

***F*-contractions of Hardy–Rogers type and application to multistage decision processes**

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Received: January 1, 2015 / **Revised:** September 17, 2015 / **Published online:** April 4, 2016

Abstract. We prove fixed point theorems for *F*-contractions of Hardy–Rogers type involving self-mappings defined on metric spaces and ordered metric spaces. An example and an application to multistage decision processes are given to show the usability of the obtained theorems.

Keywords: contraction of Hardy–Rogers type, *F*-contraction, fixed point, metric space.

1 Introduction

The study of sufficient and necessary conditions, to establishing the existence and uniqueness of fixed points for self-mappings defined in abstract spaces, is an inexhaustible research field for many researchers in mathematics and applied sciences. The key aspect of this success is the generic fixed point problem $x = Tx$, where $T : X \rightarrow X$ is a self-mapping of a space X . It is well known that a wide variety of mathematical and practical problems can be solved by reducing them to an equivalent fixed point problem. In fact, by introducing suitable operators, it is possible to solve an equilibrium problem by searching the fixed points of such operators. Moreover, the solutions of differential equations can be obtained in terms of fixed points of integro-differential operators; also the above solutions sets can be characterized by a stability analysis of fixed points sets. These facts are sufficient motivations to increase the interest of mathematicians to establishing extensions and generalizations of the celebrated Banach fixed point theorem [4], which is universally recognized as the fundamental result of metric fixed point theory, see also [20, 21]. In this paper, we continue this study by stating existence and uniqueness fixed point theorems for a self-mapping, in the setting of complete metric spaces and complete ordered metric spaces. More precisely, we work with Hardy–Rogers-type conditions, which represent one of the most interesting generalizations of Banach fixed point theorem; but we combine the original idea of Hardy and Rogers [8] with the recent concept of *F*-contraction

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provided by Wardowski [27]. In so doing, we present a weaker contractive condition with an unifying power over many contractive conditions existing in the literature. An example and an application to multistage decision processes are given to show the usability of the obtained theorems.

2 Preliminaries

The aim of this section is to present some notions and results used in the paper, according to Wardowski [27]; see also [2, 3, 10, 11, 17, 24, 26]. We denote by \mathbb{R} the set of all real numbers, by \mathbb{R}^+ the set of all positive real numbers and by \mathbb{N} the set of all positive integers.

Definition 1. Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a function satisfying:

- (F1) F is non-decreasing;
- (F2) for each sequence $\{\alpha_n\} \subset \mathbb{R}^+$ of positive numbers, $\lim_{n \rightarrow +\infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow +\infty} F(\alpha_n) = -\infty$;
- (F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

We denote with \mathcal{F} the family of all functions F that satisfy conditions (F1)–(F3) and with \mathbb{F} the family of all functions F that satisfy conditions (F1), (F2). Also we consider the family \mathbb{S} of functions $\tau : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with the following property:

- (H) $\liminf_{t \rightarrow s^+} \tau(t) > 0$ for all $s \geq 0$.

Definition 2. (See [27].) Let (X, d) be a metric space. A self-mapping T on X is called an F -contraction if there exist $F \in \mathcal{F}$ and $\sigma \in \mathbb{R}^+$ such that

$$\sigma + F(d(Tx, Ty)) \leq F(d(x, y)) \quad (1)$$

for all $x, y \in X$ with $Tx \neq Ty$.

By using contractive condition (1) with $F \in \mathcal{F}$ and $\sigma \in \mathbb{R}^+$, Wardowski established a unique fixed point result, which generalizes the Banach fixed point theorem [4]. Here we obtain some extensions of Wardowski's result by using more general contractive conditions with $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$.

Definition 3. Let (X, d) be a metric space. A self-mapping T on X is called an F -contraction of Hardy–Rogers type if there exist $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that

$$\begin{aligned} & \tau(d(x, y)) + F(d(Tx, Ty)) \\ & \leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx)) \end{aligned} \quad (2)$$

for all $x, y \in X$ with $Tx \neq Ty$, where $\alpha, \beta, \gamma, \delta, L \in [0, +\infty[$, $\alpha + \beta + \gamma + 2\delta = 1$, $\gamma \neq 1$ and $\alpha + \delta + L \leq 1$.

Moreover, T is said an F -contraction of Suzuki–Hardy–Rogers type if (2) holds for all $x, y \in X$ with $Tx \neq Ty$ and $d(x, Tx)/2 < d(x, y)$, see also [8, 25].

If we choose the function F opportunely, then we obtain some classes of contractions known in the literature; see the following example.

Example 1. Let (X, d) be a metric space and $T : X \rightarrow X$ an F -contraction of Hardy–Rogers type. Assume that T satisfies (2) with $\beta = \gamma = \delta = L = 0$ and $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ given by $F(x) = \ln x$. It is clear that F satisfies (F1), (F2). From

$$\tau(d(x, y)) + F(d(Tx, Ty)) \leq F(d(x, y)) \quad \text{for all } x, y \in X, Tx \neq Ty,$$

we obtain

$$d(Tx, Ty) \leq e^{-\tau(d(x,y))}d(x, y) \quad \text{for all } x, y \in X, Tx \neq Ty.$$

Since the inequality holds also if $Tx = Ty$, we deduce that every contraction is an F -contraction of Hardy–Rogers type.

Remark 1. From (F1) and (2) we deduce that every T , which is an F -contraction of Hardy–Rogers type, satisfies the following condition:

$$d(Tx, Ty) < \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx) \quad (3)$$

for all $x, y \in X$ with $Tx \neq Ty$, where $\alpha, \beta, \gamma, \delta, L \in [0, +\infty[$, $\alpha + \beta + \gamma + 2\delta = 1$, $\gamma \neq 1$ and $\alpha + \delta + L \leq 1$.

