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A generic method to construct new customized-shaped chaotic systems using the relative motion concept

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Abstract. Constructing chaotic systems tailored for each particular real-world application has been a long-term research desideratum. We report a solution for this problem based on the concept of relative motion. We investigate the periodic motion on a closed contour of a coordinate frame in which a chaotic system evolves. By combining these two motions (periodic on a close contour and chaotic) new customized shape trajectories are acquired. We demonstrate that these trajectories obtained in the stationary frame are also chaotic and, moreover, conserve the Lyapunov exponents of the initial chaotic system. Based on this finding, we developed an innovative method to construct new chaotic systems with customized shapes, thus fulfilling the requirements of any particular application of chaos.

Keywords: chaotic system, relative motion, Lyapunov exponents.

1 Introduction

Chaos as a fascinating nonlinear phenomenon has been intensively investigated in the last decades within diverse and multidisciplinary scientific communities. Starting with the initial effort to identify chaos in physical, mathematical, or even social systems, the research focus is moving more and more towards real-life chaos applications in a wide variety of domains including cryptology and communications [1], flow dynamics and liquid mixing [12], biomedical analysis [7], complex system optimization and prediction [8, 17], etc. Solving practical tasks requires, in many circumstances, the need to construct new chaotic systems tailored specifically for every single application.

In order to highlight the need of a methodology for developing customized chaotic systems, we present a relevant example: generating 2D or 3D chaotic trajectories, unpredictable for external observers (e.g. enemies), for autonomous mobile robots [3, 16] or unmanned aerial vehicles [20, 21] accomplishing perimeter patrolling tasks. To solve this problem, two strategies may be employed. The first method is to find an almost similar perimeter shape within the vast collection of already proven chaotic systems and adjust it using simple affine transformations. The second method requires the construction of a new chaotic system suitable for the considered application. While the first strategy can be efficiently used only for a small amount of perimeter shapes, a generic method belonging to the second strategy type to generate tailored chaotic trajectories appears to be the most appropriate line of attack.

This paper aims to generalize to the n-dimensional context and for an arbitrary chosen chaotic system, the method we developed for a particular case of a Henon system evolving in a mobile frame that pursues a 2D closed contour in a fixed referential frame [2]. For this endeavor, we started with the analysis of a general n-dimensional mobile coordinate frame in which a known chaotic system evolves. If this frame is moving with a finite velocity along a closed contour marked in a stationary n-dimensional frame, we observed that a brand-new chaotic system was constructed. As a remarkable fact, we find that the Lyapunov exponents are conserved during this transformation even the new-type trajectories have other shapes. Practically speaking, an arbitrarily-elected n-dimensional closed contour is chaotified with the means of kinematic relative motion using a given chaotic system. After offering the novel generic methodology, we exemplify and analyze it for two relevant 2D and 3D cases.

The rest of the paper is organized as follows. Section 2 provides the theoretical support for our methodology, demonstrating that the newly obtained trajectories are indeed chaotic. In Section 3, we present the overall methodology for obtaining customized chaotic shapes, accompanied by two illustrative examples, covering both continuous and discrete time systems, described in Section 4. Last section outlines the conclusions and final remarks.

2 Theoretical background

There has been much controversy regarding the proper definition of chaotic systems. Numerous definitions had been proposed, none of them receiving unanimous support. However, the definitions proposed by Kellert [9], Devaney [4], Smith [14] or Strogatz [15] are widely considered to be the most influential in the field. In what follows, due to its practicability [5], we will rely on Strogatz's definition, which states that a dynamic deterministic system is chaotic if it displays aperiodic long-term behavior exhibiting sensitive dependence on initial conditions. In other words, the trajectory of a chaotic system must not converge to fixed points, periodic or quasi-periodic orbits, and two nearby trajectories diverge exponentially fast.

The concept that quantitatively and qualitatively reflects the 'sensitivity on initial conditions' feature of chaotic systems is without doubts the Lyapunov spectrum [18]. Accordingly, the chaotic system definition can be rewritten based on three conditions, as follows:

Definition 1. A dynamical system is chaotic if it exhibits the following characteristics:

(a) the system is deterministic, any random component being forbidden;
(b) the trajectories are confined to a bounded region to avoid the trivial case of orbits escaping to infinity; and
(c) at least one of the system's Lyapunov exponents is positive.

