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Finite-time generalized synchronization of nonidentical delayed chaotic systems*

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Abstract. This paper deals with the finite-time generalized synchronization (GS) problem of drive-response systems. The main purpose of this paper is to design suitable controllers to force the drive-response systems to realize GS in a finite time. Based on the finite-time stability theory and nonlinear control theory, sufficient conditions are derived that guarantee finite-time GS. This paper extends some basic results from generalized synchronization to delayed systems. Because finite-time GS means the optimality in convergence time and has better robustness, the results in this paper are important. Numerical examples are given to show the effectiveness of the proposed control techniques.

Keywords: finite-time generalized synchronization, drive-response systems, nonlinear control.

1 Introduction

Synchronization is a very important phenomenon in nature, and has been widely applied in various fields such as chemical reactors, biological systems, information processing, secure communication, etc. There are various kinds of synchronization, such as complete synchronization (CS) [4, 11, 18, 22, 47, 54], phase synchronization [38], lag synchronization [9,23,33,41,51], anticipating synchronization [13,32], anti-synchronization [36,37], projective synchronization [43,53], generalized synchronization [1,6,12,15,20,35,40,46, 48].

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As one of the kinds of synchronization, CS requires two identical systems, but it is difficult to find two identical systems in practice because of parameter mismatch and distortion. Therefore, it is necessary to investigate generalized synchronization between two nonidentical systems. GS, which means the state of the response system synchronizes that of the drive system through a nonlinear smooth functional mapping, has been given much more attention. GS is an extension of CS, and is a robust synchronization [21, 40], that is, when the parameter of systems changed, the systems can still preserve GS. Since this character, GS has many applications in practice, such as secure communication, biological systems.

Up to now, there are two main approaches used to study GS. One is the auxiliary system approach [1, 6, 12, 15, 20, 46], introduced by Abarbanel et al. [1], which makes an identical duplication of the response system that is driven by the same driving signals. If the response system and the auxiliary system achieve CS, then the drive system and the response system can realize GS. This approach has the disadvantage that it fails to decide what kind of functional relation exists between the drive and response systems. The other is to design a controller to force two coupled systems to satisfy a prescribed functional relation [35]. If we want to know the exact functional relation between two systems, this approach is effective for studying GS of networks.

Recently, there are some papers about finite time consensus [42], stability [2, 3, 7, 44, 52], finite time boundedness [14], finite time parameter identification [33], stabilization of general control systems [16, 19, 28, 29] and finite time synchronization of networks [8, 10, 17, 34, 49, 50]. It is noticed that, most results about synchronization are related to an infinite-time asymptotical process, that is, only when the time tends to infinity, the drive-response systems can reach GS, and in theory, this will not occur in a finite time. But in practice, especially in physical and engineering systems, we often require systems achieve GS in a finite time, so it is significant to investigate the finite-time GS of networks. In [49], finite time synchronization between two different chaotic systems with uncertain parameters was investigated. Yang and Cao discussed finite-time synchronization of complex networks with stochastic disturbances [50]. In [8], Chen and Lü investigated finite time synchronization of complex dynamical networks. In [17], finite-time lag synchronization of delayed neural networks was investigated.

To the best of the authors' knowledge, the finite-time GS of two nonidentical systems has not been studied in details yet, this motivated our current research interest. In this paper, we will investigate the finite-time GS between nonidentical systems. Control laws are designed to achieve finite-time GS of drive-response systems. Based on the finite-time stability theory and nonlinear control theory, finite-time GS conditions are given. The main contribution of this paper is it provides an effective controller to realize the finite-time GS of nonidentical systems. Thus, it has more practical applications than the results of infinite-time.

The remainder of this paper is organized as follows. In Section 2, the model formulation and some preliminaries are given. The main results are stated in Section 3. Three illustrate examples are given to demonstrate the effectiveness of the proposed results in Section 4. Finally, the conclusion is made in Section 5.

2 Notations and preliminaries

In this section, some elementary notations and lemmas are introduced which play an important role in the proof of the main results in Section 3.

Notation. Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The superscript "T" denotes vector transposition. $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a matrix, $\|A\|$ denotes its operator norm, i.e. $\|A\| = \sup\{|Ax|\colon |x|=1\} = \sqrt{\lambda_{\max}(A^{\mathrm{T}}A)}$, where $\lambda_{\max}(A)$ means the largest eigenvalue of A.

Consider the following drive-response chaotic systems:

$$\dot{x}(t) = f(x), \tag{1}$$

$$\dot{y}(t) = Ay(t) + Bg\bigl(y(t)\bigr) + Cg\bigl(y\bigl(t - \tau(t)\bigr)\bigr), \tag{2}$$

where $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots, y_m)^{\mathrm{T}} \in \mathbb{R}^m$ are the state vectors of the drive system and response system, respectively. $f(\cdot)$ and $g(\cdot)$ are continuous vector functions. A, B and C are system matrices with proper dimension. $\tau(t)$ is the time-varying delay of system (2), $0 \leqslant \tau_1 \leqslant \tau(t) \leqslant \tau_2$, $\dot{\tau}(t) \leqslant h$, τ_1, τ_2, and h are constants. The initial values of systems (1) and (2) are

$$x(t_0) = x_0, \tag{3}$$

$$y(s) = \phi(s), \quad s \in [t_0 - \tau_2, t_0], \tag{4}$$

where x_0 is a vector, $\phi(\cdot) = [\phi_1(\cdot), \phi_2(\cdot), \dots, \phi_m(\cdot)]^{\mathrm{T}} \in C([t_0 - \tau_2, t_0], \mathbb{R}^m)$.