Remark 2. The function $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by $F(t) = -1/t$ for all t satisfies conditions (F1) and (F2), but it does not satisfy condition (F3) and hence $F \in \mathbb{F}$, but $F \notin \mathcal{F}$.

3 New fixed point theorems in complete metric spaces

We give some fixed point results for F -contractions of Hardy–Rogers type in the setting of complete metric spaces. We start with the following proposition.

Proposition 1. *Let (X, d) be a complete metric space and let T be a self-mapping on X . Assume that there exist $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that*

$$\begin{aligned} &\tau(d(x, Tx)) + F(d(Tx, T^2x)) \\ &\leq F((\alpha + \beta)d(x, Tx) + \gamma d(Tx, T^2x) + \delta d(x, T^2x)) \end{aligned} \quad (4)$$

for all $x \in X$ with $Tx \neq T^2x$, where $\alpha, \beta, \gamma, \delta \in [0, +\infty[$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. Then the sequence $\{d(T^{n-1}x_0, T^n x_0)\}$ is decreasing and $d(T^{n-1}x_0, T^n x_0) \rightarrow 0$ as $n \rightarrow +\infty$, for all $x_0 \in X$.

Proof. Let $x_0 \in X$ be an arbitrary point. We can assume that $x_0 \neq Tx_0$ for all $x_0 \in X$. Let $\{x_n\}$ be the Picard sequence with initial point x_0 , that is, $x_n = T^n x_0 = Tx_{n-1}$ and $d_n = d(x_n, x_{n+1}) = d(T^n x_0, T^{n+1} x_0)$ for all $n \in \mathbb{N} \cup \{0\}$. As $x_n \neq x_{n+1}$ for all

$n \in \mathbb{N} \cup \{0\}$, by using the contractive condition (4) with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} \tau(d_{n-1}) + F(d_n) &= \tau(d_{n-1}) + F(d(Tx_{n-1}, Tx_n)) \\ &\leq F((\alpha + \beta)d_{n-1} + \gamma d_n + \delta d(x_{n-1}, Tx_n)) \\ &= F((\alpha + \beta)d_{n-1} + \gamma d_n + \delta d(x_{n-1}, x_{n+1})) \\ &\leq F((\alpha + \beta)d_{n-1} + \gamma d_n + \delta[d_{n-1} + d_n]) \\ &= F((\alpha + \beta + \delta)d_{n-1} + (\gamma + \delta)d_n). \end{aligned} \quad (5)$$

First, we claim that $d_n < d_{n-1}$ for all $n \in \mathbb{N}$. Assume that there exists $n \in \mathbb{N}$ such that $d_n \geq d_{n-1}$, then from (5) we get

$$\tau(d_{n-1}) + F(d_n) \leq F(d_n).$$

This is a contradiction since $\tau(d_{n-1}) > 0$, and hence

$$d_n < d_{n-1} \quad \text{for all } n \in \mathbb{N}. \quad (6)$$

Thus the sequence $\{d_n\}$ is decreasing and hence converges to a real number $s \geq 0$. We show that $s = 0$. Assume that $s > 0$, by condition (H) there exists $r > 0$ and $n_0 \in \mathbb{N}$ such that $\tau(d_n) \geq r$ for all $n \geq n_0$. Now, from (5) and (6) we deduce

$$F(d_n) \leq F(d_{n-1}) - \tau(d_{n-1}) \leq F(d_{n-1}) - r \quad \text{for all } n > n_0.$$

This implies

$$F(d_n) \leq F(d_{n-1}) - r \leq \dots \leq F(d_{n_0}) - (n - n_0)r \quad \text{for all } n > n_0$$

and so $\lim_{n \rightarrow +\infty} F(d_n) = -\infty$. By the property (F2) we get that $d_n \rightarrow 0$ as $n \rightarrow +\infty$, and this concludes the proof. \square

Theorem 1. *Let (X, d) be a complete metric space, and let T be a self-mapping on X . Assume that there exist a continuous $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that T is an F -contraction of Hardy–Rogers type, that is, (2) holds for all $x, y \in X$ with $Tx \neq Ty$. Then T has a unique fixed point.*

Proof. First, we prove the uniqueness of the fixed point. Assume that $z, w \in X$ are fixed points of T with $z \neq w$. This means that $d(z, w) > 0$. Now, if $\alpha + \delta + L > 0$, taking $x = z$ and $y = w$ in (2), we have

$$\begin{aligned} \tau(d(z, w)) + F(d(z, w)) &= \tau(d(z, w)) + F(d(Tz, Tw)) \\ &\leq F(\alpha d(z, w) + \beta d(z, Tz) + \gamma d(w, Tw) \\ &\quad + \delta d(z, Tw) + Ld(w, Tz)) \\ &= F((\alpha + \delta + L)d(z, w)), \end{aligned}$$

which is a contradiction since $\alpha + \delta + L \leq 1$, and hence $z = w$. On the other hand, if $\alpha + \delta + L = 0$, from (3) we obtain

$$d(z, w) = d(Tz, Tw) < \beta d(z, Tz) + \gamma d(w, Tw) = 0,$$

which is a contradiction, and so $z = w$.

Next, note that T satisfies the contractive condition (4) for all $x \in X$ such that $Tx \neq T^2x$. Let $x_0 \in X$ be an arbitrary point, and let $\{x_n\}$ be the Picard sequence with initial point x_0 . If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T and the proof is complete. Assume that $d_n = d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. By Proposition 1 the sequence $\{d_n\}$ is decreasing and $d_n \rightarrow 0$ as $n \rightarrow +\infty$.

