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Best proximity point results in set-valued analysis

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Abstract. Here we introduce certain multivalued maps and use them to obtain minimum distance between two closed sets. It is a proximity point problem, which is treated here as a problem of finding global optimal solutions of certain fixed point inclusions. It is an application of set-valued analysis. The results we obtain here extend some results and are illustrated with examples. Applications are made to the corresponding single valued cases.

Keywords: set-valued analysis, multivalued α-ψ-proximal contraction, best proximity point, global optimal solution.

1 Introduction

In the paper, we address a problem of finding the distance between the two closed sets by using multivalued non-self mappings from one set to others. The problem is known as the proximity point problem, which is considered here in the context of metric spaces. Here it is an application of set-valued analysis. It is considered as a global optimization problem, which seeks a solution by finding the global best approximation solution of a fixed point inclusion. Technically, the problem is described as follows.

Let A and B be two nonempty subsets of a metric space (X, d). A pair $(a, b) \in A \times B$ is called a best proximity pair if $d(a, b) = \operatorname{dist}(A, B)= \inf\{d(x, y)\colon x \in A \text{ and } y \in B\}$. If T is a mapping from A to B, then $d(x, Tx) \geqslant \operatorname{dist}(A, B)$ for all $x \in A$. A point $p \in A$ is called a best proximity point (with respect to T) if at the point p, the function $d(x, Tx)$ attains its global minimum and the global minimum value is $\operatorname{dist}(A, B)$, that is,

$$d(p, Tp) = \operatorname{dist}(A, B). \tag{1}$$

The following are the concepts from set-valued analysis, which we use in this paper. Let (X, d) be a metric space. Then

$$\begin{aligned} N(X) &= \{A\colon\ A \text{ is a nonempty subset of } X\}, \\ B(X) &= \{A\colon\ A \text{ is a nonempty bounded subset of } X\}, \\ CB(X) &= \{A\colon\ A \text{ is a nonempty closed and bounded subset of } X\}, \end{aligned}$$

and

$$C(X) = \{A\colon\ A \text{ is a nonempty compact subset of } X\}.$$

For a nonempty subset Y of X, $CB(Y)$ denotes the set of nonempty closed and bounded subsets of Y. Clearly, $C(X) \subseteq CB(X) \subseteq B(X) \subseteq N(X)$.

$$D(x, B) = \inf\bigl\{d(x, y)\colon\ y \in B\bigr\}, \quad \text{where } B \in N(X) \text{ and } x \in X,$$

and

$$H(A, B) = \max\Bigl\{\sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A)\Bigr\}, \quad \text{where } A, B \in CB(X).$$

H is known as the Hausdorff metric on $CB(X)$ [24]. Further, if (X, d) is complete, then $(CB(X), H)$ is also complete.

Let A and B be two nonempty subsets of a metric space (X, d) and $T : A \to CB(B)$ a multivalued mapping. A point $x^* \in A$ is called a best proximity point of T if $D(x^*, Tx^*) = \inf\{d(x^*, y)\colon\ y \in Tx^*\} = \operatorname{dist}(A, B)$ [2]. This is a natural generalization of the concept defined in (1). A fixed point x of a multivalued mapping T is given by the following inclusion relation: $x \in Tx$. Now, there may not be a fixed point of the multivalued mapping in general. Here the task in the best proximity point problem is to find a global minima of the function $x \to D(x, Tx)$ by constructing an approximate solution of the inclusion relation $x \in Tx$ to satisfy $D(x, Tx) = \operatorname{dist}(A, B)$.

In the singled valued case, the problem reduces to finding an optimal approximation solution of a fixed point equation. There are several works on proximity point problems as, for examples, [4,6,9,11,12,14,15,22,25]. References [2,10,13,18,27,28] are examples of such problems, which are set-valued mappings.

Best proximity point theorems are different from best approximation theorems. The best approximation theorems provide with best approximate solutions, which need not be globally optimal. For instance, let us consider the following Ky Fan's best approximation theorem.