Assumption 1. The function g satisfies Lipschitz condition, that is, there exists a positive constant $L > 0$ such that, for all $s_1, s_2 \in \mathbb{R}^m$,

$$\bigl\|g(s_1) - g(s_2)\bigr\| \leqslant L\|s_1 - s_2\|.$$

Consider the controlled response system

$$\dot{y}(t) = Ay(t) + Bg\bigl(y(t)\bigr) + Cg\bigl(y\bigl(t - \tau(t)\bigr)\bigr) + u(t), \tag{5}$$

where $u(t)$ is a controller. Our aim is to design a proper controller to force the controlled response system to achieve GS with the drive system in a finite time.

Definition 1. Given a vector mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^m$, if there exists a constant $t^* > 0$, (t^* depends on the initial vector values $x(0)$ and $y(0)$) such that

$$\lim_{t \to t^{*-}} \bigl\|y(t) - \Phi\bigl(x(t)\bigr)\bigr\| = 0$$

and $\|y(t) - \Phi(x(t))\| = 0$ for $t > t^*$, then we say systems (1) and (5) achieve generalized synchronization in a finite time.

To obtain our main results, we need the following lemmas.

Lemma 1. (*See* [10].) *Assume that a continuous, positive-definite function* $V(t)$ *satisfies the following differential inequality*:

$$\dot{V}(t) \leqslant -\alpha V^{\eta}(t) \quad \forall t \geqslant t_0,\ V(t_0) \geqslant 0,$$

where $\alpha > 0$, $0 < \eta < 1$ *are all constants. Then, for any given* t_0, $V(t)$ *satisfies the following inequality*:

$$V^{1-\eta}(t) \leqslant V^{1-\eta}(t_0) - \alpha(1-\eta)(t-t_0), \quad t_0 \leqslant t \leqslant t_1,$$

and $V(t) \equiv 0$ *for all* $t \geqslant t_1$ *with* t_1 *given by*

$$t_1 = t_0 + \frac{V^{1-\eta}(t_0)}{\alpha(1-\eta)}.$$

Lemma 2. (*See* [45].) *If* $a_1, a_2, \ldots, a_n$ *are positive number and* $0 < r < p$, *then*

$$\left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \leqslant \left(\sum_{i=1}^{n} a_i^r\right)^{1/r}.$$

3 Main results

Let the synchronization errors between drive system (1) and controlled response system (5) be

$$e(t) = y(t) - \Phi\big(x(t)\big),$$

then the error system between (1) and (5) can be described as

$$\dot{e}(t) = Ay(t) + Bg\big(y(t)\big) + Cg\big(y\big(t-\tau(t)\big)\big) - D\Phi(x)\cdot f(x) + u(t). \tag{6}$$

Therefore, the finite-time GS problem between systems (1) and (5) is equivalent to the finite-time stability problem of the error system (6) at the origin. In order to achieve this aim, we design

$$\begin{aligned} u(t) = {} & -\Gamma e(t) - k\,\mathrm{sgn}\big(e(t)\big)\big|e(t)\big|^{\alpha} + D\Phi(x)\cdot f(x) - Bg\big(\Phi(x)\big) - A\Phi(x) \\ & - Cg\big(\phi\big(x\big(t-\tau(t)\big)\big)\big) - k\left(\int\limits_{t-\tau(t)}^{t} e^{\mathrm{T}}(s)e(s)\,\mathrm{d}s\right)^{(1+\alpha)/2} \frac{e(t)}{\|e(t)\|^2}, \end{aligned} \tag{7}$$

where

$$\big|e(t)\big|^{\alpha} = \big(\big|e_1(t)\big|^{\alpha}, \big|e_2(t)\big|^{\alpha}, \ldots, \big|e_m(t)\big|^{\alpha}\big)^{\mathrm{T}},$$
$$\mathrm{sgn}\,\big(e(t)\big) = \mathrm{diag}\big(\mathrm{sgn}\big(e_1(t)\big), \mathrm{sgn}\big(e_2(t)\big), \ldots, \mathrm{sgn}\big(e_m(t)\big)\big),$$

$\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_m) > 0$ is a positive matrix which will be determined later. $k > 0$ is a tunable constant, the real number α satisfies $0 < \alpha < 1$, $D\Phi(x)$ is the Jacobian matrix of the mapping $\Phi(x)$.

Remark 1. When $0 < \alpha < 1$, the controller $u(t)$ is a continuous function with respect t, which leads to the continuity of controlled response system (5) with respect to the system state. If $\alpha = 0$, $u(t)$ turns to be discontinuous one, which is similar to the controller that have been considered in [31]. If $\alpha = 1$ in the controller (7), then they become typical feedback control issues, which only can realize an asymptotical synchronization in an infinite time.

Substituting (7) into (6), we have

$$\begin{aligned}\dot{e}(t) &= Ae(t) - \mathit{\Gamma} e(t) + B\bigl[g\bigl(y(t)\bigr) - g\bigl(\mathit{\Phi}\bigl(x(t)\bigr)\bigr)\bigr] \\ &\quad + C\bigl[g\bigl(y\bigl(t-\tau(t)\bigr)\bigr) - g\bigl(\phi\bigl(x\bigl(t-\tau(t)\bigr)\bigr)\bigr)\bigr] \\ &\quad - k\,\mathrm{sgn}\bigl(e(t)\bigr)\bigl|e(t)\bigr|^{\alpha} - k\Biggl(\int\limits_{t-\tau(t)}^{t} e^{\mathrm{T}}(s)e(s)\,\mathrm{d}s\Biggr)^{(1+\alpha)/2} \frac{e(t)}{\|e(t)\|^2}. \end{aligned} \tag{8}$$

Theorem 1. *Suppose Assumption* 1 *holds and there exists a constant* $l > 0$ *satisfying the following conditions*:

$$\|A\| + L\|B\| + \frac{1}{2}L\|C\| - \min_i \gamma_i + l < 0,$$
$$\frac{1}{2}L\|C\| - l(1-h) < 0,$$

then under controller (7)*, the controlled drive-response systems* (1) *and* (5) *will realize finite-time GS in a finite time*

$$t^* = \frac{V(0)^{(1-\alpha)/2}}{k(1-\alpha)/2},$$

where $V(0) = \sum_{i=1}^{m} e_i^2(0)/2$, $e_i(0) = y_i(0) - \mathit{\Phi}_i(x(0))$ $(i = 1, 2, \dots, m)$.