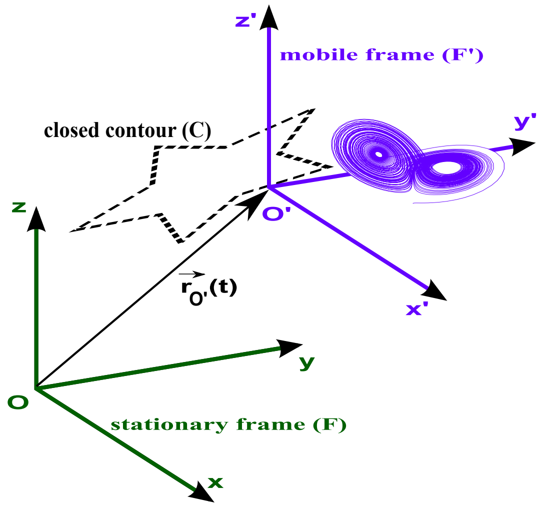

Figure 1. The context in a 3D space.

Considering these three characteristics of any chaotic system, we will center our theoretical framework on a lemma representing the basis of our method. But, first of all, let us depict the context. We assume an n-dimensional Euclidean space $\mathbb{R}^n$, where two Cartesian coordinate systems coexist. The first coordinate frame is considered to be fixed and is denoted by F. The other one (F') is mobile, its motion inside F being characterized by two specifications: a) the origin O' of the mobile frame describes a periodic motion by sliding on an arbitrary closed contour C established inside F; and b) during this motion, the axes corresponding to F' frame remain parallel and with the same orientation as the corresponding F axes (the motion of the frame F' inside F is a pure translation). An explanatory sketch for the 3D case is presented in Fig. 1.

In what follows, for notational simplicity, the variables related to the mobile frame will be denoted with prime symbols, while the variables in the fixed frame will be symbolized without prime.

In [2], we proved that, in the particular case of the Henon chaotic system evolving in a mobile frame moving on an arbitrary 2D closed contour, the obtained compound trajectory is also chaotic and, moreover, that the Lyapunov exponents are preserved. In order to demonstrate the general case, we will pursue the same line of reasoning. The demonstration proceeds in two steps. First, we will prove that the Lyapunov exponents of a generic chaotic system will be conserved through the coordinate system change. Then we show, based on the results obtained in the first step, that the new system is indeed chaotic.

The time evolution of two infinitesimally close trajectories of a dynamical system can be described on each dimension x of the state space using the exponential orbital divergence depicted by the following equation:

$$\left|\Delta x(t)\right| \approx \mathrm{e}^{\lambda_x t}\left|\Delta x(0)\right|, \tag{1}$$

where λ_x is a parameter known as Lyapunov exponent or Lyapunov characteristic exponent and, based on (1), can be defined as

$$\lambda_x = \lim_{t\to\infty} \lim_{\Delta x(0)\to 0} \frac{1}{t} \ln\left|\frac{\Delta x(t)}{\Delta x(0)}\right|. \tag{2}$$

It is worth mentioning that (2) is valid in both continuous and discrete time domains. In an n-dimensional space, there will be n Lyapunov exponents, one for each space dimension, forming the Lyapunov spectrum $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of the dynamical system under investigation, probably the most useful mean for dynamically diagnosing the chaotic systems.

Lemma. *Let $E \subseteq \mathbb{R}^n$ be an n-dimensional Euclidean space with its fixed Cartesian coordinate frame F, and let C be a chosen closed contour inside this referential. Now we construct F' to be a mobile coordinate frame with its origin O' sliding on C in a translational periodic motion (axes of F' remain parallel and with the same orientation as corresponding axes of F). If S' is an n-dimensional chaotic system evolving in F' with its set of Lyapunov exponents $\Lambda' = \{\lambda'_1, \lambda'_2, \dots, \lambda'_n\}$, then the combined evolution inside fixed frame F preserves all Lyapunov exponents of the chaotic system $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\} = \Lambda'$.*