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist a real number $\varepsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of natural numbers such that

$$n_k > m_k \geq k, \quad d(x_{n_k}, x_{m_k}) \geq \varepsilon \quad \text{and} \quad d(x_{n_k-1}, x_{m_k}) < \varepsilon \quad \text{for all } k \in \mathbb{N}.$$

From $d(x_{n_k}, x_{n_k-1}) \rightarrow 0$ as $k \rightarrow +\infty$ and

$$\varepsilon \leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-1}) + \varepsilon$$

we obtain

$$\lim_{k \rightarrow +\infty} d(x_{n_k}, x_{m_k}) = \varepsilon.$$

From

$$\varepsilon \leq d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k})$$

as $k \rightarrow +\infty$ we get

$$\varepsilon \leq \liminf_{k \rightarrow +\infty} d(x_{n_k+1}, x_{m_k+1}).$$

It follows that there exists $h \in \mathbb{N}$ such that $d(x_{n_k+1}, x_{m_k+1}) > 0$ for all $k \geq h$. Then, for all $k \geq h$, we obtain

$$\begin{aligned} & \tau(d(x_{n_k}, x_{m_k})) + F(d(Tx_{n_k}, Tx_{m_k})) \\ & \leq F(\alpha d(x_{n_k}, x_{m_k}) + \beta d(x_{n_k}, x_{n_k+1}) + \gamma d(x_{m_k}, x_{m_k+1}) \\ & \quad + \delta d(x_{n_k}, x_{m_k+1}) + Ld(x_{m_k}, x_{n_k+1})) \\ & \leq F((\alpha + \delta + L)d(x_{n_k}, x_{m_k}) + (\beta + L)d(x_{n_k}, x_{n_k+1}) \\ & \quad + (\gamma + \delta)d(x_{m_k}, x_{m_k+1})) \\ & \leq F(d(x_{n_k}, x_{m_k}) + (\beta + L)d(x_{n_k}, x_{n_k+1}) + (\gamma + \delta)d(x_{m_k}, x_{m_k+1})). \end{aligned}$$

Letting $k \rightarrow +\infty$ in the previous inequality, by the property (H) of the function τ and the continuity of the function F we get

$$F(\varepsilon) < \liminf_{k \rightarrow +\infty} \tau(d(x_{n_k}, x_{m_k})) + F(\varepsilon) \leq F(\varepsilon),$$

which is a contradiction, and hence $\{x_n\}$ is a Cauchy sequence. As (X, d) is a complete metric space, there exists $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow +\infty$. If $z = Tz$, the proof is finished. Assume that $z \neq Tz$. If $Tx_n = Tz$ for infinite values of $n \in \mathbb{N} \cup \{0\}$, then the sequence $\{x_n\}$ has a subsequence that converges to Tz and the uniqueness of the limit

implies $z = Tz$. Then we can assume that $Tx_n \neq Tz$ for all $n \in \mathbb{N} \cup \{0\}$. Now, by (3) we have

$$\begin{aligned} d(z, Tz) &\leq d(z, x_{n+1}) + d(Tx_n, Tz) \\ &< d(z, x_{n+1}) + \alpha d(x_n, z) + \beta d(x_n, Tx_n) + \gamma d(z, Tz) \\ &\quad + \delta d(x_n, Tz) + Ld(z, Tx_n) \\ &= d(z, x_{n+1}) + \alpha d(x_n, z) + \beta d(x_n, x_{n+1}) + \gamma d(z, Tz) \\ &\quad + \delta d(x_n, Tz) + Ld(z, x_{n+1}). \end{aligned}$$

Letting $n \rightarrow +\infty$ in the previous inequality, we get

$$d(z, Tz) \leq (\gamma + \delta)d(z, Tz) < d(z, Tz),$$

which is a contradiction, and hence $z = Tz$. \square

Remark 3. If in Theorem 1, we assume $\alpha + \delta + L \leq h < 1$, from

$$\varepsilon \leq \liminf_{k \rightarrow +\infty} d(x_{n_k+1}, x_{m_k+1})$$

we obtain that there exists $k_0 \in \mathbb{N}$ such that $h\varepsilon \leq d(x_{n_k+1}, x_{m_k+1})$ for all $k \geq k_0$. Then, for all $k \geq k_0$, we get

$$\begin{aligned} &\tau(d(x_{n_k}, x_{m_k})) + F(h\varepsilon) \\ &\leq \tau(d(x_{n_k}, x_{m_k})) + F(d(x_{n_k+1}, x_{m_k+1})) \\ &= \tau(d(x_{n_k}, x_{m_k})) + F(d(Tx_{n_k}, Tx_{m_k})) \\ &\leq F(\alpha d(x_{n_k}, x_{m_k}) + \beta d(x_{n_k}, x_{n_k+1}) + \gamma d(x_{m_k}, x_{m_k+1}) \\ &\quad + \delta d(x_{n_k}, x_{m_k+1}) + Ld(x_{m_k}, x_{n_k+1})) \\ &\leq F((\alpha + \delta + L)d(x_{n_k}, x_{m_k}) + (\beta + L)d(x_{n_k}, x_{n_k+1}) \\ &\quad + (\gamma + \delta)d(x_{m_k}, x_{m_k+1})) \\ &\leq F(hd(x_{n_k}, x_{m_k}) + (\beta + L)d(x_{n_k}, x_{n_k+1}) \\ &\quad + (\gamma + \delta)d(x_{m_k}, x_{m_k+1})). \end{aligned}$$

Now, if we assume that F is upper semicontinuous, letting $k \rightarrow +\infty$, we obtain

$$\begin{aligned} &\liminf_{k \rightarrow +\infty} \tau(d(x_{n_k}, x_{m_k})) + F(h\varepsilon) \\ &\leq \limsup_{k \rightarrow +\infty} F(hd(x_{n_k}, x_{m_k}) + (\beta + L)d(x_{n_k}, x_{n_k+1}) \\ &\quad + (\gamma + \delta)d(x_{m_k}, x_{m_k+1})) \\ &\leq F(h\varepsilon), \end{aligned}$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Hence Theorem 1 holds if we suppose $F \in \mathbb{F}$ upper semicontinuous and $\alpha + \delta + L < 1$.