Proof. Construct the following Lyapunov–Krasovskii functional $V(t)$ by

$$V(t) = \frac{1}{2}\bigl\|e(t)\bigr\|^2 + l\int\limits_{t-\tau(t)}^{t} e^{\mathrm{T}}(s)e(s)\,\mathrm{d}s.$$

Calculating the time derivative of $V(t)$ along the trajectory of (8), we have

$$\begin{aligned}\dot{V}(t) &= e^{\mathrm{T}}(t)Ae(t) - e^{\mathrm{T}}(t)\mathit{\Gamma} e(t) + le^{\mathrm{T}}(t)e(t) \\ &\quad - le^{\mathrm{T}}\bigl(t-\tau(t)\bigr)e\bigl(t-\tau(t)\bigr)\bigl(1-\dot{\tau}(t)\bigr) \\ &\quad + e^{\mathrm{T}}(t)B\bigl[g\bigl(y(t)\bigr) - g\bigl(\mathit{\Phi}\bigl(x(t)\bigr)\bigr)\bigr] \\ &\quad + e^{\mathrm{T}}(t)C\bigl[g\bigl(y\bigl(t-\tau(t)\bigr)\bigr) - g\bigl(x\bigl(t-\tau(t)\bigr)\bigr)\bigr] \\ &\quad - ke^{\mathrm{T}}(t)\,\mathrm{sgn}\bigl(e(t)\bigr)\bigl|e(t)\bigr|^{\alpha} - k\Biggl(\int\limits_{t-\tau(t)}^{t} e^{\mathrm{T}}(s)e(s)\,\mathrm{d}s\Biggr)^{(1+\alpha)/2}\end{aligned}$$

$$\begin{aligned}
&\leqslant \big(\|A\|+l\big)\big\|e(t)\big\|^2-\min_i\gamma_i\big\|e(t)\big\|^2\\
&\quad+\big\|e(t)\big\|\|B\|\big\|g\big(y(t)\big)-g\big(\Phi\big(x(t)\big)\big)\big\|-l(1-h)\big\|e\big(t-\tau(t)\big)\big\|^2\\
&\quad+\big\|e(t)\big\|\|C\|\big\|g\big(y\big(t-\tau(t)\big)\big)-g\big(\phi\big(x\big(t-\tau(t)\big)\big)\big)\big\|\\
&\quad-k\sum_{i=1}^m e_i^{\mathrm T}(t)\operatorname{sgn}\big(e_i(t)\big)\big|e_i(t)\big|^\alpha-k\Bigg(\int\limits_{t-\tau(t)}^t e^{\mathrm T}(s)e(s)\,\mathrm ds\Bigg)^{(1+\alpha)/2}\\
&\leqslant \big(\|A\|+l\big)\big\|e(t)\big\|^2-\min_i\gamma_i\big\|e(t)\big\|^2\\
&\quad+L\|B\|\big\|e(t)\big\|^2+L\|C\|\big\|e(t)\big\|\big\|e\big(t-\tau(t)\big)\big\|-l(1-h)\big\|e\big(t-\tau(t)\big)\big\|^2\\
&\quad-k\sum_{i=1}^m\big|e_i(t)\big|^{\alpha+1}-k\Bigg(\int\limits_{t-\tau(t)}^t e^{\mathrm T}(s)e(s)\,\mathrm ds\Bigg)^{(1+\alpha)/2}\\
&\leqslant\bigg(\|A\|+L\|B\|+\frac12L\|C\|-\min_i\gamma_i+l\bigg)\big\|e(t)\big\|^2\\
&\quad+\bigg(\frac12L\|C\|-l(1-h)\bigg)\big\|e\big(t-\tau(t)\big)\big\|^2\\
&\quad-k\sum_{i=1}^m\big|e_i(t)\big|^{\alpha+1}-k\Bigg(\int\limits_{t-\tau(t)}^t e^{\mathrm T}(s)e(s)\,\mathrm ds\Bigg)^{(1+\alpha)/2}.
\end{aligned}\tag{9}$$

From Lemma 2, we get

$$\Bigg(\sum_{i=1}^m\big|e_i(t)\big|^2\Bigg)^{1/2}\leqslant\Bigg(\sum_{i=1}^m\big|e_i(t)\big|^{\alpha+1}\Bigg)^{1/(\alpha+1)}.$$

Hence,

$$\Bigg(\sum_{i=1}^m\big|e_i(t)\big|^2\Bigg)^{(\alpha+1)/2}\leqslant\sum_{i=1}^m\big|e_i(t)\big|^{\alpha+1}.$$

Therefore,

$$\begin{aligned}
\dot V(t)&\leqslant\bigg(\|A\|+L\|B\|+\frac12L\|C\|-\min_i\gamma_i+l\bigg)\big\|e(t)\big\|^2\\
&\quad+\bigg(\frac12L\|C\|-l(1-h)\bigg)\big\|e\big(t-\tau(t)\big)\big\|^2\\
&\quad-k\Bigg(\sum_{i=1}^m\big|e_i(t)\big|^2\Bigg)^{(\alpha+1)/2}-k\Bigg(\int\limits_{t-\tau(t)}^t e^{\mathrm T}(s)e(s)\,\mathrm ds\Bigg)^{(1+\alpha)/2}
\end{aligned}$$

$$\leqslant -k\left(\sum_{i=1}^{m}\left|e_i(t)\right|^2\right)^{(\alpha+1)/2} - k\left(\int\limits_{t-\tau(t)}^{t} e^{\mathrm{T}}(s)e(s)\,\mathrm{d}s\right)^{(1+\alpha)/2}$$

$$\leqslant -kV^{(\alpha+1)/2}. \tag{10}$$

By Lemma 1, $V(t)$ converges to zero in a finite time, and the finite time is estimated by

$$t^* = \frac{V(0)^{(1-\alpha)/2}}{k(1-\alpha)/2}.$$

Hence, the error vector $e(t)$ will converge to zero within t^*. Consequently, under controller (7), systems (1) and (5) realize finite-time GS. This completes the proof of the theorem. □

