}}$, where $\alpha_2, \dots, \alpha_n$ are virtual control laws to be designed later and $\bar{z}_i = [z_1, \dots, z_i]^{\mathrm{T}}$. In the sequel, we will design an adaptive output-feedback controller by n steps.

Step 1. Choose the 2nd Lyapunov–Krasovskii function candidate as

$$V_1 = V_0 + \frac{1}{4}z_1^4 + V_Q + \frac{1}{2\Gamma_1}\tilde{\theta}^2 + \tilde{\vartheta}^{\mathrm{T}}\Gamma_2^{-1}\tilde{\vartheta}, \tag{20}$$

where $V_Q = (1-\gamma)^{-1}\int_{t-d(t)}^{t}\Pi(\|x(s)\|)\,\mathrm{d}s$, Π will be designed later, $\Gamma_1 > 0$ is a constant, $\Gamma_2 = \Gamma_2^{\mathrm{T}} > 0$ is adaptive gain matrix, $\hat{\theta}$ is the estimate of θ and $\hat{\vartheta}$ is the estimate of ϑ, $\tilde{\theta} = \theta - \hat{\theta}$ and $\tilde{\vartheta} = \vartheta - \hat{\vartheta}$ are estimation errors of θ and ϑ, respectively, and

$$\theta = \max\bigl\{N_i\|W_i^*\|^2,\, N_{i0}\|W_{i0}^*\|^2,\, i = 0, \dots, n\bigr\}, \qquad \vartheta = \bigl[\pi, h, \theta_1^{\mathrm{T}}\bigr]^{\mathrm{T}}, \tag{21}$$

Using (2), (16), (18), $\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$ and $\dot{\tilde{\vartheta}} = -\dot{\hat{\vartheta}}$, one can get

$$\begin{aligned} \mathcal{L}V_1 \leqslant{}& -c_0\|\tilde{x}\|^4 + \pi\|\bar{\psi}\|^4 y^4 + \Delta_{0d}(\cdot) + z_1^3\bigl(h\hat{x}_2 + h\tilde{x}_2 + hf_1 + \theta_1^{\mathrm{T}}\psi_1\bigr) \\ &+ \frac{3}{2}h^2 z_1^2 g_1 g_1^{\mathrm{T}} - \frac{1}{\Gamma_1}\tilde{\theta}\dot{\hat{\theta}} - \tilde{\vartheta}^{\mathrm{T}}\Gamma_2^{-1}\dot{\hat{\vartheta}} + f_0\bigl(\|x\|\bigr) - \frac{1-\dot{d}(t)}{1-\gamma}\Pi(\cdot), \end{aligned} \tag{22}$$

where $f_0(\|x\|) = 4n\sum_{i=1}^{n}(f_{i1}^4(\|x\|) + g_{i1}^4(\|x\|)) + (1-\gamma)^{-1}\Pi(\|x\|)$.

In the following, the estimates for terms of (22) are given. From (3), for any given $0 < \varepsilon_{10} < 1$, $0 < \varepsilon_0 < 1$, there exist $W_{10}^{*\mathrm{T}}S_{10}$, $W_0^{*\mathrm{T}}S_0$ such that

$$\begin{aligned} f_0\bigl(\|x\|\bigr) &= W_0^{*\mathrm{T}}S_0\bigl(\|x\|\bigr) + \delta_0\bigl(\|x\|\bigr), \quad \bigl|\delta_0\bigl(\|x\|\bigr)\bigr| \leqslant \varepsilon_0, \\ g_{11}^2\bigl(\|x\|\bigr) &= W_{10}^{*\mathrm{T}}S_{10}\bigl(\|x\|\bigr) + \delta_{10}\bigl(\|x\|\bigr), \quad \bigl|\delta_{10}\bigl(\|x\|\bigr)\bigr| \leqslant \varepsilon_{10}, \end{aligned} \tag{23}$$

where $x \in S_x$ and S_x is the compact set through which the state trajectories may travel. According to $\|W_0^{*\mathrm{T}}\|^2\|S_0\|^2 \leqslant \|W_0^{*\mathrm{T}}\|^2 N_0 \leqslant \theta$, $\|W_{10}^{*\mathrm{T}}\|^2\|S_{10}\|^2 \leqslant \|W_{10}^{*\mathrm{T}}\|^2 N_{10} \leqslant \theta$, (5), (9), (23) and Lemma 1, one obtains

$$\begin{aligned} hz_1^3 f_1 &\leqslant hz_1^3\big(f_{11}(\|x\|) + f_{12}\big(\big\|x\big(t-d(t)\big)\big\|\big)\big) \\ &\leqslant hz_1^3 f_{11}\big(\|x\|\big) + \frac{3}{4}z_1^4 + \frac{h_M^4}{4} f_{12}^4\big(\big\|x\big(t-d(t)\big)\big\|\big), \end{aligned} \tag{24}$$

$$f_0\big(\|x\|\big) \leqslant \varrho_0^2 + \frac{\sqrt{1+\theta^2}}{2\varrho_0^2} + \frac{\varepsilon_0^2}{2\varrho_0^2}, \tag{25}$$

