ISSN 1392-5113 Nonlinear Analysis: Modelling and Control, 2016, Vol. 21, No. 3, 424–436
http://dx.doi.org/10.15388/NA.2016.3.9

α-admissible Prešić type operators and fixed points*

Satish Shukla[a,1], **Naseer Shahzad**[b]

[a]Department of Applied Mathematics,
Shri Vaishnav Institute of Technology & Science,
Gram Baroli, Sanwer Road, Indore (M.P.) 453331, India
satishmathematics@yahoo.co.in

[b]Department of Mathematics, King Abdulaziz University,
P.O. Box 80203, Jeddah, 21859, Saudi Arabia
nshahzad@kau.edu.sa

Received: September 29, 2014 / **Revised:** May 19, 2015 / **Published online:** February 7, 2016

Abstract. In this paper, we introduce α-admissible mappings on product spaces and obtain fixed point results for α-admissible Prešić type operators. Our results extend, unify and generalize some known results of the literature. We also provide examples, which illustrate the results proved herein and show that how the new results are different from the existing ones.

Keywords: α-admissible operator, Prešić type operator, fixed point.

1 Introduction

Let X be a nonempty set and $f : X \to X$ be a mapping. An element $x^* \in X$ is called a fixed point of f if $fx^* = x^*$. The famous Banach contraction principle ensures the existence and uniqueness of the fixed point of a mapping defined on a complete metric space. It states that:

Theorem 1 [Banach contraction principle]. *Let (X, d) be a complete metric space with a contraction mapping $f : X \to X$, that is, f satisfies the following condition*:

$$d(fx, fy) \leqslant \lambda d(x, y) \quad \text{for all } x, y \in X,$$

where $\lambda \in [0, 1)$. Then f admits a unique fixed point $x^* \in X$, that is, $fx^* = x^*$.

The Banach contraction principle has many applications in various branches of mathematics. There are several interesting generalizations of this eminent principle. In 1965,

*This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The second author acknowledges with thanks DSR for financial support.

[1]Corresponding author.

Prešić [24, 25] in his work generalized this principle for the mappings defined on the product spaces and applied the obtained results to ensure the convergence of a particular type of sequences. Prešić proved the following theorem:

Theorem 2 [Prešić's theorem]. *Let (X, d) be a complete metric space, k a positive integer and $T : X^k \to X$ be a mapping satisfying the following contractive type condition*:

$$d\big(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})\big) \leqslant \sum_{i=1}^{k} q_i d(x_i, x_{i+1}) \tag{1}$$

for every $x_1, x_2, \dots, x_{k+1} \in X$, where $q_1, q_2, \dots, q_k$ are nonnegative constants such that $q_1 + q_2 + \dots + q_k < 1$. Then there exists a unique point $x \in X$ such that $T(x, x, \dots, x) = x$. Moreover, if $x_1, x_2, \dots, x_k$ are arbitrary points in X and, for $n \in \mathbb{N}$, $x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \dots, \lim x_n)$.

A mapping $T : X^k \to X$ satisfying (1) is called a Prešić operator. A point $x \in X$ such that $T(x, x, \dots, x) = x$ is called a fixed point of T. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$. Note that the Banach contraction principle is a particular case (when $k = 1$) of Prešić's theorem.

Let I be an interval of real numbers. Consider a nonlinear difference equation of kth order

$$x_{n+k} = T(x_1, x_2, \dots, x_{n+k-1}), \quad n = 1, 2, \dots. \tag{2}$$

A point $u \in I$ is said to be an equilibrium point of difference equation (2) if $T(u, u, \dots, u) = u$. Note that, $u \in I$ is an equilibrium point of the difference equation (2) if and only if u is a fixed point of the mapping $T : I^k \to I$. Thus, Prešić's theorem insures the existence and uniqueness of the equilibrium point of nonlinear difference equation of kth order given by (2). For some more applications of Prešić's theorem, the reader is referred to [8, 13, 15, 24, 25, 30, 34]. Some important and interesting generalizations of Prešić's theorem can be found in [9, 10, 17, 22, 23, 31, 32, 33, 35, 36].