Theorem 1. (*See* [16].) *Let A be a nonempty compact convex subset of a normed linear space X and $T : A \to X$ be a continuous function. Then there exists $x \in A$ such that*

$$\|x - Tx\| = d(Tx, A) = \inf\bigl\{\|Tx - a\|\colon\ a \in A\bigr\}.$$

The element x in the above theorem need not give the optimum value of $\|x - Tx\|$. On the other hand, the best proximity point theorems assert that the approximate solution is also optimal, that is, a best proximity point theorem explores the possibility of finding the global minima of the real valued function $x \to d(x, Tx)$ by constraining an approximate solution of $x = Tx$, or $x \in Tx$ in the multivalued case, to satisfy $d(x, Tx) = \operatorname{dist}(A, B)$.

In this paper, we define multivalued α-proximal admissible mapping and multivalued α-ψ-proximal contraction. We investigate some proximity point problems with the help of multivalued α-ψ-proximal contractive mappings. We use weak P-property in our results. Supporting examples are also discussed. We give applications of our theorems to obtain results in the single-valued cases.

Recently, in two papers, the P-property and the weak P-property have been utilized for proving proximity point results in [3] and [5], respectively. The authors of these papers have proved their results through applications of fixed point theorems. Our approach as mentioned in the above is different from those in [3] and [5]. Moreover, our result is not related to those in the above two papers.

2 Mathematical preliminaries

Let A and B be two nonempty subsets of a metric space (X, d). Then A_0 and B_0 are given by

$$A_0 = \bigl\{x \in A\colon\ d(x, y) = \operatorname{dist}(A, B) \text{ for some } y \in B\bigr\}$$

and

$$B_0 = \bigl\{y \in B\colon\ d(x, y) = \operatorname{dist}(A, B) \text{ for some } x \in A\bigr\}.$$

It is to be noted that for every $x \in A_0$, there exists $y \in B_0$ such that $d(x, y) = \operatorname{dist}(A, B)$ and, conversely, for every $y \in B_0$, there exists $x \in A_0$ such that $d(x, y) = \operatorname{dist}(A, B)$.

In the following, we give the definitions of P-property and weak P-property.

Definition 1. (See [25].) Let A and B be two nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the *P-property* if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\begin{aligned} &d(x_1, y_1) = \operatorname{dist}(A, B), \quad d(x_2, y_2) = \operatorname{dist}(A, B) \\ &\quad \Longrightarrow \quad d(x_1, x_2) = d(y_1, y_2). \end{aligned}$$

In [1], Abkar and Gabeleh show that every nonempty, bounded, closed and convex pair of subsets of a uniformly convex Banach space has the P-property. Some nontrivial examples of a nonempty pair of subsets, which satisfies the P-property are given in [1].

The notion of weak P-property was first introduced by Gabeleh [17].

Definition 2. (See [17].) Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the *weak P-property* if and only if for $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\begin{aligned} &d(x_1, y_1) = \operatorname{dist}(A, B), \quad d(x_2, y_2) = \operatorname{dist}(A, B) \\ &\quad \Longrightarrow \quad d(x_1, x_2) \leqslant d(y_1, y_2). \end{aligned}$$

In the study of proximity point problems, P-property and weak P-property for a pair of subsets are used in a number of works [11, 12, 17, 19, 25, 29].

Nadler [24] established the following lemma.

Lemma 1. (*See* [24].) *Let* (X, d) *be a metric space and* $A, B \in CB(X)$. *Let* $q > 1$. *Then for every* $x \in A$, *there exists* $y \in B$ *such that* $d(x, y) \leqslant qH(A, B)$.

The following is a consequence of Lemma 1.

Lemma 2. *Let* A *and* B *be two nonempty subset of a metric space* (X, d) *and* $T : A \to CB(B)$ *be a multivalued mapping. Let* $q > 1$. *Then for* $a, b \in A$ *and* $x \in Ta$, *there exists* $y \in Tb$ *such that* $d(x, y) \leqslant qH(Ta, Tb)$.

In [24], Nadler stated that Lemma 1 is also valid for $q \geqslant 1$ if $A, B \in C(X)$. Here we present the lemma with a proof.

Lemma 3. *Let* (X, d) *be a metric space and* $A, B \in C(X)$. *Let* $q \geqslant 1$. *Then for every* $x \in A$, *there exists* $y \in B$ *such that* $d(x, y) \leqslant qH(A, B)$.