$$\begin{aligned} \frac{3}{2}h^2 z_1^2 g_1 g_1^{\mathrm{T}} &\leqslant 3h_M^2 z_1^2\big(g_{11}^2\big(\|x\|\big) + g_{12}^2\big(\big\|x\big(t-d(t)\big)\big\|\big)\big) \\ &\leqslant 3h_M^2 z_1^2\big(W_{10}^{*\mathrm{T}} S_{10} + \delta_{10}\big) + 3h_M^2 z_1^2 g_{12}^2\big(\big\|x\big(t-d(t)\big)\big\|\big) \\ &\leqslant \frac{3}{2\varrho_{11}^2} z_1^4\theta + \frac{3}{2}h_M^4\varrho_{11}^2 + 3z_1^4 + \frac{3}{2}h_M^4\varepsilon_{10}^2 + \frac{3}{2}h_M^4 g_{12}^4\big(\big\|x(\cdot)\big\|\big), \end{aligned} \tag{26}$$

where $\varrho_0 > 0$ and $\varrho_{11} > 0$ are design constants. Furthermore, from Lemma 1, the definition of ϑ and (17), it can be verified that

$$\begin{aligned} &\pi\|\bar\psi\|^4 y^4 + hz_1^3\tilde{x}_2 + z_1^3\theta_1^{\mathrm{T}}\psi_1(y) \\ &\quad\leqslant \pi\|\bar\psi\|^4 y^4 + \frac{\varrho^4}{4}\|\tilde{x}_2\|^4 + \frac{3h_M^{4/3}}{4\varrho^{4/3}} z_1^4 + z_1^3\theta_1^{\mathrm{T}}\psi_1(y) \leqslant \frac{\varrho^4}{4}\|\tilde{x}\|^4 + z_1^3\vartheta^{\mathrm{T}}\Psi_1(y), \end{aligned} \tag{27}$$

where ϱ is a positive constant and $\Psi_1(y) = [(\|\bar\psi\|^4+1)y, 0, y^{\mathrm{T}}\bar\psi_1^{\mathrm{T}}(y)]^{\mathrm{T}}$. By Assumption 2, (19) and substituting (24)–(27) into (22), one yields

$$\begin{aligned} \mathcal{L}V_1 \leqslant{}& -c_1\|\tilde{x}\|^4 + hz_1^3(z_2+\alpha_2) + z_1^3\bar{f}_1(X_1) - \frac{3}{4}z_1^4 + \frac{3}{2\varrho_{11}^2}z_1^4\theta \\ &+ z_1^3\vartheta^{\mathrm{T}}\Psi_1(y) + \frac{3}{2}h_M^4\varrho_{11}^2 + \Delta_0 + \frac{3}{2}h_M^4\varepsilon_{10}^2 + \Delta_{1d}\big(\big\|x\big(t-d(t)\big)\big\|\big) \\ &- \Pi\big(\big\|x\big(t-d(t)\big)\big\|\big) - \frac{1}{\Gamma_1}\tilde\theta\dot{\hat\theta} - \tilde\vartheta^{\mathrm{T}}\Gamma_2^{-1}\dot{\hat\vartheta}, \end{aligned} \tag{28}$$

where $c_1 = c_0 - \varrho^4/4$, $\bar{f}_1 = hf_{11}(\|x\|) + (9/2)z_1$, $\Delta_0 = \varrho_0^2 + \sqrt{1+\theta^2}/(2\varrho_0^2) + \varepsilon_0^2/(2\varrho_0^2)$ and $\Delta_{1d}(\|x(t-d(t))\|) = \Delta_{0d}(\|x(t-d(t))\|) + (h_M^4/4)f_{12}^4(\|x(t-d(t))\|) + (3/2)h_M^4 g_{12}^4(\|x(t-d(t))\|)$.

Define a nonlinear function $\beta_1(X_1) = -k_1 z_1 - \bar{f}_1$, from (3), for any given $0 < \varepsilon_1 < 1$, there exists $W_1^{*\mathrm{T}}S_1(X_1)$ such that

$$\beta_1(X_1) = W_1^{*\mathrm{T}}S_1(X_1) + \delta_1(X_1), \quad \big|\delta_1(X_1)\big| \leqslant \varepsilon_1, \tag{29}$$

where k_1 is a positive number, and $X_1 \in S_{X_1} = \{X_1 \mid X_1 \in S_x\}$. With the use of (21), $\|W_1^{*\mathrm{T}}\|^2\|S_1\|^2 \leqslant \|W_1^{*\mathrm{T}}\|^2 N_1 \leqslant \theta$ holds, together with Lemma 1 and (29) leads to

$$-z_1^3\beta_1 \leqslant \frac{1}{2\epsilon_1^2}z_1^6\theta + \frac{1}{2}\epsilon_1^2 + \frac{3}{4}z_1^4 + \frac{1}{4}\varepsilon_1^4, \tag{30}$$

where $\epsilon_1 > 0$ is a constant. Substituting $\bar{f}_1 = -\beta_1 - k_1 z_1$ and (30) into (28), and choosing the first virtual control law

$$\alpha_2 = -\operatorname{sgn} h\left(\frac{3}{2\varrho_{11}^2} z_1 \hat{\theta} + \frac{1}{2\epsilon_1^2} z_1^3 \hat{\theta} + \hat{\vartheta}^{\mathrm{T}} \Psi_1(y)\right), \tag{31}$$

one obtains

$$\begin{aligned} \mathcal{L}V_1 \leqslant{}& -c_1\|\tilde{x}\|^4 + h z_1^3 z_2 - k_1 z_1^4 - \frac{\tilde{\theta}}{\Gamma_1}(\dot{\hat{\theta}} - \tau_1) - \tilde{\vartheta}^{\mathrm{T}} \Gamma_2^{-1}(\dot{\hat{\vartheta}} - \sigma_1) \\ & + \Delta_{1d}\big(\big\|x\big(t-d(t)\big)\big\|\big) - \Pi\big(\big\|x\big(t-d(t)\big)\big\|\big) + \Delta_1, \end{aligned} \tag{32}$$

where $\tau_1 = 3\Gamma_1(2\varrho_{11}^2)^{-1} z_1^4 + \Gamma_1(2\epsilon_1^2)^{-1} z_1^6$, $\sigma_1 = z_1^3 \Gamma_2 \Psi_1$ and $\Delta_1 = \Delta_0 + (3/2)h_M^4 \varrho_{11}^2 + (3/2)h_M^4 \varepsilon_{10}^2 + (1/2)\epsilon_1^2 + (1/4)\varepsilon_1^4$.