Proof. We start the proof from the compound vector equation of motion

$$\vec{r}(t) = \vec{r}\,'(t) + \vec{r}_{O'}(t), \tag{3}$$

where $\vec{r}(t)$ and $\vec{r}\,'(t)$ are the position vectors corresponding to the t moment in time in the fixed and the mobile frames, respectively, and $\vec{r}_{O'}(t)$ is the position vector of the origin of F' inside F. Equation (3) can be decomposed for an arbitrary axis x of the n-dimensional space as

$$x(t) = x'(t) + x_{O'}(t). \tag{4}$$

Now, let us calculate the displacement of two trajectories on x axis at a given moment in time t:

$$\begin{aligned}\Delta x(t) &= x_2(t) - x_1(t)\\ &= \bigl(x'_2(t) + x_{O'}(t)\bigr) - \bigl(x'_1(t) + x_{O'}(t)\bigr) = \Delta x'(t).\end{aligned} \tag{5}$$

With this result in mind, we can write the formula of the Lyapunov exponent corresponding to the arbitrarily chosen axis x:

$$\begin{aligned}\lambda_x &= \lim_{t\to\infty} \lim_{\Delta x(0)\to 0} \frac{1}{t} \ln\left|\frac{\Delta x(t)}{\Delta x(0)}\right|\\ &= \lim_{t\to\infty} \lim_{\Delta x(0)\to 0} \frac{1}{t} \ln\left|\frac{\Delta x'(t)}{\Delta x'(0)}\right| = \lambda'_x.\end{aligned} \tag{6}$$

Result (6) is valid for all the n dimensions of the considered state space and, as a consequence, the Lyapunov spectrum $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is preserved ($\Lambda = \Lambda'$). $\square$

Let us observe that this lemma is applicable for either continuous or discrete systems.

Theorem. *Let $E \subseteq \mathbb{R}^n$ be an n-dimensional Euclidean space with its fixed Cartesian coordinate frame F, and let C be a chosen closed contour inside this referential. Now we construct F' to be a mobile coordinate frame with its origin O' sliding on C in a translational periodic motion (axes of F' remain parallel and with the same orientation as corresponding axes of F). If S' is an n-dimensional chaotic system evolving in F', then the trajectories observed in the fixed frame F will correspond to a new chaotic system (S) having the same Lyapunov spectrum as the original chaotic system S'.*

Proof. We will start the demonstration by formalizing the system S in both continuous and discrete time domains. In continuous time domain, the newly obtained chaotic system S will be described as follows:

$$\begin{aligned} \frac{\mathrm{d}X'(t)}{\mathrm{d}t} &= G\bigl(X'(t), t\bigr), \\ X(t) &= X'(t) + H(t), \end{aligned} \tag{7}$$

where the first equation describes an autonomous chaotic system evolving in F', the second equation describes the periodic motion of F' inside F, $X'(t) \in V' \subset \mathbb{R}^n$ represents the coordinates of the trajectory point inside frame F', V' is a bounded volume in F' and $H(t) = H(t+T)$ is a periodic function that describes the motion of O' inside F along the closed contour C.

Similarly, in the discrete time domain, the differential equations (7) will be changed into recurrent equations as follows:

$$\begin{aligned} X'_{k+1} &= G(X'_k, k), \\ X_k &= X'_k + H(k), \end{aligned} \tag{8}$$

where $X'_k \in V' \subset \mathbb{R}^n$ represents the coordinates of the trajectory point inside frame F' and $H(k) = H(k+\tau)$ is a periodic discrete function that describes the motion of O' inside F along the closed contour C.

In order to demonstrate that the new system obtained in the fixed frame F is chaotic, we will prove one-by-one the three chaotic system characteristics described by the definition:

(a) *The system is deterministic.* In both continuous- (7) and discrete-time system (8) variants the newly obtained chaotic system does not contain any sort of random variables, the state of the system at any given moment in time t being computed based only on the initial state of the system using deterministic functions. Thus, the new trajectories have only a deterministic nature.

(b) *The trajectories are confined in a bounded region of the space.* Let us have a look at the second equation from (7) and (8), respectively. We know that each of the n-coordinates included in the vector X' are bounded (S' is chaotic so its orbits are bounded) and the function H, which represents the motion on a closed contour is also bounded, so the result (vector X) can only be bounded. If the original chaotic system evolves in the bounded region V' of the n-dimensional space, we can obtain a bounded region in which the new system (S) evolves by considering the limits of each coordinates. For an arbitrary coordinate x, we will have $x \in [x'_{\min} + x_{C,\min}, x'_{\max} + x_{C,\max}]$, where $x'_{\min}$ and $x'_{\max}$ are the inferior and superior limit of the coordinate x' for the original chaotic system S' inside F' and $x_{C,\min}$ and $x_{C,\max}$ are the limits of coordinate x for the closed contour C inside F.