As a first corollary of Theorem 1, taking $\alpha = 1$ and $\beta = \gamma = \delta = L = 0$, we obtain a generalization of Theorem 2.1 of Piri and Kumam [17]. Further, putting $\alpha = \delta = L = 0$ and $\beta + \gamma = 1$ and $\beta \neq 0$, from Theorem 1 and Remark 3 we obtain the following version of Kannan’s result [9].

Corollary 1. *Let (X, d) be a complete metric space, and let T be a self-mapping on X . Assume that there exist an upper semicontinuous $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that*

$$\tau(d(x, y)) + F(d(Tx, Ty)) \leq F(\beta d(x, Tx) + \gamma d(y, Ty))$$

for all $x, y \in X, Tx \neq Ty$, where $\beta, \gamma \in [0, +\infty[, \beta + \gamma = 1, \gamma \neq 1$. Then T has a unique fixed point in X .

Also, a version of the Chatterjea [7] fixed point theorem is obtained from Theorem 1 putting $\alpha = \beta = \gamma = 0$ and $\delta = L = 1/2$.

Corollary 2. *Let (X, d) be a complete metric space, and let T be a self-mapping on X . Assume that there exist a continuous $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that*

$$\tau(d(x, y)) + F(d(Tx, Ty)) \leq F\left(\frac{1}{2}d(x, Ty) + \frac{1}{2}d(y, Tx)\right)$$

for all $x, y \in X, Tx \neq Ty$. Then T has a unique fixed point in X .

Finally, if we choose $\delta = L = 0$, we obtain a Reich-type theorem, see [19].

Corollary 3. *Let (X, d) be a complete metric space, and let T be a self-mapping on X . Assume that there exist a continuous $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that*

$$\tau(d(x, y)) + F(d(Tx, Ty)) \leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty))$$

for all $x, y \in X, Tx \neq Ty$, where $\alpha, \beta, \gamma \in [0, +\infty[, \alpha + \beta + \gamma = 1, \gamma \neq 1$. Then T has a unique fixed point in X .

We present the following illustrative example.

Example 2. Let ρ be a positive real number. Consider the sequence $\{x_n\}$ defined as follows:

$$x_0 = \rho \quad \text{and} \quad x_n = x_{n-1} + n \quad \text{for all } n \in \mathbb{N}.$$

Let $X = \{x_n: n \in \mathbb{N}\}$ be endowed with the metric $d : X \times X \rightarrow [0, +\infty[$ given by $d(x, y) = |x - y|$ for all $x, y \in X$, so that (X, d) is a complete metric space. Next, we consider the self-mapping $T : X \rightarrow X$ defined by $Tx_1 = x_1$ and $Tx_n = x_{n-1}$ for all $n \in \mathbb{N} \setminus \{1\}$. We distinguish the following two cases:

1. If $1 = m < 3 \leq n$, we have

$$Tx_n - Tx_1 = x_{n-1} - x_1 < x_{n-1} - x_1 + n = x_n - x_1.$$

2. If $1 < m < n$, we have

$$Tx_n - Tx_m = x_{n-1} - x_{m-1} < x_{n-1} - x_{m-1} + n - m = x_n - x_m.$$

From the previous inequalities, since $x_{n-1} - x_{m-1} < x_n - x_m$, we deduce

$$\begin{aligned} & \frac{1}{2(x_n - x_m)} - \frac{1}{x_{n-1} - x_{m-1}} + x_{n-1} - x_{m-1} \\ & < \frac{1}{2(x_n - x_m)} - \frac{1}{x_n - x_m} + x_{n-1} - x_{m-1} \\ & \leq \frac{1}{2} - \frac{1}{x_n - x_m} + x_{n-1} - x_{m-1} \\ & \leq -\frac{1}{x_n - x_m} + x_{n-1} - x_{m-1} + n - m \\ & = -\frac{1}{x_n - x_m} + x_n - x_m \end{aligned}$$

for all $1 \leq m < n$ such that $Tx_n \neq Tx_m$.

Notice that the function $F : \mathbb{R}^+ \rightarrow \mathbb{R}$, given by $F(t) = -1/t + t$ for all $t \in \mathbb{R}^+$, is a continuous function belonging to \mathbb{F} and the function $\tau : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, given by $\tau(t) = (2t)^{-1}$ for all $t \in \mathbb{R}^+$, belongs to \mathbb{S} . Consequently, one has that T satisfies the contractive condition

$$\tau(d(x_n, x_m)) + F(d(Tx_n, x_m)) \leq F(d(x_n, x_m))$$

for all $1 \leq m < n$ such that $Tx_n \neq Tx_m$.

We conclude that all the conditions of Theorem 1 are satisfied taking $\alpha = 1$ and $\beta = \gamma = \delta = L = 0$, and hence T has a unique fixed point.