Step i ($2 \leqslant i \leqslant n-1$). At this step, we state the result in Proposition 1.

Proposition 1. *For the Lyapunov function candidate* $V_i = V_0 + (1/4)\sum_{j=1}^{i} z_j^4 + (1/2) \times \Gamma_1 \tilde{\theta}^2 + \tilde{\vartheta}^{\mathrm{T}} \Gamma_2^{-1} \tilde{\vartheta} + V_Q$, *there exists the ith virtual control law* α_{i+1} *in the following form*:

$$\alpha_{i+1} = -\left(\frac{3}{2\varrho_{i1}^2} z_i \hat{\theta} + \frac{1}{2\epsilon_i^2} z_i^3 \hat{\theta} + \hat{\vartheta}^{\mathrm{T}} \Psi_i(\cdot)\right) \tag{33}$$

such that

$$\begin{aligned} \mathcal{L}V_i \leqslant{}& -c_i\|\tilde{x}\|^4 + h z_1^3 z_2 + \sum_{j=2}^{i} z_j^3 z_{j+1} - \sum_{j=1}^{i} k_j z_j^4 + \Delta_i - \tilde{\vartheta}^{\mathrm{T}} \Gamma_2^{-1}(\dot{\hat{\vartheta}} - \sigma_i) \\ & - \frac{\tilde{\theta}}{\Gamma_1}(\dot{\hat{\theta}} - \tau_i) + \Delta_{id}\big(\big\|x\big(t-d(t)\big)\big\|\big) - \Pi\big(\big\|x\big(t-d(t)\big)\big\|\big), \end{aligned} \tag{34}$$

where $\Psi_i = [(1 + (\partial\alpha_i/\partial y)^2)^{2/3} z_i,\ -(\partial\alpha_i/\partial y)\hat{x}_2,\ -(\partial\alpha_i/\partial y) y^{\mathrm{T}} \bar{\psi}_1^{\mathrm{T}}]^{\mathrm{T}}$, $\varrho_{i1} > 0$, $\epsilon_i > 0$ *are design constants,* $c_i = c_{i-1} - \varrho^4/4$, $\tau_i = \tau_{i-1} + 3\Gamma_1(2\varrho_{i1}^2)^{-1} z_i^4 + \Gamma_1(2\epsilon_i^2)^{-1} z_i^6$, $\sigma_i = \sigma_{i-1} + z_i^3 \Gamma_2 \Psi_i$, $\Delta_{id} = \Delta_{i-1,d} + (h_M^8/4)(\partial^2\alpha_i/\partial y^2)^4 g_{12}^8(\|x(t-d(t))\|) + (3/2)h_M^4(\partial\alpha_i/\partial y)^4 g_{12}^4(\|x(t-d(t))\|)$ *and* $\Delta_i = \Delta_{i-1} + (3/2)h_M^4 \varrho_{i1}^2 + (3/2)h_M^4 \varepsilon_{i0}^2 + (1/2)\epsilon_i^2 + (1/4)\varepsilon_i^4$.

Proof. See the Appendix. □

According to the recursive steps, at step n, choosing the Lyapunov function

$$V_n = \frac{1}{2}\big(\tilde{x}^{\mathrm{T}} P \tilde{x}\big)^2 + \frac{1}{4}\sum_{i=1}^{n} z_i^4 + \frac{1}{2\Gamma_1}\tilde{\theta}^2 + \frac{1}{2}\tilde{\vartheta}^{\mathrm{T}} \Gamma_2^{-1} \tilde{\vartheta} + V_Q, \tag{35}$$

and constructing the adaptive control law as

$$u = -\underline{h}^{-n}\left(\frac{3}{2\varrho_{n1}^2} z_n \hat{\theta} + \frac{1}{2\epsilon_n^2} z_n^3 \hat{\theta} + \hat{\vartheta}^{\mathrm{T}} \Psi_n(\cdot)\right), \tag{36}$$

$$\dot{\hat{\theta}} = \sum_{i=1}^{n} \frac{3\Gamma_1}{2\varrho_{i1}^2} z_i^4 + \sum_{i=1}^{n} \frac{\Gamma_1}{2\epsilon_i^2} z_i^6 - \hat{\theta}, \qquad \dot{\hat{\vartheta}} = \sum_{i=1}^{n} z_i^3 \Gamma_2 \Psi_i - \hat{\vartheta}, \tag{37}$$

yield

$$\mathcal{L}V_n \leqslant -c_n\|\tilde{x}\|^4 - \sum_{i=1}^{n} k_i z_i^4 + hz_1^3 z_2 + \sum_{i=2}^{n} z_i^3 z_{i+1} + \frac{1}{\Gamma_1}\tilde{\theta}\hat{\theta} + \tilde{\vartheta}^{\mathrm{T}}\Gamma_2^{-1}\hat{\vartheta}$$
$$+ \Delta_{nd}\big(\big\|x\big(t-d(t)\big)\big\|\big) - \Pi\big(\big\|x\big(t-d(t)\big)\big\|\big) + \Delta_n, \tag{38}$$