If $C = 0$, then the controlled response system becomes

$$\dot{y}(t) = Ay(t) + Bg\bigl(y(t)\bigr) + u(t), \tag{11}$$

we design

$$u(t) = -\mathit{\Gamma} e(t) - k\,\mathrm{sgn}\bigl(e(t)\bigr)\bigl|e(t)\bigr|^{\alpha} + D\mathit{\Phi}(x)\cdot f(x) \\ - Bg\bigl(\mathit{\Phi}\bigl(x(t)\bigr)\bigr) - A\mathit{\Phi}\bigl(x(t)\bigr), \tag{12}$$

and the error system becomes

$$\dot{e}(t) = Ae(t) - \mathit{\Gamma} e(t) + B\bigl[g\bigl(y(t)\bigr) - g\bigl(\mathit{\Phi}\bigl(x(t)\bigr)\bigr)\bigr] \\ - k\,\mathrm{sgn}\bigl(e(t)\bigr)\bigl|e(t)\bigr|^{\alpha}. \tag{13}$$

Theorem 2. *Suppose Assumption* 1 *and the following condition hold*:

$$\|A\| + L\|B\| - \min_i \gamma_i < 0,$$

then under controller (12)*, the controlled drive-response systems* (1) *and* (11) *will realize finite-time GS in a finite time*

$$t^* = \frac{V(0)^{(1-\alpha)/2}}{k2^{(1+\alpha)/2}(1-\alpha)/2},$$

where $V(0) = \sum_{i=1}^{m} e_i^2(0)/2$, $e_i(0) = y_i(0) - \mathit{\Phi}_i(x(0))$ $(i = 1, 2, \dots, m)$.

Proof. Construct the following Lyapunov–Krasovskii functional $V(t)$ by

$$V(t) = \frac{1}{2}e^{\mathrm{T}}(t)e(t) = \frac{1}{2}\bigl\|e(t)\bigr\|^2.$$

Calculating the time derivative of $V(t)$ along the trajectory of (13), we have

$$\dot{V}(t) = e^{\mathrm{T}}(t)Ae(t) - e^{\mathrm{T}}(t)\mathit{\Gamma} e(t) \\ + e^{\mathrm{T}}(t)B\bigl[g\bigl(y(t)\bigr) - g\bigl(\mathit{\Phi}\bigl(x(t)\bigr)\bigr)\bigr] - ke^{\mathrm{T}}(t)\,\mathrm{sgn}\bigl(e(t)\bigr)\bigl|e(t)\bigr|^{\alpha}$$

$$
\begin{aligned}
&\leqslant \|A\|\big\|e(t)\big\|^2 - \min_i \gamma_i \big\|e(t)\big\|^2 + \big\|e(t)\big\|\|B\|\big\|g\big(y(t)\big) - g\big(\Phi\big(x(t)\big)\big)\big\| \\
&\quad - k\sum_{i=1}^{m} e_i^{\mathrm{T}}(t)\operatorname{sgn}\big(e_i(t)\big)\big|e_i(t)\big|^{\alpha} \\
&\leqslant \|A\|\big\|e(t)\big\|^2 - \min_i \gamma_i \big\|e(t)\big\|^2 + L\|B\|\big\|e(t)\big\|^2 - k\sum_{i=1}^{m}\big|e_i(t)\big|^{\alpha+1} \\
&= \big(\|A\| + L\|B\| - \min_i \gamma_i\big)\big\|e(t)\big\|^2 - k\sum_{i=1}^{m}\big|e_i(t)\big|^{\alpha+1}. \qquad (14)
\end{aligned}
$$

From Lemma 2, we get

$$
\left(\sum_{i=1}^{m}\big|e_i(t)\big|^2\right)^{1/2} \leqslant \left(\sum_{i=1}^{m}\big|e_i(t)\big|^{\alpha+1}\right)^{1/(\alpha+1)}.
$$

Hence,

$$
\left(\sum_{i=1}^{m}\big|e_i(t)\big|^2\right)^{(\alpha+1)/2} \leqslant \sum_{i=1}^{m}\big|e_i(t)\big|^{\alpha+1}.
$$

Therefore,

$$
\begin{aligned}
\dot{V}(t) &\leqslant \big(\|A\| + L\|B\| - \min_i \gamma_i\big)\big\|e(t)\big\|^2 - k\left(\sum_{i=1}^{m}\big|e_i(t)\big|^2\right)^{(\alpha+1)/2} \\
&\leqslant -k\left(\sum_{i=1}^{m}\big|e_i(t)\big|^2\right)^{(\alpha+1)/2} = -k(2V)^{(\alpha+1)/2} = -k2^{(\alpha+1)/2}V^{(\alpha+1)/2}. \qquad (15)
\end{aligned}
$$

By Lemma 1, $V(t)$ converges to zero in a finite time, and the finite time is estimated by

$$
t^* = \frac{V(0)^{(1-\alpha)/2}}{k2^{(1+\alpha)/2}(1-\alpha)/2}.
$$

Hence, the error vector $e(t)$ will converge to zero within t^*. Consequently, under controller (12), systems (1) and (11) realize finite-time GS. This completes the proof of the theorem. □

Remark 2. The sufficient conditions given in Theorems 1 and 2 can avoid the problem that networks only realize generalized synchronization when time tends to infinity efficiently, and this has significant and basic meanings in real engineering applications of network synchronization.