In the following result we use a contractive condition of Suzuki type.

Theorem 2. *Let (X, d) be a complete metric space, and let T be a self-mapping on X . Assume that there exist a continuous $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that T is an F -contraction of Suzuki-Hardy-Rogers type, that is, (2) holds for all $x, y \in X$ with $Tx \neq Ty$ and $d(x, Tx)/2 < d(x, y)$. Then T has a unique fixed point.*

Proof. The uniqueness part is obtained by proceeding as in the proof of Theorem 1 and hence, to avoid repetition, we omit the details.

Now, note that T satisfies the contractive condition (4) for all $x \in X$ such that $Tx \neq T^2x$. Let $x_0 \in X$ be an arbitrary point, and let $\{x_n\}$ be the Picard sequence with initial point x_0 . If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T and the proof is completed. Assume that $d_n = d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. By Proposition 1 the sequence $\{d_n\}$ is decreasing and $d_n \rightarrow 0$ as $n \rightarrow +\infty$.

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist a real number $\varepsilon > 0$ and two sequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of natural numbers such that

$$n_k > m_k \geq k, \quad d(x_{n_k}, x_{m_k}) \geq \varepsilon \quad \text{and} \quad d(x_{n_k-1}, x_{m_k}) < \varepsilon \quad \text{for all } k \in \mathbb{N}.$$

From the proof of Theorem 1 we write

$$\lim_{k \rightarrow +\infty} d(x_{n_k}, x_{m_k}) = \varepsilon \quad \text{and} \quad \varepsilon \leq \liminf_{k \rightarrow +\infty} d(x_{n_k+1}, x_{m_k+1}). \quad (7)$$

Thus there is $N \in \mathbb{N}$ such that

$$\frac{1}{2}d(x_{n_k}, Tx_{n_k}) < \frac{1}{2}\varepsilon < d(x_{n_k}, x_{m_k}) \quad \text{for all } k \geq N.$$

From (7) it follows there exists $h \in \mathbb{N}$ with $h \geq N$, so that $d(x_{n_{k+1}}, x_{m_{k+1}}) > 0$ for all $k \geq h$. Then, for all $k \geq h$, we obtain

$$\begin{aligned} &\tau(d(x_{n_k}, x_{m_k})) + F(d(Tx_{n_k}, Tx_{m_k})) \\ &\leq F(\alpha d(x_{n_k}, x_{m_k}) + \beta d(x_{n_k}, x_{n_{k+1}}) + \gamma d(x_{m_k}, x_{m_{k+1}}) \\ &\quad + \delta d(x_{n_k}, x_{m_{k+1}}) + Ld(x_{m_k}, x_{n_{k+1}})) \\ &\leq F((\alpha + \delta + L)d(x_{n_k}, x_{m_k}) + (\beta + L)d(x_{n_k}, x_{n_{k+1}}) \\ &\quad + (\gamma + \delta)d(x_{m_k}, x_{m_{k+1}})) \\ &\leq F(d(x_{n_k}, x_{m_k}) + (\beta + L)d(x_{n_k}, x_{n_{k+1}}) \\ &\quad + (\gamma + \delta)d(x_{m_k}, x_{m_{k+1}})). \end{aligned}$$

Letting $k \rightarrow +\infty$ in the previous inequality, by the property (H) of the function τ and the continuity of the function F we get

$$F(\varepsilon) < \liminf_{k \rightarrow +\infty} \tau(d(x_{n_k}, x_{m_k})) + F(\varepsilon) \leq F(\varepsilon),$$

which is a contradiction, and hence $\{x_n\}$ is a Cauchy sequence. As (X, d) is a complete metric space, there exists $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow +\infty$. If $z = Tz$, the proof is finished. Assume that $z \neq Tz$. If $Tx_n = Tz$ for infinite values of $n \in \mathbb{N} \cup \{0\}$, then the sequence $\{x_n\}$ has a subsequence that converges to Tz and the uniqueness of the limit implies $z = Tz$. Then we can assume that $Tx_n \neq Tz$ for all $n \in \mathbb{N} \cup \{0\}$. Now, we claim that

$$\frac{1}{2}d(x_n, Tx_n) < d(x_n, z) \quad \text{or} \quad \frac{1}{2}d(x_{n+1}, Tx_{n+1}) < d(x_{n+1}, z) \tag{8}$$

for all $n \in \mathbb{N}$. In fact, if there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2}d(x_m, Tx_m) \geq d(x_m, z) \quad \text{and} \quad \frac{1}{2}d(x_{m+1}, Tx_{m+1}) \geq d(x_{m+1}, z),$$

then

$$2d(x_m, z) \leq d(x_m, Tx_m) \leq d(x_m, z) + d(Tx_m, z).$$

This implies

$$d(x_m, z) \leq d(Tx_m, z) \leq \frac{1}{2}d(Tx_m, T^2x_m).$$

From

$$\begin{aligned} d(Tx_m, T^2x_m) &< d(x_m, Tx_m) \leq d(x_m, z) + d(Tx_m, z) \\ &\leq \frac{1}{2}d(Tx_m, T^2x_m) + \frac{1}{2}d(Tx_m, T^2x_m) \end{aligned}$$

we obtain a contradiction, and so (8) holds.