where $\Psi_n(\cdot) = [(1+(\partial\alpha_n/\partial y)^2)^{2/3} z_n, -(\partial\alpha_n/\partial y)\hat{x}_n, -(\partial\alpha_n/\partial y)y^{\mathrm{T}}\bar{\psi}_1^{\mathrm{T}}(y)]^{\mathrm{T}}$, $\varrho_{n1} > 0$, $\epsilon_n > 0$ are design constants, $c_n = c_0 - n\varrho^4/4$, $\Delta_{nd} = \Delta_{1d} + (h_M^8/4)\sum_{i=2}^{n}(\partial^2\alpha_i/\partial y^2)^4 \times g_{12}^8(\|x(t-d(t))\|) + (3/2)h_M^4\sum_{i=2}^{n}(\partial\alpha_i/\partial y)^4 g_{12}^4(\|x(t-d(t))\|)$ and $\Delta_n = \Delta_0 + \sum_{i=1}^{n}(3\varrho_{i1}^2/2 + 3\varepsilon_{i0}^2/2)h_M^4 + (1/2)\sum_{i=1}^{n}\epsilon_i^2 + (1/4)\sum_{i=1}^{n}\varepsilon_i^4$.

5 Stability analysis

We now state the main theorem in this paper.

Theorem 1. *For stochastic nonlinear system* (4) *satisfying Assumptions* 1–3*, the adaptive control laws* (31)*,* (33) *and* (36)*,* (37) *guarantee that all the signals in the closed-loop system* (4)*,* (7)*,* (10)*,* (19)*,* (31)*,* (33) *and* (36)*,* (37) *are* 4*-moment* (*or mean square*) *semi-globally uniformly ultimately bounded* (*SGUUB*)*. The bounds of* z_i*,* $\tilde{\theta}$ *and* $\tilde{\vartheta}$ *remain in*

$$\Omega_1 := \left\{ z_i \;\Big|\; \sum_{i=1}^{n} \mathbf{E}|z_i|^4 \leqslant 4(1+\delta)\frac{a_2}{a_1} \right\}, \tag{39}$$

$$\Omega_2 := \left\{ \tilde{\theta} \;\Big|\; \mathbf{E}|\tilde{\theta}|^2 \leqslant 2\Gamma_1(1+\delta)\frac{a_2}{a_1} \right\}, \tag{40}$$

$$\Omega_3 := \left\{ \tilde{\vartheta} \;\Big|\; \mathbf{E}\|\tilde{\vartheta}\|^2 \leqslant \frac{1}{\lambda_{\min}(\Gamma_2^{-1})}(1+\delta)\frac{a_2}{a_1} \right\}, \tag{41}$$

where δ*,* a_1 *and* a_2 *are positive design constants.*

Proof. Firstly, we estimate the terms on the right-hand side of (38). In terms of Lemma 1 and (5), one obtains

$$hz_1^3 z_2 \leqslant \frac{3}{4}z_1^4 + \frac{h_M^4}{4}z_2^4, \qquad z_i^3 z_{i+1} \leqslant \frac{3}{4}z_i^4 + \frac{h_M^4}{4}z_{i+1}^4, \qquad \tilde{\theta}\hat{\theta} \leqslant -\frac{1}{2}\tilde{\theta}^2 + \frac{1}{2}\theta^2,$$
$$\tilde{\vartheta}^{\mathrm{T}}\Gamma_2^{-1}(\vartheta - \tilde{\vartheta}) \leqslant -\frac{1}{2}\tilde{\vartheta}^{\mathrm{T}}\Gamma_2^{-1}\tilde{\vartheta} + \frac{1}{2}\lambda_{\max}\big(\Gamma_2^{-1}\big)\|\vartheta\|^2. \tag{42}$$

Substituting (42) into (38) and choosing $\Pi(\cdot) = \Delta_{nd}(\cdot)$, we have

$$\mathcal{L}V_n \leqslant -c_n\|\tilde{x}\|^4 - \sum_{i=1}^{n} k_i z_i^4 + \frac{3}{4}\sum_{i=1}^{n} z_i^4 + \frac{h_M^4}{4}\sum_{i=2}^{n} z_i^4 - \frac{1}{2\Gamma_1}\tilde{\theta}^2$$
$$- \frac{1}{2}\tilde{\vartheta}^{\mathrm{T}}\Gamma_2^{-1}\tilde{\vartheta} + \Delta_n + \frac{1}{2\Gamma}\theta^2 + \frac{1}{2}\lambda_{\max}\big(\Gamma_2^{-1}\big)\|\vartheta\|^2. \tag{43}$$

Furthermore, from (35), it holds

$$V_n - V_Q \leqslant \frac{1}{2}\|\tilde{x}\|^4\lambda_{\max}^2(P) + \frac{1}{4}\sum_{i=1}^{n} z_i^4 + \frac{1}{2\Gamma}\tilde{\theta}^2 + \frac{1}{2}\tilde{\vartheta}^{\mathrm{T}}\Gamma_2^{-1}\tilde{\vartheta}. \tag{44}$$