(c) *At least one Lyapunov exponent is positive.* Considering the results of the lemma proved before, the entire spectrum of Lyapunov exponents is conserved in the case of a translational motion on a close contour of the frame F' inside a fixed frame F. Because S' is a chaotic system at least one its Lyaponov exponents is positive. Coupling these two facts, we may conclude that at least one Lyapunov exponent of the newly obtained system S is positive.

By proving all the three conditions for a system to be chaotic, we can conclude that the newly obtained system S is indeed chaotic. □

Observation 1. The converse theorem is also true, that is: If the orbit of a proved chaotic system S can be decomposed in two components: a) a periodic motion on a closed contour C described in the fixed coordinate system F; and b) an orbit of a system S' in a mobile frame F', with F' traveling along C in a translational motion; then the S' system is also chaotic and preserves the Lyapunov exponents corresponding to S.

The proof of the converse theorem is immediate, from the definition adopted for the chaotic systems and the already proved lemma, by following the same line of thinking as in the case of the mentioned theorem.

It is worthwhile to remark that the converse theorem may represent a particularly useful tool in proving the chaotic behavior of new systems using the mentioned decomposition of trajectories.

Other three useful observations about the theorem presented above are only listed, their demonstration being immediate:

Observation 2. The theorem is applicable for both continuous- and discrete-time systems.

Observation 3. The n-dimensional closed contour C may have any shape, from simple polygons to complex, self-intersecting curves.

Observation 4. Any periodic or asymptotically periodic orbit of S' is converted into periodic or asymptotically periodic orbits of S, respectively.

3 Proposed method

In order to obtain customized chaotic shapes using the relative motion concept, in our view, a series of seven steps must be followed:

Step 1. Select the dimension of the space. By knowing the dimensionality of the state space, where the trajectory that must be chaotified exists, we will select this parameter as the dimension n of the space.

Step 2. Choose the shape of the closed curve C. Analyzing the trajectory in the state space that needs to be chaotified, we will approximate it using a closed curve. The simplest way to perform this step is to use either a polygonal shape or a closed curve obtained by interpolating a set of points.

Step 3. Choose an already proven chaotic system. The initial chaotic system selection is governed by the shape of the attractor, its attracting basin and its intrinsic parameters (e.g. Lyapunov exponents). Basically, any known chaotic system evolving in the chosen n-dimensional space can be a potential candidate.

Step 4. Adjust the orbit of the chaotic shape if necessary using affine transformations. In order to control the region around the closed curve C, where the chaotic trajectory will evolve, we can fine-tune the initial chaotic system by applying tailored affine transformations. This step can be accomplished by choosing the n-dimensional vicinity around an arbitrary point of the closed curve, where the chaotic system may take values.

Step 5. Select an initial point of the trajectory in the mobile frame F' for which the orbit of S' will be chaotic. Equilibrium points and points on periodic orbits must be strictly avoided.

Step 6. Select the period T. Essentially, this parameter characterizes the speed with which the closed curve C is covered. Its selection is affected by the speed of initial chaotic system evolution inside the mobile frame and can be selected based on expertise or simulations, in conjunction with step 7.

Step 7. Construct the compound trajectories. Having all the necessary settings completed in previous steps, we can construct the new chaotic system and implicitly its trajectories using the relative motion concept using either (7) or (8).

4 Illustrative case studies

We illustrate the proposed methodology by two case studies: i) one in discrete-time inside the two-dimensional space based on the 2D logistic map [19]; and ii) one in continuous-time inside the three-dimensional space using the classical Lorenz chaotic system [11].

4.1 Case study using 2D logistic map

We adopt a variant of the 2D logistic map [10, 19] to be the primary source of chaotic behavior. This chaotic system is generated using the following set of equations:

$$\begin{aligned} \tilde{x}'_{n+1} &= r(3\tilde{y}'_n + 1)\tilde{x}'_n(1 - \tilde{x}'_n), \\ \tilde{y}'_{n+1} &= r(3\tilde{x}'_{n+1} + 1)\tilde{y}'_n(1 - \tilde{y}'_n), \end{aligned} \tag{9}$$

where $\tilde{x}'_i$ and $\tilde{y}'_i$ are the Cartesian coordinates of the ith point of the trajectory in the mobile frame F', while the parameter r must take values in the interval $[1.11, 1.19]$ to obtain the much-needed chaotic behavior [19]. Choosing $r = 1.19$ and the initial point $(0.8909; 0.3342)$, we obtained the attractor presented in Fig. 2, while its bifurcation diagram (Fig. 3) confirms the chaotic dynamics for $r \in [1.11, 1.19]$.