Let \mathbb{J} be the set of $n \in \mathbb{N}$ such that the first inequality in (8) holds. Assume that the set \mathbb{J} is infinite. Then, for all $n \in \mathbb{J}$, by (3) we have

$$\begin{aligned} d(z, Tz) &\leq d(z, x_{n+1}) + d(Tx_n, Tz) \\ &< d(z, x_{n+1}) + \alpha d(x_n, z) + \beta d(x_n, Tx_n) + \gamma d(z, Tz) \\ &\quad + \delta d(x_n, Tz) + Ld(z, Tx_n) \\ &= d(z, x_{n+1}) + \alpha d(x_n, z) + \beta d(x_n, x_{n+1}) + \gamma d(z, Tz) \\ &\quad + \delta d(x_n, Tz) + Ld(z, x_{n+1}). \end{aligned}$$

Letting $n \in \mathbb{J} \rightarrow +\infty$ in the previous inequality, we get the contradiction

$$d(z, Tz) \leq (\gamma + \delta)d(z, Tz) < d(z, Tz)$$

and hence $z = Tz$. The same holds if $\mathbb{N} \setminus \mathbb{J}$ is an infinite set. \square

The next theorem is inspired by Secelean [23]. Precisely, the following result establishes that the limit of a sequence of F -contractions of Hardy–Rogers type is an F -contraction of Hardy–Rogers type.

Theorem 3. *Let (X, d) be a complete metric space, and let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of self-mappings on X . Assume that there exist a continuous $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that*

$$\begin{aligned} &\tau(d(x, y)) + F(d(T_n x, T_n y)) \\ &\leq F(\alpha d(x, y) + \beta d(x, T_n x) + \gamma d(y, T_n y) + \delta d(x, T_n y) + Ld(y, T_n x)) \quad (9) \end{aligned}$$

for all $x, y \in X$ with $T_n x \neq T_n y$ and $n \in \mathbb{N}$, where $\alpha, \beta, \gamma, \delta, L \in [0, +\infty[$, $\alpha + \beta + \gamma + 2\delta = 1$, $\gamma \neq 1$ and $\alpha + \delta + L \leq 1$. Assume that:

- (i) the sequence $\{T_n\}$ converges pointwise to a map $T : X \rightarrow X$;
- (ii) the sequence $\{z_n\}$ converges to $z \in X$, where z_n is the fixed point of T_n .

Then T is an F -contraction of Hardy–Rogers type and $Tz = z$.

Proof. Let $x, y \in X$ be such that $Tx \neq Ty$. Since $d(Tx, Ty) > 0$, for every $\varepsilon < d(Tx, Ty)$, there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned} d(Tx, Ty) - \varepsilon &< d(T_n x, T_n y) < d(Tx, Ty) + \varepsilon, \\ d(x, T_n x) &< d(x, Tx) + \varepsilon, \quad d(y, T_n y) < d(y, Ty) + \varepsilon, \\ d(x, T_n y) &< d(x, Ty) + \varepsilon, \quad d(y, T_n x) < d(y, Tx) + \varepsilon \end{aligned} \quad (10)$$

for all $n \geq n(\varepsilon)$. By (9) and (10), for each $n \geq n(\varepsilon)$, we get

$$\begin{aligned} &\tau(d(x, y)) + F(d(Tx, Ty)) \\ &\leq \tau(d(x, y)) + F(d(T_n x, T_n y) + \varepsilon) \\ &\leq F(\alpha d(x, y) + \beta d(x, T_n x) + \gamma d(y, T_n y) + \delta d(x, T_n y) + Ld(y, T_n x)) \\ &\quad + F(d(T_n x, T_n y) + \varepsilon) - F(d(T_n x, T_n y)) \\ &\leq F(\alpha d(x, y) + \beta [d(x, Tx) + \varepsilon] + \gamma [d(y, Ty) + \varepsilon] + \delta [d(x, Ty) + \varepsilon] \\ &\quad + L [d(y, Tx) + \varepsilon]) + F(d(Tx, Ty) + 2\varepsilon) - F(d(Tx, Ty) - \varepsilon). \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, by the continuity of F we obtain

$$\begin{aligned} &\tau(d(x, y)) + F(d(Tx, Ty)) \\ &\leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx)), \end{aligned}$$

and hence T is an F -contraction of Hardy–Rogers type.

At last, from

$$\begin{aligned} d(z, Tz) &\leq d(z, z_n) + d(T_n z_n, T_n z) + d(T_n z, Tz) \\ &\leq d(z, z_n) + d(T_n z, Tz) + \alpha d(z_n, z) + \beta d(z_n, z_n) \\ &\quad + \gamma d(z, T_n z) + \delta d(z_n, T_n z) + Ld(z, z_n), \end{aligned}$$

letting $n \rightarrow +\infty$, we deduce that $d(z, Tz) \leq (\gamma + \delta)d(z, Tz)$ and hence $d(z, Tz) = 0$. Thus z is a fixed point of T . \square

4 New fixed point theorems in complete ordered metric spaces

Fixed point theory for self-mappings on partially ordered sets has been initiated by Ran and Reurings [18], in dealing with matrix equations, and continued by many mathematicians [1, 6, 12, 13, 14, 15, 22], particularly in dealing with differential equations.