Choosing $\bar{a}_1 = \min\{2c_n/\lambda^2_{\max}(P), 4(k_i - (3 + h^4_M)/4), 1\}$, $a_2 = \varDelta_n + \theta^2/(2\varGamma) + (1/2)\lambda_{\max}(\varGamma_2^{-1})\|\vartheta\|^2$, and using (43)–(44), one has $\mathcal{L}V_n \leqslant -\bar{a}_1(V_n - V_Q) + a_2$. Then, there must exist a positive constant a_1 such that

$$\mathcal{L}V_n \leqslant -a_1 V_n + a_2. \tag{45}$$

Multiplying (45) by $\mathrm{e}^{a_1 t}$ and taking expectations on both sides, one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{a_1 t}\mathbf{E}(V_n)\big) \leqslant a_2 \mathrm{e}^{a_1 t}. \tag{46}$$

Integrating (46) on $[0, t]$ yields $\mathbf{E}(V_n(t)) \leqslant \mathrm{e}^{-a_1 t}V_n(0) + a_2/a_1$ for all $t > 0$. Hence, there exists $T = \max\{0, (1/a_1)\ln(a_1 V_n(0)/(\delta a_2))\}$ for some small $\delta > 0$ such that $\mathbf{E}(V_n(t)) \leqslant (1+\delta)a_2/a_1$ for all $t > T$. Using (35), we obtain

$$\begin{aligned}
\mathbf{E}\big(\|\tilde{x}\|^4\big) \leqslant \frac{2}{\lambda_{\min}(P)}\mathbf{E}\big(V_n(t)\big) &\leqslant \frac{2}{\lambda_{\min}(P)}(1+\delta)\frac{a_2}{a_1},\\
\sum_{i=1}^{n}\mathbf{E}|z_i|^4 \leqslant 4\mathbf{E}\big(V_n(t)\big) &\leqslant 4(1+\delta)\frac{a_2}{a_1},\\
\mathbf{E}|\tilde{\theta}|^2 \leqslant 2\varGamma_1\mathbf{E}\big(V_n(t)\big) &\leqslant 2\varGamma_1(1+\delta)\frac{a_2}{a_1},\\
\mathbf{E}\|\tilde{\vartheta}\|^2 \leqslant \frac{1}{\lambda_{\min}(\varGamma_2^{-1})}\mathbf{E}\big(V_n(t)\big) &\leqslant \frac{1}{\lambda_{\min}(\varGamma_2^{-1})}(1+\delta)\frac{a_2}{a_1}.
\end{aligned}$$

Thus, from Definition 1, $\|\tilde{x}\|$ and z_i are 4-moment SGUUB, $\tilde{\theta}$ and $\tilde{\vartheta}$ are mean square SGUUB and (39)–(41) hold. Furthermore, from $\tilde{x} = x - \hat{x}$, (7) and (19), η is 4-moment SGUUB, i.e. all the signals in the closed-loop system (4), (7), (10), (19), (31), (33) and (36), (37) are 4-moment (or mean square) SGUUB. □

Remark 2. We give a further explanation on how to design parameters. By choosing larger NN nodes N, the approximation error ε in (3) can be reduced, which may improve the approximation accuracy. Furthermore, smaller $\varepsilon_0, \dots, \varepsilon_n$, ϱ_{i1}, ϵ_{i0}, ϵ_i together with larger c_n, k_i $(i = 1, \dots, n)$ will reduce a_2/a_1, which leads to smaller converging region. Hence, one can reduce the bounded compact sets $\varOmega_1$, $\varOmega_2$ and $\varOmega_3$ by appropriately regulating the parameters.

6 A simulation example

Consider the following stochastic nonlinear time-delay system:

$$\begin{aligned}
\mathrm{d}\eta_1 &= h_1\eta_2\,\mathrm{d}t + \big(\eta_1^2 - 10\eta_2\big)\,\mathrm{d}t + \theta_1^{\mathrm{T}}\psi_1(y)\,\mathrm{d}t\\
&\quad + \eta_1^2\big(t - d(t)\big)\eta_2\big(t - d(t)\big)\,\mathrm{d}\omega,\\
\mathrm{d}\eta_2 &= h_2 u\,\mathrm{d}t + \eta_1^3\cos\big(\eta_2\big(t - d(t)\big)\big)\,\mathrm{d}t + \theta_2^{\mathrm{T}}\psi_2(y)\,\mathrm{d}t\\
&\quad + \eta_1\sin\big(\eta_2\big(t - d(t)\big)\big)\,\mathrm{d}\omega,\\
y &= \eta_1,
\end{aligned} \tag{47}$$

where h_1 and h_2 are unknown with known signs and satisfying $0.8 = \underline{h} \leqslant |h_i| \leqslant \bar{h} = 2$ $(i = 1, 2)$, $d(t) = 0.2(1+\sin(t))$ is time-varying delay, $\psi_1(y) = y^2$ and $\psi_2(y) = y$. From Lemma 2, one can get that $\psi(y) = [y^2, y]^{\mathrm{T}}$ and $\bar{\psi}(y) = [y, 1]^{\mathrm{T}}$. Thus, Assumptions 1–3 hold with $\gamma = 0.2$.