Proof. Let $A, B \in C(X)$ and $x \in A$. Since $A, B \in C(X)$ implies $A, B \in CB(X)$, by Lemma 1 the result is true if $q > 1$. So, we shall prove the result for $q = 1$. Now,

$$H(A, B) = \max\Big\{\sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A)\Big\}.$$

From the definition, $p = D(x, B) = \inf\{d(x, b)\colon b \in B\} \leqslant H(A, B)$. Then there exists a sequence $\{y_n\}$ in B such that $d(x, y_n) \to p$ as $n \to \infty$. Since B is compact, $\{y_n\}$ has a convergent subsequence $\{y_{n(k)}\}$. Hence there exists $y \in X$ such that $y_{n(k)} \to y$ as $k \to \infty$. As B is compact, it is closed. Then $y \in B$. Now, $\lim_{n\to\infty} d(x, y_n) = p$ implies that $\lim_{k\to\infty} d(x, y_{n(k)}) = p$, that is, $d(x, y) = p = D(x, B) \leqslant H(A, B)$. Hence the proof is completed. □

The following is a consequence of Lemma 2.

Lemma 4. *Let* A *and* B *be two nonempty compact subsets of a metric space* (X, d) *and* $T : A \to C(B)$ *be a multivalued mapping. Let* $q \geqslant 1$. *Then for* $a, b \in A$ *and* $x \in Ta$, *there exists* $y \in Tb$ *such that* $d(x, y) \leqslant qH(Ta, Tb)$.

In [26], Samet et al. introduced the concept of α-admissible mappings and utilized these mappings to prove some fixed point results in metric spaces. α-admissible mappings has been used in several fixed point and best proximity point results [7, 8, 20, 23]. Jleli and Samet [21] extend the concept of α-admissible mappings to α-proximal admissible mappings.

Definition 3. (See [21].) Let A, B be two nonempty subsets of a metric space (X, d). Let $T : A \to B$ and $\alpha : A \times A \to [0, \infty)$. The mapping T is called *α-proximal admissible* if for all $x_1, x_2, u_1, u_2 \in A$,

$$\alpha(x_1, x_2) \geqslant 1, \quad d(u_1, Tx_1) = \operatorname{dist}(A, B), \quad d(u_2, Tx_2) = \operatorname{dist}(A, B) \implies \alpha(u_1, u_2) \geqslant 1.$$

Definition 4. (See [21].) Let A, B be two nonempty subsets of a metric space (X,d). Let $T : A \to B$ and $\alpha : A \times A \to [0,\infty)$. Let $\psi : [0,\infty) \to [0,\infty)$ be a nondecreasing and continuous function with $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ and $\psi(t) < t$ for each $t > 0$. The mapping T is called *α-ψ-proximal contraction* if

$$\alpha(x,y)d(Tx,Ty) \leqslant \psi\bigl(d(x,y)\bigr) \quad \text{for all } x,y \in A.$$

In the following, we define two definitions in which we extend the idea of α-proximal admissible mapping and α-ψ-proximal contraction to the set-valued cases.

Definition 5. Let A, B be two nonempty subsets of a metric space (X,d). Let $T : A \to N(B)$ and $\alpha : A \times A \to [0,\infty)$. The multivalued mapping T is called *multivalued α-proximal admissible* if for $x_1, x_2, u_1, u_2 \in A$, $y_1 \in Tx_1$ and $y_2 \in Tx_2$,

$$\begin{aligned} &\alpha(x_1,x_2) \geqslant 1, \quad d(u_1,y_1) = \operatorname{dist}(A,B), \quad d(u_2,y_2) = \operatorname{dist}(A,B) \\ &\quad \Longrightarrow \quad \alpha(u_1,u_2) \geqslant 1. \end{aligned}$$

Definition 6. Let A, B be two nonempty subsets of a metric space (X,d). Let $T : A \to C(B)$ and $\alpha : A \times A \to [0,\infty)$. Let $\psi : [0,\infty) \to [0,\infty)$ be a nondecreasing and continuous function with $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ and $\psi(t) < t$ for each $t > 0$. The multivalued mapping T is called *multivalued α-ψ-proximal contraction* if

$$\alpha(x,y)H(Tx,Ty) \leqslant \psi\bigl(d(x,y)\bigr) \quad \text{for all } x,y \in A.$$

Remark 1. Definitions 5 and 6 reduce to Definitions 3 and 4, respectively, when Tx is singleton set for every $x \in A$.