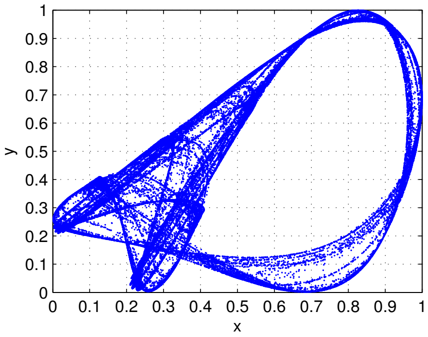

Figure 2. 2D Logistic map.

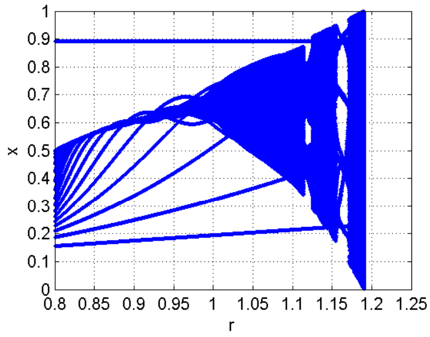

Figure 3. Bifurcation diagram.

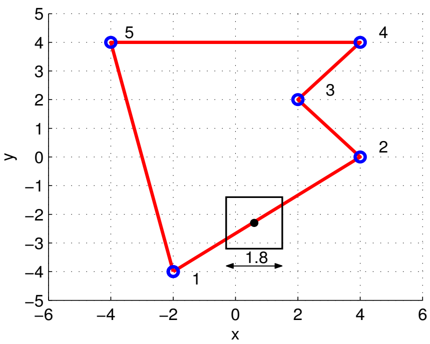

Figure 4. Closed 2D polygonal contour with five chosen vertices.

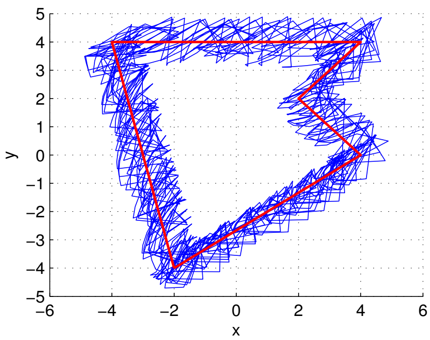

Figure 5. The new chaotic attractor (5 laps for T=196).

We also assume a closed contour C inside the fixed frame F with a pentagonal shape (Fig. 4) having the following vertices in Cartesian coordinates: $(-2, -4)$, $(4, 0)$, $(2, 2)$, $(4, 4)$, and $(-4, 4)$. This contour will be covered with a periodicity of T time samples. In order to obtain the chaotic system S' mentioned in our methodology, we adapted the system described by (9) using the following affine transformation:

$$x'_n = 1.8\,\tilde{x}'_n - 0.9,$$
$$y'_n = 1.8\,\tilde{y}'_n - 0.9, \tag{10}$$

that will force x'_n and y'_n to vary in the interval $[-0.9, 0.9]$. By this, the chaotic system S' given by (9) coupled with (10) will provide a point inside the square vicinity presented in Fig. 4 for each considered point of the closed contour. Having the chaotic system S' and the closed contour C, by using our method we will obtain the trajectory described in Fig. 5.

The obtained trajectory that chaotifies the closed contour C depends on the period T as presented in Fig. 6. As a supplementary proof of the chaotic nature of the newly obtained system S, we present the bifurcation diagram for $T = 16$ (Fig. 7), that shows the chaotic behavior for the same values of parameter r ($r \in [1.11, 1.19]$).

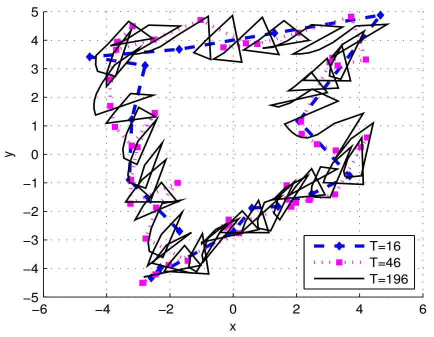

Figure 6. Chaotic trajectories for diverse values of the period T.