By exactly following the design procedure in Section 4, we construct the controller as

$$
\begin{aligned}
&x_1 = \frac{\underline{h}^2}{h_1 h_2}\eta_1, \qquad x_2 = \frac{\underline{h}^2}{h_2}\eta_2,\\
&\dot{\hat{x}}_1 = \hat{x}_2 + b_1(x_1 - \hat{x}_1), \qquad \dot{\hat{x}}_2 = \underline{h}^n u + b_2(x_1 - \hat{x}_1),\\
&z_1 = y, \qquad z_2 = \hat{x}_2 - \alpha_2,\\
&\alpha_2 = -\operatorname{sgn} h\left(\frac{3}{2\varrho_{11}^2}z_1\hat{\theta} + \frac{1}{2\epsilon_1^2}z_1^3\hat{\theta} + \hat{\vartheta}^{\mathrm{T}}\Psi_1(y)\right),\\
&u = -\underline{h}^{-2}\left(\frac{3}{2\varrho_{21}^2}z_2\hat{\theta} + \frac{1}{2\epsilon_2^2}z_2^3\hat{\theta} + \hat{\vartheta}^{\mathrm{T}}\Psi_2(\cdot)\right),\\
&\dot{\hat{\theta}} = \sum_{i=1}^{2}\frac{3\Gamma_1}{2\varrho_{i1}^2}z_i^4 + \sum_{i=1}^{2}\frac{\Gamma_1}{2\epsilon_i^2}z_i^6 - \hat{\theta}, \qquad \dot{\hat{\vartheta}} = \sum_{i=1}^{2} z_i^3\Gamma_2\Psi_i - \hat{\vartheta},
\end{aligned}
\tag{48}
$$

where $\varrho_{11}, \epsilon_1, \varrho_{21}, \epsilon_2, \Gamma_1 > 0$ are constants, Γ_2 is a gain matrix, $\Psi_1(y) = [(1+\|\bar{\psi}\|^4)y, 0, \psi_1(y)]^{\mathrm{T}}$ and $\Psi_2(\cdot) = [(1 + (\partial\alpha_2/\partial y)^2)^{2/3}z_2, -(\partial\alpha_2/\partial y)\hat{x}_2, -(\partial\alpha_2/\partial y)\psi_1^{\mathrm{T}}(y)]^{\mathrm{T}}$.

In the simulation, the design parameters are chosen as $\theta_1 = 1$, $\theta_2 = 1$, $\varrho_{11} = 0.5$, $\epsilon_1 = 0.5$, $\varrho_{21} = 0.5$, $\epsilon_2 = 0.5$, $b_1 = 1$, $b_2 = 1$, $\Gamma_1 = 100$, $\Gamma_2 = I$. To get the simulation results, we choose $h_1 = 1.2$ and $h_2 = 1$, then $h = h_1 h_2/\underline{h}^2 = -1.875$. The initial states are chosen as $\eta_1(0) = 0$, $\eta_2(0) = 0.5$, $\hat{x}_1(0) = 0$, $\hat{x}_2(0) = 0$, $\hat{\theta}(0) = 0$, $\vartheta(0) = [5, 7.5, -1.5]^{\mathrm{T}}$. Figure 1 shows the effectiveness of the control scheme.

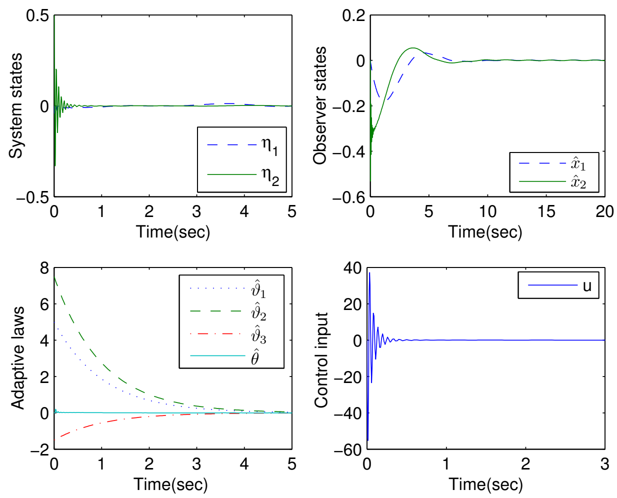

Figure 1. The responses of the closed-loop system (47)–(48).

7 Conclusions

This paper solves the adaptive NN output-feedback control problem for a class of stochastic time-delay nonlinear systems with unknown control coefficients and perturbations. By using RBF NN, tuning function approach and backstepping technique, the proposed control scheme requires fewer parameters and guarantees all the signals in the closed-loop system to be 4-moment(or mean-square) SGUUB. An important issue under investigation is how to extend the design scheme to high-order stochastic nonlinear time-delay systems.

Appendix: Proof of Proposition 1

We will prove it by induction. Assume that for the Lyapunov function candidate $V_{i-1} = V_0 + (1/4)\sum_{j=1}^{i-1} z_j^4 + (2\mathit{\Gamma}_1)^{-1}\tilde{\theta}^2 + \tilde{\vartheta}^{\mathrm{T}}\mathit{\Gamma}_2^{-1}\tilde{\vartheta} + V_Q$, there exist a series of virtual control laws

$$\alpha_{j+1} = -\left(\frac{3}{2\varrho_{j1}^2} z_i\hat{\theta} + \frac{1}{2\epsilon_j^2} z_i^3\hat{\theta} + \hat{\vartheta}^{\mathrm{T}}\mathit{\Psi}_i(\cdot)\right), \quad j = 2, \dots, i-1, \tag{49}$$

such that

$$\begin{aligned}\mathcal{L}V_{i-1} \leqslant{}& -c_{i-1}\|\tilde{x}\|^4 + hz_1^3 z_2 + \sum_{j=2}^{i-1} z_j^3 z_{j+1} - \sum_{j=1}^{i-1} k_j z_j^4 \\ &+ \mathit{\Delta}_{i-1} - \tilde{\vartheta}^{\mathrm{T}}\mathit{\Gamma}_2^{-1}(\dot{\hat{\vartheta}} - \sigma_{i-1}) - \frac{\tilde{\theta}}{\mathit{\Gamma}_1}(\dot{\hat{\theta}} - \tau_{i-1}) \\ &+ \mathit{\Delta}_{i-1,d}\big(\big\|x\big(t-d(t)\big)\big\|\big) - \mathit{\Pi}\big(\big\|x\big(t-d(t)\big)\big\|\big),\end{aligned} \tag{50}$$