Remark 3. Our results have more real meanings than that of [35], and the result in [35] is invalid for delayed systems, while Theorem 1 is still valid for delayed systems, so it is an important and useful extension of [35].

4 Numerical examples

Three numerical examples are presented to demonstrate the effectiveness of the proposed control laws:

Example 1. Consider the drive Rössler system [15, 39]

$$\begin{aligned}\dot{x}_1(t) &= -(x_2 + x_3),\\ \dot{x}_2(t) &= x_1 + 0.2x_2,\\ \dot{x}_3(t) &= 0.2 + x_3(x_1 - 5.7),\end{aligned} \tag{16}$$

the response system is the following neural network [30]:

$$\dot{y}(t) = Ay(t) + Bg\big(y(t)\big) + Cg\big(y\big(t - \tau(t)\big)\big) + u(t), \tag{17}$$

where $y(t) = (y_1(t), y_2(t))^{\mathrm{T}}$, $g(y(t)) = (g(y_1(t)), g(y_2(t)))^{\mathrm{T}}$, $k = 5$, $\alpha = 0.8$, $u(t)$ is the form of (7),

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2.0 & -0.1 \\ -5.0 & 4.5 \end{bmatrix},$$

$$C = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -4 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix},$$

$$g(x) = \tanh(x).$$

When $\tau(t) = 1$, $u(t) = 0$, system (17) has a chaotic attractor, as shown in Fig. 1.

First, letting $\Phi(x) = (x_1^3 + 0.5, x_3^2 - 0.3)^{\mathrm{T}}$. Figure 1 indicates systems (16) and (17) are two different chaotic attractors. It is easy to know $\|A\| = 1$, $\|B\| = 6.9099$, $L = 1$,

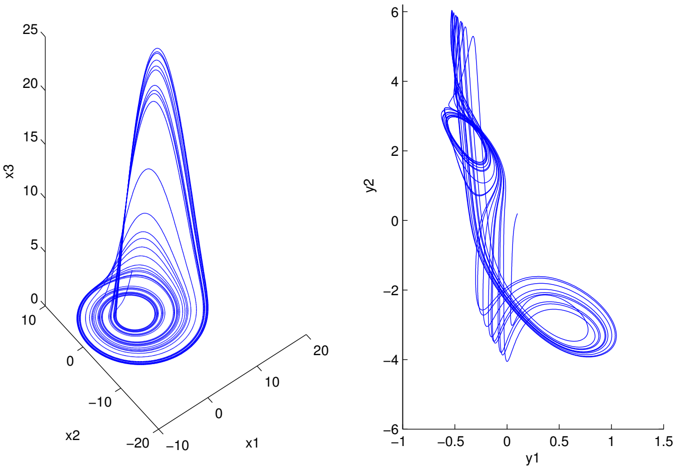


Figure 1. Chaotic attractors of (16) and (17).

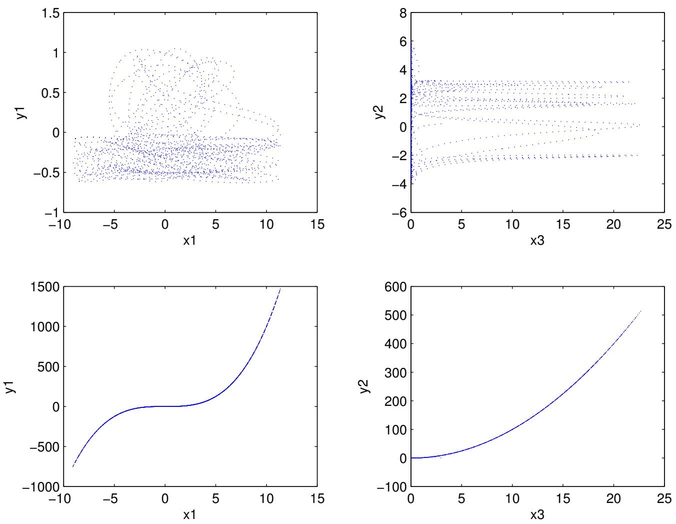


Figure 2. Relationship between the corresponding variables of systems (16) and (17).

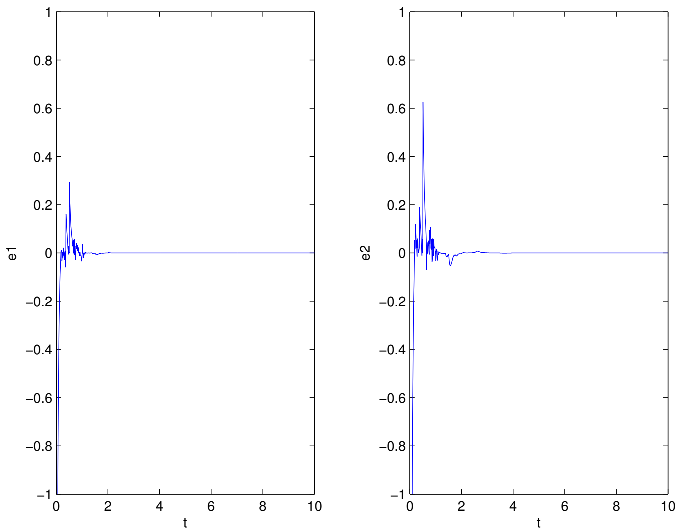
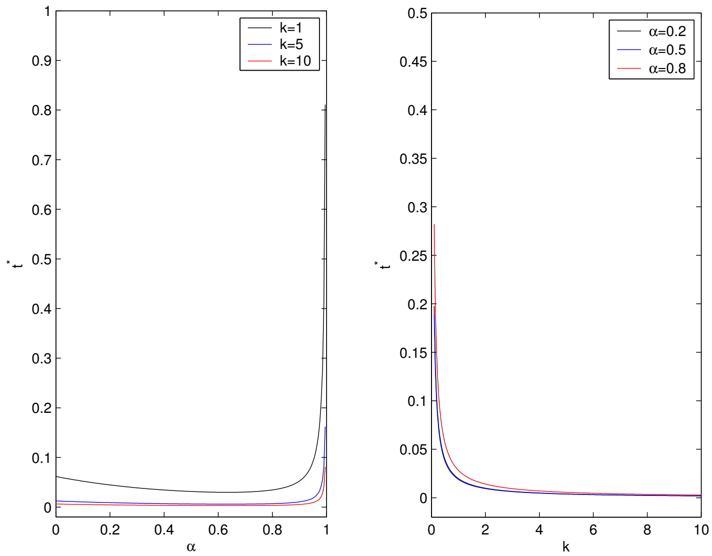