Ran and Reurings [26] and Nieto and Rodríguez-López [20, 21] generalized the Banach contraction principle into the metric spaces equipped with a partial order. Recently, Samet et al. [29] generalized the results of Ran and Reurings [26] and Nieto and Rodríguez-López [20, 21] and extended the Banach contraction principle by introducing the notion of α-admissible mappings. Subsequently, several authors proved fixed point results for contraction mappings via the concept of α-admissible mapping in metric spaces and other spaces (see [1, 2, 3, 4, 5, 6, 7, 11, 12, 14, 16, 19, 27, 28] and references therein).

In this paper, we define α-admissible mappings on product spaces and prove some fixed point results for α-admissible Prešić type operators. Our results generalize, unify and extend the results of Prešić [24, 25], Ran and Reurings [26], Nieto and Rodríguez-López [20, 21] and a particular case of the recent result of Samet et al. [29]. Some examples are provided, which illustrate the results proved herein and show the applicability of results.

2 Main results

First, we state some definitions and properties, which will be useful in the sequel.

Definition 1. Let (X,d) be a metric space, k a positive integer and $T: X^k \to X$ be a mapping. Then the mapping T is called an α-Prešić operator if there exist a function $\alpha: X^2 \to [0,\infty)$ and nonnegative constants α_i such that $\sum_{i=1}^{k} \alpha_i < 1$ and

$$\min\{\alpha(x_i,x_{i+1})\colon 1 \leqslant i \leqslant k\} d\big(T(x_1,\dots,x_k), T(x_2,\dots,x_{k+1})\big) \leqslant \sum_{i=1}^{k} \alpha_i d(x_i,x_{i+1}) \tag{3}$$

for all $x_1,x_2,\dots,x_k,x_{k+1} \in X$.

Remark 1. If $T: X^k \to X$ is a Prešić operator, then it is an α-Prešić operator with $\alpha(x,y)=1$ for all $x,y \in X$.

Remark 2. For $k=1$, an α-Prešić operator reduces into an α-ψ-contractive mapping with $\alpha_1 = k \in [0,1)$, $\psi(t)=kt$ for all $t \geqslant 0$ (see [29]). Therefore, α-Prešić operators extend this particular case of Samet et al. [29] into product spaces.

Definition 2. Let X be a nonempty set, k a positive integer and $T: X^k \to X$ be a mapping. Then the mapping T is called an α-admissible operator if there exists a function $\alpha: X^2 \to [0,\infty)$ such that

$$\min\{\alpha(x_i,x_{i+1})\colon 1 \leqslant i \leqslant k\} \geqslant 1 \implies \alpha\big(T(x_1,\dots,x_k), T(x_2,\dots,x_{k+1})\big) \geqslant 1$$

for all $x_1,x_2,\dots,x_k,x_{k+1} \in X$.

Remark 3. Let X be a nonempty set with a partial order $\preccurlyeq$ on it. Let k be a positive integer and $T: X^k \to X$ be a mapping such that, for any finite sequence $\{x_n\}_{n=1}^{k+1}$ with $x_i \preccurlyeq x_{i+1}$, $i=1,2,\dots,k$, we have $T(x_1,\dots,x_k) \preccurlyeq T(x_2,\dots,x_{k+1})$. Let a,b be two reals such that $a \geqslant 1, b<1$, and define $\alpha: X^2 \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} a & \text{if } x \preccurlyeq y; \\ b & \text{otherwise.} \end{cases}$$

Then T is an α-admissible operator.

Example 1. Let $X=[0,\infty)$ with usual partial order $\leqslant$ and $k=2$. Define the mappings $T: X^2 \to X$ and $\alpha: X^2 \to [0,\infty)$ by $T(x_1,x_2)=\sqrt{x_1+x_2}$ for all $x_1,x_2 \in X$ and

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \leqslant y; \\ 0 & \text{otherwise.} \end{cases}$$

If $x_1 \leqslant x_2 \leqslant x_3$ for some $x_1,x_2,x_3 \in X$, then $\sqrt{x_1+x_2} \leqslant \sqrt{x_2+x_3}$, that is, $T(x_1,x_2) \leqslant T(x_2,x_3)$. Therefore, by Remark 3 T is an α-admissible operator.