where $\mathit{\Psi}_j(\cdot) = [(1 + (\partial\alpha_j/\partial y)^2)^{2/3} z_j, -(\partial\alpha_j/\partial y)\hat{x}_2, -(\partial\alpha_j/\partial y)y^{\mathrm{T}}\bar{\psi}_1^{\mathrm{T}}(y)]^{\mathrm{T}}$, c_{i-1}, ϱ_{j1}, $\epsilon_j > 0$ are constants, $\tau_{i-1} = \sum_{j=1}^{i-1}(3\mathit{\Gamma}_1(2\varrho_{j1}^2)^{-1}z_j^4 + \mathit{\Gamma}_1(2\epsilon_j^2)^{-1}z_j^6)$, $\sigma_{i-1} = \sum_{j=1}^{i-1} z_j^3\mathit{\Gamma}_2\mathit{\Psi}_j$, $\mathit{\Delta}_{i-1,d} = \mathit{\Delta}_{1d} + (h_M^8/4)\sum_{j=2}^{i-1}((\partial^2\alpha_j/\partial y^2)^4 g_{12}^8(\|x(t-d(t))\|) + (3/2)\times h_M^4\sum_{j=2}^{i-1}(\partial\alpha_j/\partial y)^4 g_{12}^4(\|x(t-d(t))\|)$ and $\mathit{\Delta}_{i-1} = \mathit{\Delta}_0 + \sum_{j=1}^{j-1}(3\varrho_{j1}^2/2 + 3\varepsilon_{j0}^2/2)h_M^4 + (1/2)\sum_{j=1}^{i-1}\epsilon_j^2 + (1/4)\sum_{j=1}^{i-1}\varepsilon_j^4$. In the sequel, we prove (50) still holds at step i.

From (18), (19) and (49), one has

$$\begin{aligned}\mathrm{d}z_i ={}& \left(\hat{x}_{i+1} + F_i - \frac{\partial\alpha_i}{\partial y}\big(h\hat{x}_2 + h\tilde{x}_2 + \theta_1^{\mathrm{T}}\psi_1\big) - \frac{1}{2}\frac{\partial^2\alpha_i}{\partial y^2}h^2 g_1 g_1^{\mathrm{T}}\right)\mathrm{d}t \\ &+ h\frac{\partial\alpha_i}{\partial y}g_1\,\mathrm{d}\omega,\end{aligned} \tag{51}$$

where $F = b_i\tilde{x}_1 - \sum_{j=2}^{i-1}(\partial\alpha_i/\partial\hat{x}_j)(\hat{x}_{j+1} + b_j\tilde{x}_1) - (\partial\alpha_i/\partial\hat{\theta})\dot{\hat{\theta}} - (\partial\alpha_i/\partial\hat{\vartheta})\dot{\hat{\vartheta}}$. Applying (2), (50), (51) and $V_i = V_{i-1} + (1/4)z_i^4$ leads to

$$\begin{aligned}\mathcal{L}V_i \leqslant{}& \mathcal{L}V_{i-1} + z_i^3\left(\hat{x}_{i+1} + F_i - \frac{\partial\alpha_i}{\partial y}\big(h\hat{x}_2 + h\tilde{x}_2 + \theta_1^{\mathrm{T}}\psi_1(y)\big)\right. \\ &\left. - \frac{1}{2}\frac{\partial^2\alpha_i}{\partial y^2}h^2 g_1 g_1^{\mathrm{T}}\right) + \frac{3}{2}h^2 z_i^2\left(\frac{\partial\alpha_i}{\partial y}\right)^2 g_1 g_1^{\mathrm{T}}.\end{aligned} \tag{52}$$

Now, we start to estimate the right-hand side terms of (52). Considering Lemma 1, (3), (5), (9) and (21), one gets

$$
\begin{aligned}
-&\frac{1}{2}z_i^3\frac{\partial^2\alpha_i}{\partial y^2}h^2g_1g_1^{\mathrm{T}}\\
&\leqslant z_i^3\frac{\partial^2\alpha_i}{\partial y^2}h_M^2\big(g_{11}^2\big(\|x\|\big)+g_{12}^2\big(\big\|x\big(t-d(t)\big)\big\|\big)\big)\\
&\leqslant z_i^3\frac{\partial^2\alpha_i}{\partial y^2}h_M^2g_{11}^2\big(\|x\|\big)+\frac{3}{4}z_i^4+\frac{h_M^8}{4}\bigg(\frac{\partial^2\alpha_i}{\partial y^2}\bigg)^4g_{12}^8\big(\big\|x\big(t-d(t)\big)\big\|\big),
\end{aligned}\tag{53}
$$