Figure 3. T he generalized synchronization error of systems (16) and (17).

$\|C\| = 4.0094$. Let $l = 5$, then $\|A\| + L\|B\| + L\|C\|/2 - \min_i \gamma_i + l = -5.0854 < 0$, $L\|C\|/2 - l = -2.9953 < 0$, from Theorem 1, systems (16) and (17) realize finite-time GS. The initial values are taken as $x(0) = (1.5, 2.0, 3.0)$, $y(0) = (0.1, 0.2)$. Figure 2 shows the relationships between the corresponding state variables of the drive-response systems. Obviously, they are generalized synchronization not complete synchronization. Figure 3 shows the generalized synchronization errors between the drive-response



(a) (b)

Figure 4. (a) The generalized synchronization time versus α for three different values of k. (b) The generalized synchronization time versus k for three different values of α.

systems. Figure 4a presents the generalized synchronization time versus α for three different values of k. Figure 4b presents the generalized synchronization time versus k for three different values of α. As is shown in Fig. 3, e_1, e_2, e_3 have been stabilized to zero at finite time, that is, systems (16) and (17) reach finite-time GS.

Example 2. Consider the drive Rössler system

$$\begin{aligned}\dot{x}_1(t) &= -(x_2 + x_3),\\ \dot{x}_2(t) &= x_1 + 0.2x_2,\\ \dot{x}_3(t) &= 0.2 + x_3(x_1 - 5.7),\end{aligned} \tag{18}$$

the response system is three-dimensional cell neural network [5]

$$\dot{y}(t) = Ay(t) + Bg\big(y(t)\big) + u(t), \tag{19}$$

where $y(t) = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}}$, $g(y(t)) = (g(y_1(t)), g(y_2(t)), g(y_3(t)))^{\mathrm{T}}$, $k = 5$, $\alpha = 0.8$, $u(t)$ is the form of (12),

$$A = \begin{bmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1.25 & -3.2 & -3.2\\ -3.2 & 1.1 & -4.4\\ -3.2 & 4.4 & 1.0 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} 8.1 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 10 \end{bmatrix},$$

$$g(y) = \frac{1}{2}\big(|y+1| - |y-1|\big).$$

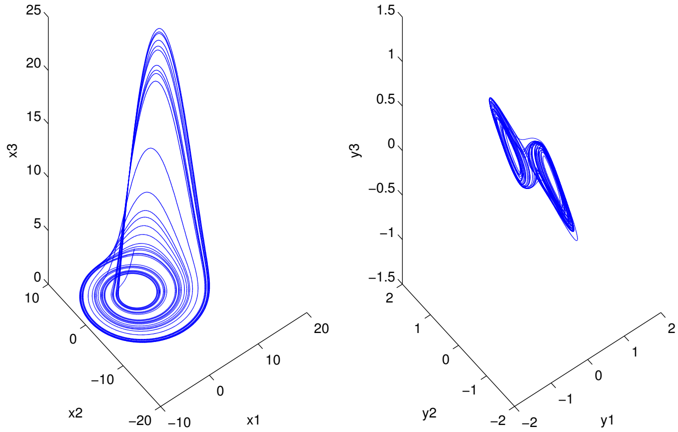


Figure 5. Chaotic attractors of (18) and (19).

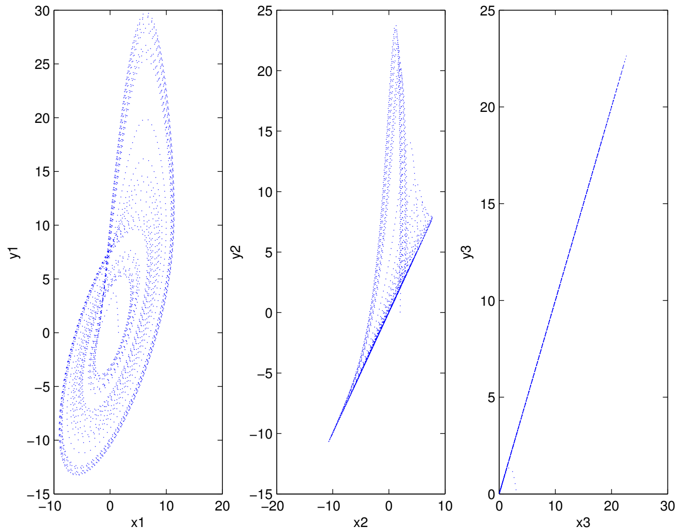


Figure 6. Relationship between the corresponding variables of systems (18) and (19).