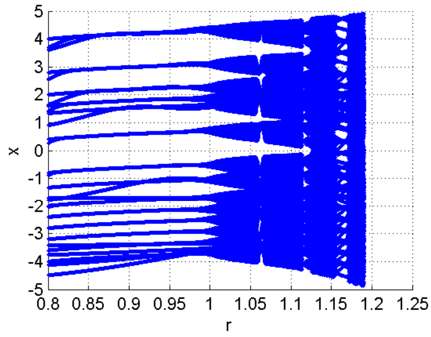

Figure 7. The bifurcation diagram of the new chaotic system ($T = 16$).

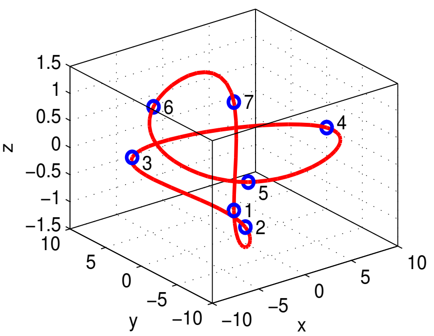

Figure 8. Closed 3D contour obtained by interpolating the set of seven points.

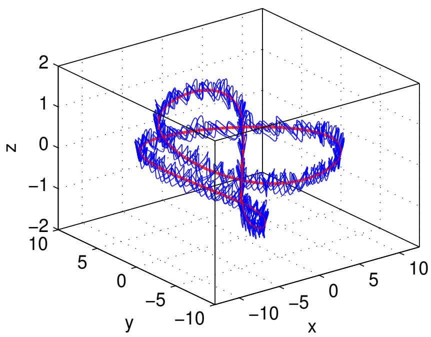

Figure 9. The new chaotic attractor and its guiding contour.

4.2 Case study using Lorenz system

In order to exemplify how our methodology works in the 3D case, we established a set of seven points having the following Cartesian coordinates: $(0, 0, -1)$, $(-3.72, -6.45, -0.66)$, $(-4.71, 8.16, -0.33)$, $(10, 0, 0)$, $(-4.71, -8.16, 0.33)$, $(-3.72, 6.45, 0.66)$ and $(0, 0, 1)$. We used natural cubic spline interpolation method [13] to obtain the guiding closed contour C (Fig. 8). The original chaotic system used in this example was the well known Lorenz system [11], which was adjusted by a combined affine transform [6] having the following transformation matrix in homogeneous coordinates:

$$T = \begin{bmatrix} 0.05 & 0 & 0 & -0.51 \\ 0 & 0.025 & 0 & -0.52 \\ 0 & 0 & 0.016 & -24.42 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

The resulted attractor is presented in Fig. 9.

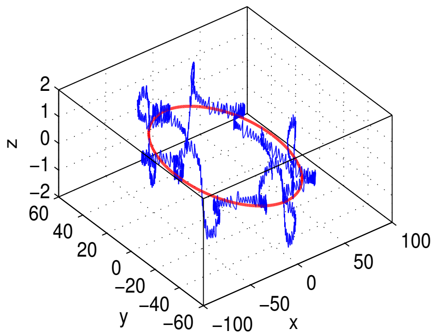

Figure 10. The chaotic attractor obtained by translating the attractor from Fig. 9 along a circle (one lap).

A chaotic system obtained through our methodology can be further used as an initial chaotic system for the same methodology. In Fig. 10, we present the chaotic attractor obtained by translating the attractor from Fig. 9 along a circle of radius 50 situated in the xOy-plane, with the center in the origin of the fixed frame.

5 Conclusions

We have demonstrated that new chaotic shapes can be obtained using the relative motion concept in a simple and efficient manner. We started with the periodic motion on a closed contour of a reference frame in which an already-known chaotic system evolves. We proved that the compound trajectories obtained in the fixed frame are also chaotic and, furthermore, preserve the chaotic properties of the original chaotic system. Based on this result, we developed an original method to create new chaotic systems with customized shapes. Practically speaking, a seed of chaos (the original system) can bloom into an infinite variety of new chaotic systems based on applications' needs.

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