$$
\begin{aligned}
&\frac{3}{2}h^2z_i^2\bigg(\frac{\partial\alpha_i}{\partial y}\bigg)^2g_1g_1^{\mathrm{T}}\\
&\leqslant 3h^2z_i^2\bigg(\frac{\partial\alpha_i}{\partial y}\bigg)^2\big(g_{11}^2\big(\|x\|\big)+g_{12}^2\big(\big\|x\big(t-d(t)\big)\big\|\big)\big)\\
&\leqslant 3h_M^2z_i^2\big(W_{i0}^{*\mathrm{T}}S_{i0}+\delta_{i0}\big)+3h_M^2z_i^2\bigg(\frac{\partial\alpha_i}{\partial y}\bigg)^2g_{12}^2\big(\big\|x\big(t-d(t)\big)\big\|\big)\\
&\leqslant \frac{3}{2\varrho_{i1}^2}z_i^4\theta+\frac{3h_M^4}{2}\varrho_{i1}^2+3z_i^4+\frac{3}{2}h_M^4\varepsilon_{i0}^2\\
&\quad+\frac{3}{2}h_M^4\bigg(\frac{\partial\alpha_i}{\partial y}\bigg)^4g_{12}^4\big(\big\|x\big(t-d(t)\big)\big\|\big),
\end{aligned}\tag{54}
$$

where $\varrho_{i1}>0$ is a design constant, $(\partial\alpha_i/\partial y)^2g_{11}^2(\|x\|)=W_{i0}^{*\mathrm{T}}S_{i0}+\delta_{i0}$, $\delta_{i0}\leqslant\varepsilon_{i0}$ and $\|W_{i0}^{*\mathrm{T}}\|^2\|S_{i0}\|^2\leqslant\|W_{i0}^{*\mathrm{T}}\|^2N_{i0}\leqslant\theta$. Furthermore, from Lemma 1, the definition of ϑ and (17), it holds

$$
\begin{aligned}
-&\frac{\partial\alpha_i}{\partial y}h\hat{x}_2z_i^3-\frac{\partial\alpha_i}{\partial y}h\tilde{x}_2z_i^3-\frac{\partial\alpha_i}{\partial y}\theta_1^{\mathrm{T}}\psi_1(y)z_i^3\\
&\leqslant -\frac{\partial\alpha_i}{\partial y}h\hat{x}_2z_i^3+\frac{\varrho^4}{4}\|\tilde{x}\|^4+\frac{3h_M^{4/3}}{4\varrho^4}\bigg(1+\bigg(\frac{\partial\alpha_i}{\partial y}\bigg)^2\bigg)^{2/3}z_i^4-\frac{\partial\alpha_i}{\partial y}\theta_1^{\mathrm{T}}\psi_1(y)z_i^3\\
&\leqslant\frac{\varrho^4}{4}\|\tilde{x}\|^4+z_i^3\vartheta^{\mathrm{T}}\Psi_i(\cdot).
\end{aligned}\tag{55}
$$

Substituting (53)–(55) into (52) and using (19) yields

$$
\begin{aligned}
\mathcal{L}V_i\leqslant{}&-c_i\|\tilde{x}\|^4+hz_1^3z_2+z_i^3\bar{f}_i-\frac{3}{4}z_i^3+z_i^3\alpha_{i+1}+\sum_{j=2}^{i}z_j^3z_{j+1}-\sum_{j=1}^{i-1}k_jz_j^4\\
&-\frac{\tilde{\theta}}{\Gamma_1}(\dot{\hat{\theta}}-\tau_{i-1})-\tilde{\vartheta}^{\mathrm{T}}\Gamma_2^{-1}(\dot{\hat{\vartheta}}-\sigma_{i-1})+\Delta_{i-1}+\Delta_{i,d}\big(\big\|x\big(t-d(t)\big)\big\|\big)\\
&-\Pi\big(\big\|x\big(t-d(t)\big)\big\|\big)+z_i^3\vartheta^{\mathrm{T}}\Psi_i(\cdot)+\frac{3h_M^4}{2}\varrho_{i1}^2+\frac{3}{2}h_M^4\varepsilon_{i0}^2,
\end{aligned}\tag{56}
$$

where $c_i=c_{i-1}-\varrho^4/4$, $\bar{f}_i=F_i+(\partial^2\alpha_i/\partial y^2)h_M^2g_{11}^2(\|x\|)+(9/2)z_i$ and $\Delta_{id}=\Delta_{i-1,d}+(h_M^8/4)(\partial^2\alpha_i/\partial y^2)^4g_{12}^8(\|x(t-d(t))\|)+(3/2)h_M^4(\partial\alpha_i/\partial y)^4g_{12}^4(\|x(t-d(t))\|)$.

Define a nonlinear function $\beta_i(X_i) = -k_i z_i - \bar{f}_i$, from (3), for any given $0 < \varepsilon_i < 1$, there exists $W_i^{*\mathrm{T}} S_i(X_i)$ such that $\beta_i(X_i) = W_i^{*\mathrm{T}} S_i(X_i) + \delta_1(X_i)$, $|\delta_1(X_i)| \leqslant \varepsilon_i$, where k_i is a positive number, and $X_i \in S_{X_i} = \{X_i \mid X_i \in S_x\}$. With the use of (21), $\|W_i^{*\mathrm{T}}\|^2 \|S_i\|^2 \leqslant \|W_i^{*\mathrm{T}}\|^2 N_i \leqslant \theta$ holds, together with Lemma 1, one gets

$$-z_i^3 \beta_i \leqslant \frac{1}{2\epsilon_i^2} z_i^6 \theta + \frac{1}{2}\epsilon_i^2 + \frac{3}{4} z_i^4 + \frac{1}{4}\varepsilon_i^4, \tag{57}$$

where $\epsilon_i > 0$ is a constant. Substituting $\bar{f}_i = -\beta_i - k_i z_i$ and (57) into (56), and choosing the ith virtual control law as (33) yield (34). The proof is completed.

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