Definition 7. Let $T : X \to CB(Y)$ be a multivalued mapping, where (X,ρ), (Y,d) are two metric spaces and H is the Hausdorff metric on $CB(Y)$. The mapping T is said to be continuous at $x \in X$ if $H(Tx,Tx_n) \to 0$ whenever $\rho(x,x_n) \to 0$ as $n \to \infty$.

Definition 8. Let A, B be two nonempty subsets of a metric space (X,d) and $T : A \to N(B)$ a multivalued mapping. A point $x^* \in A$ is called *best proximity point of T if* $D(x^*,Tx^*) = \operatorname{dist}(A,B)$.

3 Main results

Theorem 2. *Let (X,d) be a complete metric space and (A,B) be a pair of nonempty closed subsets of X such that A_0 is nonempty. Let $T : A \to C(B)$ be a continuous multi-valued mapping such that*:

(i) *$Tx \subseteq B_0$ for each $x \in A_0$ and (A,B) satisfies the weak P-property*;
(ii) *T is multivalued α-proximal admissible mapping*;
(iii) *there exist $x_0, x_1 \in A_0$ and $y_0 \in Tx_0 \subseteq B_0$ such that $d(x_1,y_0) = \operatorname{dist}(A,B)$ and $\alpha(x_0,x_1) \geqslant 1$*;
(iv) *T is multivalued α-ψ-proximal contraction.*

Then T has a best proximity point in A.

Proof. By condition (iii), there exist $x_0, x_1 \in A_0$ and $y_0 \in Tx_0 \subseteq B_0$ such that

$$d(x_1, y_0) = \operatorname{dist}(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geqslant 1.$$

By Lemma 4 and condition (iv), corresponding to $y_0 \in Tx_0$, there exists $y_1 \in Tx_1$ such that

$$d(y_0, y_1) \leqslant \alpha(x_0, x_1)H(Tx_0, Tx_1) \leqslant \psi\big(d(x_0, x_1)\big).$$

Since $y_1 \in Tx_1 \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(x_2, y_1) = \operatorname{dist}(A, B)$. Now, $x_0, x_1, x_2 \in A_0 \subseteq A$ and $y_0 \in Tx_0$, $y_1 \in Tx_1$ such that $\alpha(x_0, x_1) \geqslant 1$, $d(x_1, y_0) = \operatorname{dist}(A, B)$, $d(x_2, y_1) = \operatorname{dist}(A, B)$. Then it follows from condition (ii) that $\alpha(x_1, x_2) \geqslant 1$. Thus, we have

$$d(x_2, y_1) = \operatorname{dist}(A, B) \quad \text{and} \quad \alpha(x_1, x_2) \geqslant 1.$$

Again, by Lemma 4 and condition (iv), corresponding to $y_1 \in Tx_1$, there exists $y_2 \in Tx_2$ such that

$$d(y_1, y_2) \leqslant \alpha(x_1, x_2)H(Tx_1, Tx_2) \leqslant \psi\big(d(x_1, x_2)\big).$$

Since $y_2 \in Tx_2 \subseteq B_0$, there exists $x_3 \in A_0$ such that $d(x_3, y_2) = \operatorname{dist}(A, B)$. So, $x_1, x_2, x_3 \in A_0 \subseteq A$ and $y_1 \in Tx_1$, $y_2 \in Tx_2$ such that $\alpha(x_1, x_2) \geqslant 1$, $d(x_2, y_1) = \operatorname{dist}(A, B)$, $d(x_3, y_2) = \operatorname{dist}(A, B)$. Then it follows from condition (ii) that $\alpha(x_2, x_3) \geqslant 1$. Thus, we have

$$d(x_3, y_2) = \operatorname{dist}(A, B) \quad \text{and} \quad \alpha(x_2, x_3) \geqslant 1.$$

By Lemma 4 and condition (iv), corresponding to $y_2 \in Tx_2$, there exists $y_3 \in Tx_3$ such that

$$d(y_2, y_3) \leqslant \alpha(x_2, x_3)H(Tx_2, Tx_3) \leqslant \psi\big(d(x_2, x_3)\big).$$

Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ respectively in $A_0 \subseteq A$ and $B_0 \subseteq B$ such that for $n = 0, 1, 2, \dots,$

$$y_n \in Tx_n, \quad (x_{n+1}, y_n) = \operatorname{dist}(A, B) \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geqslant 1,$$

and also

$$d(y_n, y_{n+1}) \leqslant \alpha(x_n, x_{n+1})H(Tx_n, Tx_{n+1}) \leqslant \psi\big(d(x_n, x_{n+1})\big). \tag{2}$$