Let (X, d) be a metric space and (X, \preceq) be a partially ordered non-empty set, therefore (X, d, \preceq) is called an ordered metric space. Moreover, two elements $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. A self-mapping T on a partially ordered set (X, \preceq) is called non-decreasing if $Tx \preceq Ty$ whenever $x \preceq y$ for all $x, y \in X$. Also, an ordered metric space (X, d, \preceq) is called regular if for every non-decreasing sequence $\{x_n\}$ in X , convergent to some $x \in X$, we get $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Theorem 4. *Let (X, d, \preceq) be a complete ordered metric space, and let T be a non-decreasing self-mapping on X . Assume that there exist a continuous $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that T is an ordered F -contraction of Hardy–Rogers type, that is,*

$$\begin{aligned} &\tau(d(x, y)) + F(d(Tx, Ty)) \\ &\leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx)) \end{aligned} \quad (11)$$

for all comparable $x, y \in X$ with $Tx \neq Ty$, where $\alpha, \beta, \gamma, \delta, L \in [0, +\infty]$, $\alpha + \beta + \gamma + 2\delta = 1$, $\gamma \neq 1$ and $\alpha + \delta + L \leq 1$. If the following conditions are satisfied:

- (i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
- (ii) X is regular;

then T has a fixed point. Moreover, the set of fixed points of T is well-ordered if and only if T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point such that (i) holds, and let $\{x_n\}$ be the Picard sequence of initial point x_0 , that is, $x_n = T^n x_0 = Tx_{n-1}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$,

then x_n is a fixed point of T and this ensures the existence of a fixed point of T in X . Now, assume that $d_n = d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. As T is non-decreasing, we deduce that

$$x_0 \prec x_1 \prec \cdots \prec x_n \prec \cdots, \quad (12)$$

that is, x_{n-1} and x_n are comparable and $Tx_{n-1} \neq Tx_n$ for all $n \in \mathbb{N}$.

Proceeding as in the proofs of Proposition 1 and Theorem 1, we obtain that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow +\infty$. If $z = Tz$, the proof is finished. Assume that $z \neq Tz$. Since X is regular, from (12) we deduce that x_n and z are comparable and $Tx_n \neq Tz$ for all $n \in \mathbb{N} \cup \{0\}$. Now, by using (3), we obtain

$$\begin{aligned} d(z, Tz) &\leq d(z, x_{n+1}) + d(Tx_n, Tz) \\ &< d(z, x_{n+1}) + \alpha d(x_n, z) + \beta d(x_n, x_{n+1}) + \gamma d(z, Tz) \\ &\quad + \delta d(x_n, Tz) + Ld(z, x_{n+1}). \end{aligned}$$

Letting $n \rightarrow +\infty$ in the previous inequality, we get

$$d(z, Tz) \leq (\gamma + \delta)d(z, Tz) < d(z, Tz),$$

which is a contradiction, and hence $z = Tz$.

Next, we assume that the set of fixed points of T is well-ordered. We claim that the fixed point of T is unique. Assume on the contrary that there exists another fixed point w in X such that $z \neq w$. Then, by using the condition (11) with $x = z$ and $y = w$, we get

$$\begin{aligned} &\tau(d(z, w)) + F(d(z, w)) \\ &= \tau(d(z, w)) + F(d(Tz, Tw)) \\ &\leq F(\alpha d(z, w) + \beta d(z, Tz) + \gamma d(w, Tw) + \delta d(z, Tw) + Ld(w, Tz)) \\ &= F((\alpha + \delta + L)d(z, w)) \leq F(d(z, w)), \end{aligned}$$

which is a contradiction, and hence $z = w$. Conversely, if T has a unique fixed point, then the set of fixed points of T , being a singleton, is well-ordered. \square

The following theorem is analogous to the previous one.

Theorem 5. *Let (X, d) be a complete ordered metric space and let T be a non-decreasing self-mapping on X . Assume that there exist a continuous $F \in \mathbb{F}$ and $\tau \in \mathbb{S}$ such that T is an ordered F -contraction of Hardy–Rogers type. If the following conditions are satisfied:*

- (i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
- (ii) *X is regular;*

then T has a fixed point. Moreover, if $\alpha + 2\gamma + \delta + L < 1$ and the following condition holds:

- (iii) *for all $z, w \in X$ there exists $v \in X$ such that z and v are comparable and w and v are comparable;*

then T has a unique fixed point.

Proof. The existence of a fixed point of T is a consequence of Theorem 4. Now, let $z \in X$ be a fixed point of T . For all $v \in X$ comparable with z such that $Tz \neq Tv$, we have

$$\begin{aligned} &\tau(d(z, v)) + F(d(Tz, Tv)) \\ &\leq F(\alpha d(z, v) + \beta d(z, Tz) + \gamma d(v, Tv) + \delta d(z, Tv) + Ld(v, Tz)) \\ &\leq F(\alpha d(z, v) + \gamma(d(v, z) + d(z, Tv)) + \delta d(z, Tv) + Ld(v, z)) \\ &= F((\alpha + \gamma + L)d(z, v) + (\gamma + \delta)d(z, Tv)). \end{aligned}$$

Since F is non-decreasing, we deduce

$$d(z, Tv) < (\alpha + \gamma + L)d(z, v) + (\gamma + \delta)d(z, Tv)$$

and hence

$$(1 - \gamma - \delta)d(z, Tv) < (\alpha + \gamma + L)d(z, v).$$

Since $1 - \gamma - \delta > 0$, we get

$$d(z, Tv) < \frac{\alpha + \gamma + L}{1 - \gamma - \delta} d(z, v).$$

As T is non-decreasing, we have that z and $T^n v$ are comparable for all $n \in \mathbb{N}$. If $z \neq T^n v$ for all $n \in \mathbb{N}$, then

$$d(z, T^n v) < \lambda^n d(z, v) \quad \text{for all } n \in \mathbb{N},$$

where $\lambda = (\alpha + \gamma + L)/(1 - \gamma - \delta) < 1$. From the previous inequality we obtain $d(z, T^n v) \rightarrow 0$ as $n \rightarrow +\infty$. Now, if z, w are two fixed points of T , by the condition (iii) there exists $v \in X$ such that z and v are comparable and w and v are comparable. If $z = T^n v$ or $w = T^n v$ for some $n \in \mathbb{N}$, then z and w are comparable and the uniqueness of the fixed point follows since T is an F -contraction of Hardy–Rogers type. Assume that $z \neq T^n v$ and $w \neq T^n v$ for all $n \in \mathbb{N}$. Then