First, letting $\Phi(x) = (x_1 + x_2 + x_3, x_2 + x_3, x_3)^{\mathrm{T}}$. Figure 5 indicates systems (18) and (19) are two different chaotic attractors. It is easy to know $\|A\| = 1$, $\|B\| = 7.0099$, $L = 1$, $\|A\| + L\|B\| - \min_i \gamma_i = -0.0001 < 0$, from Theorem 2, systems (18) and (19) realize finite-time GS. The initial values are taken as $x(0) = (1.5, 2.0, 3.0)$, $y(0) = (-0.001, 0.01, 0.2)$. Figure 6 shows the relationships between the corresponding state variables of the drive-response systems. Obviously, they are generalized synchronization not complete synchronization. Figure 7 shows the generalized synchronization errors between the drive-response systems. Figure 8a presents the generalized synchronization time versus α for three different values of k. Figure 8b presents the generalized

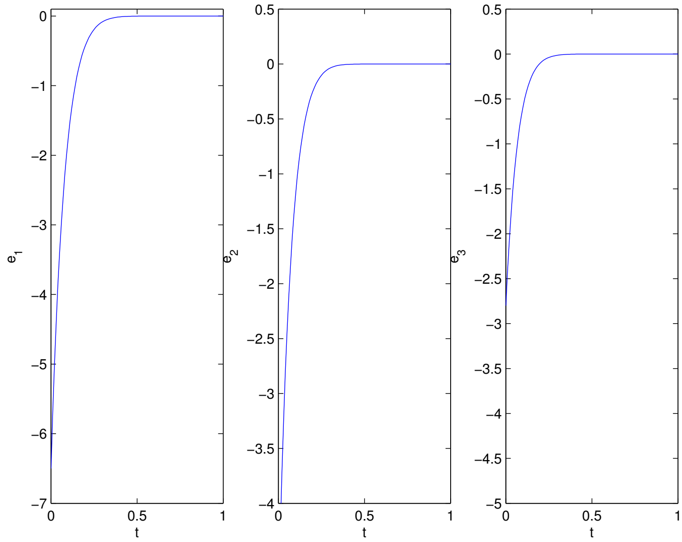
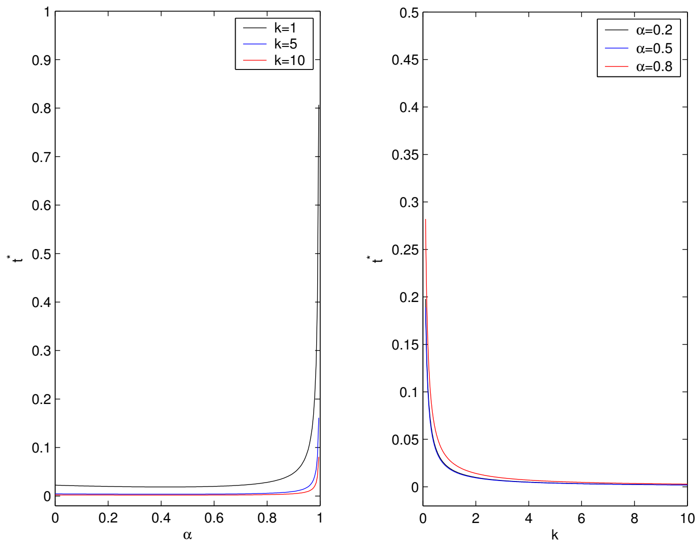


Figure 7. The generalized synchronization error of system (18) and (19).



(a) (b)

Figure 8. (a) The generalized synchronization time versus α for three different values of k. (b) The generalized synchronization time versus k for three different values of α.

synchronization time versus k for three different values of α. As is shown in Fig. 7, e_1, e_2, e_3 have been stabilized to zero at a finite time, that is, systems (18) and (19) reach finite-time GS.

Example 3. Consider the drive Lorenz system

$$\begin{aligned}\dot{x}_1(t) &= 10(x_2 - x_1),\\ \dot{x}_2(t) &= (28 - x_3)x_1 - x_2,\\ \dot{x}_3(t) &= x_1x_2 - \frac{8}{3}x_3,\end{aligned} \tag{20}$$

the response system is Chen system

$$\dot{y}(t) = Ay(t) + Bg\bigl(y(t)\bigr) + u(t), \tag{21}$$

where $y(t) = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}}$, $g(y(t)) = (g(y_1(t)), g(y_2(t)), g(y_3(t)))^{\mathrm{T}}$, $k = 50$, $\alpha = 0.8$, $u(t)$ is the form of (12),

$$A = \begin{bmatrix} -35 & 35 & 0\\ -7 & 28 & 0\\ 0 & 0 & -3\end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

$$\varGamma = \begin{bmatrix} 60 & 0 & 0\\ 0 & 70 & 0\\ 0 & 0 & 80\end{bmatrix}, \qquad g(y) = \begin{bmatrix} 0\\ y_1y_3\\ y_1y_2\end{bmatrix}.$$

First, letting $\varPhi(x) = (-1.5x_1 + 1, x_2 + 2x_3, 2x_3)^{\mathrm{T}}$, Fig. 9 indicates systems (20) and (21) are two different chaotic attractors. It is easy to know $\|A\| = 55.7607$, $\|B\| = 1$, $L = 1$, $\|A\| + L\|B\| - \min_i \gamma_i = -3.2393 < 0$, from Theorem 2, systems (20) and (21) realize finite-time GS. The initial values are taken as $x(0) = (1.0, 2.0, 3.0)$, $y(0) = (1, 0.2, 3.0)$. Figure 10 shows the relationships between the corresponding state variables of the drive-response systems. Obviously, they are generalized synchronization

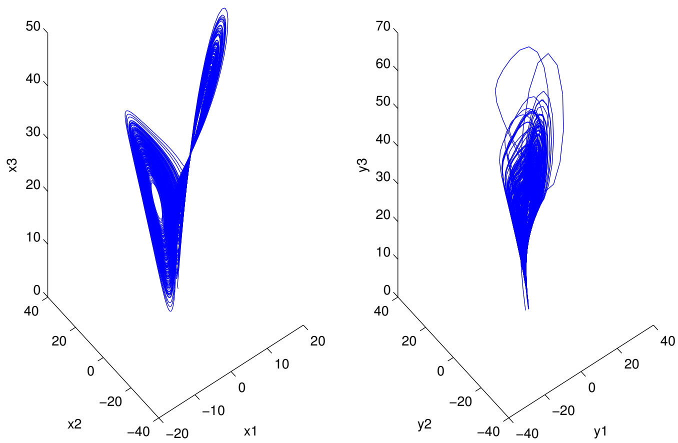


Figure 9. Chaotic attractors of (20) and (21).