Since, $d(x_{n+1}, y_n) = \operatorname{dist}(A, B)$ and $d(x_n, y_{n-1}) = \operatorname{dist}(A, B)$ for all $n \geqslant 1$, it follows by the weak P-property of the pair (A, B) that

$$d(x_n, x_{n+1}) \leqslant d(y_{n-1}, y_n) \quad \text{for all } n \in N. \tag{3}$$

Applying (2) repeatedly and using (3) and the monotone property of ψ, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant d(y_{n-1}, y_n) \leqslant \alpha(x_{n-1}, x_n)H(Tx_{n-1}, Tx_n) \\ &\leqslant \psi\big(d(x_{n-1}, x_n)\big) \leqslant \psi\big(d(y_{n-2}, y_{n-1})\big) \\ &\leqslant \psi\big(\alpha(x_{n-2}, x_{n-1})H(Tx_{n-2}, Tx_{n-1})\big) \\ &\leqslant \psi^2\big(d(x_{n-2}, x_{n-1})\big) \leqslant \cdots \leqslant \psi^n\big(d(x_0, x_1)\big). \end{aligned}$$

Then by a property of ψ, we have

$$\sum_n d(x_n, x_{n+1}) \leqslant \sum_n \psi^n\big(d(x_0, x_1)\big) < \infty.$$

This shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. From the completeness of X, there exist $x^*, y^* \in X$ such that

$$x_n \to x^* \quad \text{and} \quad y_n \to y^* \quad \text{as } n \to \infty. \tag{4}$$

Since A and B are closed and $\{x_n\}$ and $\{y_n\}$ are sequences in A and B, respectively, we have $x^* \in A$ and $y^* \in B$.

Now,

$$d(x_{n+1}, y_n) = \operatorname{dist}(A, B) \quad \text{for all } n \in N.$$

Taking limit as $n \to \infty$, we obtain

$$d(x^*, y^*) = \operatorname{dist}(A, B). \tag{5}$$

Now, we claim that $y^* \in Tx^*$.

Since $y_n \in Tx_n$, we have

$$D(y_n, Tx^*) \leqslant H(Tx_n, Tx^*).$$

Taking limit as $n \to \infty$ in the above inequality, and using (4) and the continuity of T, we have

$$D(y^*, Tx^*) = \lim_{n\to\infty} D(y_n, Tx^*) \leqslant \lim_{n\to\infty} H(Tx_n, Tx^*) = 0.$$

Since $Tx^* \in C(B)$, Tx^* is compact and hence Tx^* is closed, that is, $\overline{Tx^*} = Tx^*$. Now, $D(y^*, Tx^*) = 0$ implies $y^* \in \overline{Tx^*} = Tx^*$.

Now, using (5), we have

$$D(x^*, Tx^*) \leqslant d(x^*, y^*) = \operatorname{dist}(A, B) \leqslant D(x^*, Tx^*),$$

, which implies that $D(x^*, Tx^*) = \operatorname{dist}(A, B)$, that is, x^* is a best proximity point of T in A. □

In the next theorem, we replace the continuity assumption of T with another condition involving α.

Theorem 3. *Let (X, d) be a complete metric space and (A, B) be a pair of nonempty closed subsets of X such that A_0 is nonempty. Assume that if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geqslant 1$ for all k. Let $T : A \to C(B)$ be a multi-valued mapping such that conditions* (i)–(iv) *of Theorem* 2 *are satisfied. Then T has a best proximity point in A.*

Proof. Arguing like in the proof of Theorem 2, we construct two sequences $\{x_n\}$ and $\{y_n\}$ respectively in A and B such that:

$$\alpha(x_n, x_{n+1}) \geqslant 1,$$
$$x_n \to x^* \in A \quad \text{and} \quad y_n \to y^* \in B \quad \text{as } n \to \infty$$

and

$$d(x^*, y^*) = \mathrm{dist}(A, B).$$

By the assumption, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geqslant 1$ for all k. Since $y_{n(k)} \in Tx_{n(k)}$ for all $k \geqslant 1$, applying condition (iv), we get

$$\begin{aligned} D\bigl(y_{n(k)}, Tx^*\bigr) &\leqslant H\bigl(Tx_{n(k)}, Tx^*\bigr) \leqslant \alpha\bigl(x_{n(k)}, x^*\bigr) H\bigl(Tx_{n(k)}, Tx^*\bigr) \\ &\leqslant \psi\bigl(d\bigl(x_{n(k)}, x^*\bigr)\bigr). \end{aligned}$$