$$d(z, w) \leq d(z, T^n v) + d(w, T^n v) \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

and hence $d(z, w) = 0$, that is, $z = w$. □

By choosing opportunely the function F in Theorems 4 and 5, we obtain some results of fixed point in the setting of ordered metric spaces known in the literature. For example, if we choose $F(x) = \ln x$ in Theorem 4 and putting $\beta = \gamma = \delta = L = 0$ and $\tau(t) = 1$ for all $t \in \mathbb{R}^+$, then we obtain Theorem 2.2 of [13].

5 Application to functional equation

We get a typical application of the results in previous sections in mathematical optimization. More precisely, we prove the existence of solutions of some functional equations,

arising in dynamic programming of multistage decision processes. From a mathematical point of view we consider the problem of finding a function u such that

$$u(x) = \sup_{y \in D} \{f(x, y) + G(x, y, u(\eta(x, y)))\}, \quad x \in W, \quad (13)$$

where $f : W \times D \rightarrow \mathbb{R}$ and $G : W \times D \times \mathbb{R} \rightarrow \mathbb{R}$ are bounded, $\eta : W \times D \rightarrow W$, W and D are Banach spaces; precisely, W is a state space and D is a decision space. We refer the reader to [5, 16] for more details and examples.

Let $B(W)$ denote the set of all bounded real-valued functions on W . The pair $(B(W), \|\cdot\|)$, where

$$\|h\| = \sup_{x \in W} \{\min\{|h(x)|, s\}\}, \quad h \in B(W),$$

with $s \in]0, 1[$ is a Banach space.

Mathematical optimization is one of the fields in which the methods of fixed point theory are widely used, then, to show the existence of a solution of (13), we define the self-operator $T : B(W) \rightarrow B(W)$ by

$$(Th)(x) = \sup_{y \in D} \{f(x, y) + G(x, y, h(\eta(x, y)))\} \quad (14)$$

for all $h \in B(W)$ and $x \in W$. Obviously, T is well-defined since f and G are bounded.

Then we state and prove the following result of existence of fixed points for the operator T , which is equivalent to establishing the existence of solutions of the functional equation (13).

Theorem 6. *Let T be the self-operator on $B(W)$ given by (14), and assume that the following condition holds:*

(i) *there exists $s \in]0, 1[$ such that*

$$\min\{|G(x, y, h(z)) - G(x, y, k(z))|, s\} \leq \frac{\min\{|h(x) - k(x)|, s\}}{1 - \|h - k\| \ln(\|h - k\|)}$$

for all $h, k \in B(W)$ with $h \neq k$, all $y \in D$, $x \in W$ and $z = \eta(x, y)$.

Then the operator T has a unique fixed point.

Proof. Let $\lambda \in \mathbb{R}^+$ be arbitrary, $x \in W$ and $h, k \in B(W)$ with $Th \neq Tk$. Then there exist $y_1, y_2 \in D$ such that

$$(Th)(x) < f(x, y_1) + G(x, y_1, h(\eta(x, y_1))) + \lambda, \quad (15)$$

$$(Tk)(x) < f(x, y_2) + G(x, y_2, k(\eta(x, y_2))) + \lambda. \quad (16)$$

Also, we have

$$(Th)(x) \geq f(x, y_2) + G(x, y_2, h(\eta(x, y_2))), \quad (17)$$

$$(Tk)(x) \geq f(x, y_1) + G(x, y_1, k(\eta(x, y_1))). \quad (18)$$

Then by (15) and (18) we have

$$(Th)(x) - (Tk)(x) < G(x, y_1, h(\eta(x, y_1))) - G(x, y_1, k(\eta(x, y_1))) + \lambda,$$

and by (16) and (17) we have

$$(Tk)(x) - (Th)(x) < G(x, y_2, k(\eta(x, y_2))) - G(x, y_2, h(\eta(x, y_2))) + \lambda.$$

By using the condition (i) and last two inequalities, we get

$$\begin{aligned} & \min\{|(Th)(x) - (Tk)(x)|, s\} \\ & \leq \frac{\min\{|h(x) - k(x)|, s\}}{1 - \|h - k\| \ln(\|h - k\|)} \leq \frac{\|h - k\|}{1 - \|h - k\| \ln(\|h - k\|)}. \end{aligned}$$

From the last inequality we get

$$\begin{aligned} \|Th - Tk\| &= \sup_{x \in W} \{ \min\{|(Th)(x) - (Tk)(x)|, s\} \} \\ &\leq \frac{\|h - k\|}{1 - \|h - k\| \ln(\|h - k\|)} \end{aligned}$$

and so

$$-\ln(\|h - k\|) - \frac{1}{\|Th - Tk\|} \leq -\frac{1}{\|h - k\|}.$$

Now, if we choose $\tau : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ given by

$$\tau(t) = \begin{cases} -\ln t & \text{for } t \in]0, s[, \\ \ln s & \text{for } t \geq s \end{cases}$$

and $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ given by $F(t) = -1/t$ for all $t \in \mathbb{R}^+$, we conclude that T is an F -contraction of Hardy–Rogers type with $\beta = \gamma = \delta = L = 0$. All the conditions of Theorem 1 are satisfied, and so T has a unique fixed point, that is, the functional equation (13) admits a unique solution. \square

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