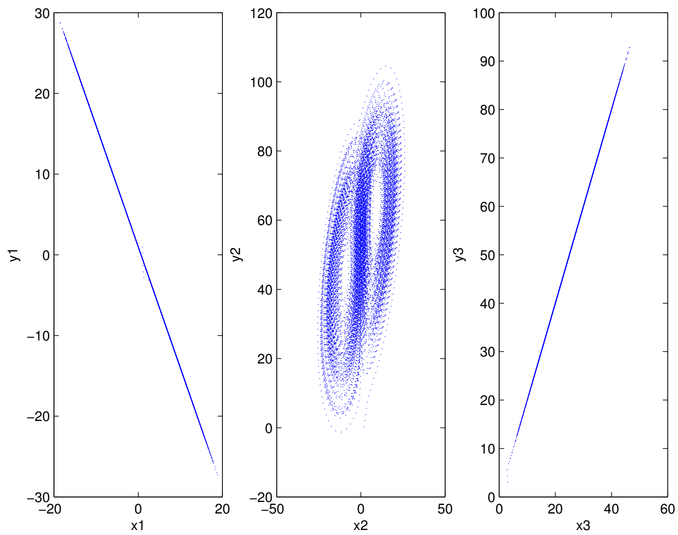


Figure 10. Relationship between the corresponding variables of systems (20) and (21).

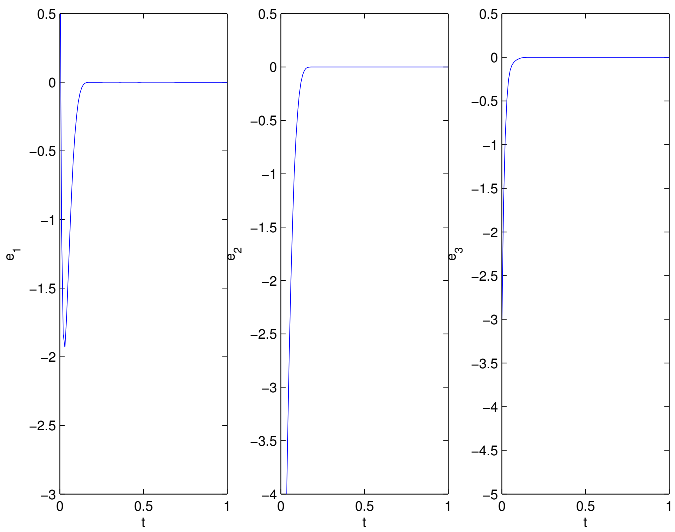
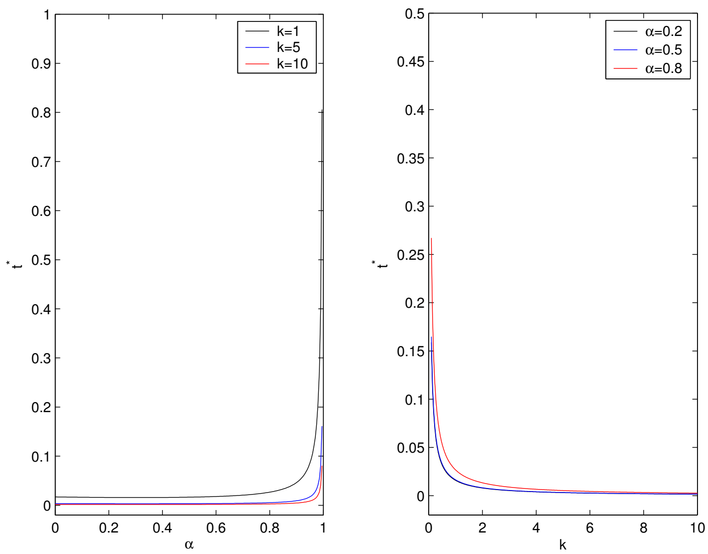


Figure 11. The generalized synchronization error of systems (20) and (21).

not complete synchronization. Figure 11 shows the generalized synchronization errors between the drive-response systems. Figure 12a presents the generalized synchronization time versus α for three different values of k. Figure 12b presents the generalized synchronization time versus k for three different values of α. As is shown in Fig. 11, e_1, e_2, e_3 have been stabilized to zero at a finite time, that is, systems (20) and (21) reach finite-time GS.



(a) (b)

Figure 12. (a) The generalized synchronization time versus α for three different values of k. (b) The generalized synchronization time versus k for three different values of α.

5 Conclusions

In this paper, by using a Lyapunov–Krasovskii functional, we investigate the finite-time generalized synchronization problem of nonidentical delayed chaotic systems. Control laws are designed to realized the finite-time generalized synchronization of two chaotic systems. The main contribution of this paper is, we can realize GS in a finite-time.

Furthermore, we would like to point out that, it is still a challenging work to investigate the finite-time synchronization of complex networks and fractional neural networks with and without time delays and it might be possible to extend the current results to stochastic chaotic systems with discontinuous dynamic behaviors, which is inspired by the [24, 25, 26, 27]. These will be considered in next papers.

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