Taking the limit as $k \to \infty$ in the above inequality and using the continuity of ψ, we obtain $D(y^*, Tx^*) = 0$. Then arguing like in the proof of Theorem 2, we have that $D(x^*, Tx^*) = \mathrm{dist}(A, B)$, that is, x^* is a best proximity point of T in A. □

Since every pair (A, B) having the weak P-property, also satisfies the P-property, we have the following corollaries.

Corollary 1. *Let (X, d) be a complete metric space and (A, B) be a pair of nonempty closed subsets of X such that A_0 is nonempty. Let $T : A \to C(B)$ be a continuous multi-valued mapping such that*:

(i) *$Tx \subseteq B_0$ for each $x \in A_0$ and (A, B) satisfies the P-property*;
(ii) *T is multivalued α-proximal admissible mapping*;
(iii) *there exist $x_0, x_1 \in A_0$ and $y_0 \in Tx_0 \subseteq B_0$ such that $d(x_1, y_0) = \mathrm{dist}(A, B)$ and $\alpha(x_0, x_1) \geqslant 1$*;
(iv) *T is multivalued α-ψ-proximal contraction.*

Then T has a best proximity point in A.

Corollary 2. *Let (X, d) be a complete metric space and (A, B) be a pair of nonempty closed subsets of X such that A_0 is nonempty. Assume that if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geqslant 1$ for all k. Let $T : A \to C(B)$ be a multi-valued mapping such that conditions* (i)–(iv) *of Corollary* 1 *are satisfied. Then T has a best proximity point in A.*

Example 1. Let $X = \mathbb{R}^2$. The metric d on X is given by

$$d(x, y) = |x_1 - x_2| + |y_1 - y_2|, \quad x = (x_1, y_1),\ y = (x_2, y_2) \in X.$$

Let

$$\begin{aligned} A &= \bigl\{(-1, 1), (1, 1), (0, 2)\bigr\}, \\ B &= \bigl\{(-4, v)\colon -4 \leqslant v \leqslant 0\bigr\} \cup \bigl\{(4, v)\colon -4 \leqslant v \leqslant 0\bigr\} \\ &\quad \cup \bigl\{(u, v)\colon -4 < u < 4,\ v = -4\bigr\}, \\ A_0 &= \bigl\{(-1, 1), (1, 1)\bigr\} \quad \text{and} \quad B_0 = \bigl\{(-4, 0), (4, 0)\bigr\}. \end{aligned}$$

Notice that

$$d\big((-1,1),(-4,0)\big) = \operatorname{dist}(A,B) = 4, \qquad d((1,1),(4,0)) = \operatorname{dist}(A,B) = 4$$

and

$$d\big((-1,1),(1,1)\big) = 2, \qquad d\big((-4,0),(4,0)\big) = 8.$$

This shows that the pair (A,B) satisfies the weak P-property.

Let $T : A \to C(B)$ be defined as follows:

$$Tx = \begin{cases} \{(-4,0)\} & \text{if } x = (-1,1), \\ \{(4,0)\} & \text{if } x = (1,1), \\ \{(u,v)\colon -4 \leqslant u \leqslant 4,\ v = -4\} & \text{if } x = (0,2). \end{cases}$$

Let $\alpha : A \times A \to [0,\infty)$ be defined as follows:

$$\alpha(x,y) = \begin{cases} 2 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Let $\psi : [0,\infty) \to [0,\infty)$ be defined as $\psi(t) = ct$, $0 < c < 1$. Then all the conditions of Theorem 2 are satisfied. Here $(-1,1)$, $(1,1)$ are the best proximity points of T in A.

Example 2. We take the metric space (X,d) as considered in Example 1. Let

$$\begin{aligned} A &= \big\{(u,v)\colon u = -1,\ 1 \leqslant v \leqslant 2\big\} \cup \big\{(u,v)\colon -1 \leqslant u \leqslant 1,\ v = 2\big\} \\ &\quad \cup \big\{(u,v)\colon u = 1,\ 1 \leqslant v \leqslant 2\big\}, \\ B &= \big\{(u,v)\colon 4 \leqslant u \leqslant 8,\ -4 \leqslant v \leqslant 0\big\} \cup \big\{(u,v)\colon -8 \leqslant u \leqslant -4,\ -4 \leqslant v \leqslant 0\big\}, \\ A_0 &= \big\{(-1,1),(1,1)\big\} \quad \text{and} \quad B_0 = \big\{(-4,0),(4,0)\big\}. \end{aligned}$$

Noticed that

$$d\big((-1,1),(-4,0)\big) = \operatorname{dist}(A,B) = 4, \qquad d\big((1,1),(4,0)\big) = \operatorname{dist}(A,B) = 4$$

and

$$d\big((-1,1),(1,1)\big) = 2, \qquad d\big((-4,0),(4,0)\big) = 8.$$

This shows that the pair (A,B) satisfies the weak P-property.

Define the mapping $T : A \to C(B)$ by

$$Tx = \begin{cases} \{(-4,0)\} & \text{if } x = (-1,1), \\ \{(4,0)\} & \text{if } x = (1,1), \\ \{(u,v)\colon u = -8,\ -4 \leqslant v \leqslant 0\} & \text{if } x \in \big\{(a,b)\colon a = -1,\ 1 < b \leqslant 2\big\}, \\ \{(u,v)\colon -8 \leqslant u \leqslant -4,\ v = 4\} & \text{if } x \in \{(a,b)\colon -1 < a \leqslant 1,\ b = 2\}, \\ \{(u,v)\colon u = 8,\ -4 \leqslant v \leqslant 0\} & \text{if } x \in \{(a,b)\colon a = 1,\ 1 < b < 2\}. \end{cases}$$

We define the mapping $\alpha : A \times A \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 2 & \text{if } x = y \in \{(-1,1),(1,1)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\psi : [0,\infty) \to [0,\infty)$ be defined as $\psi(t) = ct$, $0 < c < 1$. Here all the conditions of Theorem 3 are satisfied and it is seen that $(-1,1)$ and $(1,1)$ are the best proximity points of T in A.

Remark 2. In each of the above examples (Examples 1 and 2), the pair (A,B) does not satisfy the P-property and hence Corollaries 1 and 2 are not applicable to Examples 1 and 2, respectively. Therefore, Theorems 2 and 3 properly contain Corollaries 1 and 2, respectively.

4 Application to single-valued mappings

In this section, we obtain some best proximity point results for single-valued mappings by an application of the corresponding results of previous section.

Theorem 4. *Let (X,d) be a complete metric space and (A,B) be a pair of nonempty closed subsets of X such that A_0 is nonempty. Let $T : A \to B$ be a continuous mapping such that*:

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the weak P-property*;
(ii) *T is α-proximal admissible mapping*;
(iii) *there exist $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = \operatorname{dist}(A,B)$ and $\alpha(x_0,x_1) \geqslant 1$*;
(iv) *T is α-ψ-proximal contraction.*

Then T has a best proximity point in A.

Proof. We know that for every $x \in X$, $\{x\}$ is compact in X. Now, we define multivalued mapping $S : A \to C(B)$ as $Sx = \{Tx\}$ for $x \in A$. The continuity of T implies that S is continuous.

By condition (i) of the theorem, $Sx = \{Tx\} \subseteq B_0$ for each $x \in A_0$.

Let $x_1, x_2, u_1, u_2 \in A$, $y_1 \in Sx_1 = \{Tx_1\}$ and $y_2 \in Sx_2 = \{Tx_2\}$ such that

$$\alpha(x_1,x_2) \geqslant 1, \quad d(u_1,y_1) = \operatorname{dist}(A,B) \quad \text{and} \quad d(u_2,y_2) = \operatorname{dist}(A,B).$$

Then it follows that for $x_1, x_2, u_1, u_2 \in A$,

$$\alpha(x_1,x_2) \geqslant 1, \quad d(u_1,Tx_1) = \operatorname{dist}(A,B) \quad \text{and} \quad d(u_2,Tx_2) = \operatorname{dist}(A,B).$$

By condition (ii) of the theorem, we have $\alpha(u_1,u_2) \geqslant 1$. So, we have that for x_1, x_2, $u_1, u_2 \in A$, $y_1 \in Sx_1$ and $y_2 \in Sx_2$,

$$\alpha(x_1,x_2) \geqslant 1, \quad d(u_1,y_1) = \operatorname{dist}(A,B), \quad d(u_2,y_2) = \operatorname{dist}(A,B) \\ \implies \quad \alpha(u_1,u_2) \geqslant 1,$$

that is, S is a multivalued α-proximal admissible mapping.

Suppose there exist $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = \mathrm{dist}(A,B)$ and $\alpha(x_0, x_1) \geqslant 1$. Let $y_0 \in Sx_0 = \{Tx_0\} \subseteq B_0$. Then $d(x_1, Tx_0) = \mathrm{dist}(A,B)$ means $d(x_1, y_0) = \mathrm{dist}(A,B)$. Therefore, by condition (iii) of the theorem, we have that there exist $x_0, x_1 \in A_0$ and $y_0 \in Sx_0 \subseteq B_0$ such that $d(x_1, y_0) = \mathrm{dist}(A,B)$ and $\alpha(x_0, x_1) \geqslant 1$.

Let $x, y \in A$. Then using condition (iv) of the theorem, we have

$$\alpha(x,y)H(Sx,Sy) = \alpha(x,y)d(Tx,Ty) \leqslant \psi\big(d(x,y)\big),$$

that is, S is a multivalued α-ψ-proximal contraction.

So, all the conditions of Theorem 2 are satisfied and hence S has a best proximity point x^* in A. Then $D(x^*, Sx^*) = \mathrm{dist}(A,B)$, that is, $d(x^*, Tx^*) = \mathrm{dist}(A,B)$, that is, x^* is a best proximity point of T in A. □

Theorem 5. *Let (X,d) be a complete metric space and (A,B) be a pair of nonempty closed subsets of X such that A_0 is nonempty. Assume that if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geqslant 1$ for all k. Let $T : A \to B$ be a mapping such that conditions* (i)–(iv) *of Theorem* 4 *are satisfied. Then T has a best proximity point in A.*

Proof. Like in the proof of Theorem 4, we define the multivalued mapping $S : A \to C(B)$. Arguing similarly as in the proof of Theorem 4, we prove that

- $Sx \subseteq B_0$ for each $x \in A_0$;
- S is a multivalued α-proximal admissible mapping;
- there exist $x_0, x_1 \in A_0$ and $y_0 \in Sx_0 \subseteq B_0$ such that $d(x_1, y_0) = \mathrm{dist}(A,B)$ and $\alpha(x_0, x_1) \geqslant 1$;
- S is a multivalued α-ψ-proximal contraction.

So, all the conditions of Theorem 3 are satisfied and hence S has a best proximity point x^* in A. Then $D(x^*, Sx^*) = \mathrm{dist}(A,B)$, that is, $d(x^*, Tx^*) = \mathrm{dist}(A,B)$, that is, x^* is a best proximity point of T in A. □

Corollary 3. *Let (X,d) be a complete metric space and (A,B) be a pair of nonempty closed subsets of X such that A_0 is nonempty. Let $T : A \to B$ be a continuous mapping such that*:

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the P-property*;
(ii) *T is α-proximal admissible mapping*;
(iii) *there exist $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = \mathrm{dist}(A,B)$ and $\alpha(x_0, x_1) \geqslant 1$*;
(iv) *T is α-ψ-proximal contraction.*

Then T has a best proximity point in A.

Proof. The proof is similar to that of Theorem 4. □

Corollary 4. *Let (X,d) be a complete metric space and (A,B) be a pair of nonempty closed subsets of X such that A_0 is nonempty. Assume that if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists* *a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geqslant 1$ *for all* k*. Let* $T : A \to B$ *be a mapping such that conditions* (i)–(iv) *of Corollary* 3 *are satisfied. Then* T *has a best proximity point in* A*.*

Proof. The proof is similar to that of Theorem 5. □

Remark 3. Corollaries 3 and 4 are respectively Theorems 3.1 and 3.2 of Jleli and Samet [21].

5 Conclusion

This paper is an application of multifunctions to a global optimality problem. Fixed point methods in set-valued analysis is used since this problem is viewed here as that of finding an optimal approximate solution of a fixed point inclusion. For this purpose, new types of set-valued contractive functions are introduced. This approach to finding minimum distance between two sets may be adopted in future works.

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