Definition 3. Let (X,d) be a metric space, k a positive integer and $T : X^k \to X$ be a mapping. Let $\alpha : X^2 \to X$ be a function and $\{x_n\}$ be a sequence in X. Then the sequence $\{x_n\}$ is called a termwise α-sequence if $\alpha(x_i, x_{i+1}) \geqslant 1$ for all $i = 1, 2, \ldots$. T is said to be diagonally continuous on X, if $T(x_n, x_{n+1}, \ldots, x_{n+k-1}) \to T(u, \ldots, u)$ as $n \to \infty$ whenever $\{x_n\}$ is a sequence in X with $x_n \to u \in X$ as $n \to \infty$. T is said to be diagonally α-continuous on X, if $T(x_n, x_{n+1}, \ldots, x_{n+k-1}) \to T(u, \ldots, u)$ as $n \to \infty$ whenever $\{x_n\}$ is a termwise α-sequence in X with $x_n \to u \in X$ as $n \to \infty$.

Remark 4. It is obvious that every diagonally continuous mapping $T : X^k \to X$ is diagonally α-continuous on X. But the converse of this fact is not true in general (see Example 2 of this paper).

The following theorem is an existence theorem for α-admissible operators on product spaces.

Theorem 3. *Let (X,d) be a complete metric space, k a positive integer and $T : X^k \to X$ be an α-Prešić operator. Suppose, the following conditions are satisfied*:

(i) *T is an α-admissible operator*;
(ii) *there exist $x_1, x_2, \ldots, x_k \in X$ such that*

$$\min\bigl\{\alpha(x_i, x_{i+1}), \alpha\bigl(x_k, T(x_1, \ldots, x_k)\bigr)\colon 1 \leqslant i \leqslant k-1\bigr\} \geqslant 1;$$

(iii) *T is diagonally α-continuous.*

Then T has a fixed point in X.

Proof. Suppose $x_1, x_2, \ldots, x_k \in X$ be such that

$$\min\bigl\{\alpha(x_i, x_{i+1}), \alpha\bigl(x_k, T(x_1, \ldots, x_k)\bigr)\colon 1 \leqslant i \leqslant k\bigr\} \geqslant 1.$$

Define a sequence $\{x_n\}$ in X by

$$x_{n+k} = T(x_n, \ldots, x_{n+k-1}) \quad \text{for all } n \in \mathbb{N}.$$

We shall show that $\{x_n\}$ is a termwise α-sequence in X. By definition of sequence and the assumption we have $\alpha(x_i, x_{i+1}) \geqslant 1$, $1 \leqslant i \leqslant k$. Because T is an α-admissible operator, by assumption we have

$$\alpha\bigl(T(x_1, \ldots, x_k), T(x_2, \ldots, x_{k+1})\bigr) \geqslant 1,$$

that is, $\alpha(x_{k+1}, x_{k+2}) \geqslant 1$. So, $\min\{\alpha(x_i, x_{i+1})\colon 2 \leqslant i \leqslant k+1\} \geqslant 1$. Again, as T is an α-admissible operator, we have $\alpha(T(x_2, \ldots, x_{k+1}), T(x_3, \ldots, x_{k+2})) \geqslant 1$, that is, $\alpha(x_{k+2}, x_{k+3}) \geqslant 1$. Following a similar process, we obtain $\alpha(x_i, x_{i+1}) \geqslant 1$ for all $n \in \mathbb{N}$. Thus, $\{x_n\}$ is a termwise α-sequence.

For notational convenience, suppose $\theta = \{\sum_{i=1}^k \alpha_i\}^{1/k}$, $d_n = d(x_n, x_{n+1})$, $n \in \mathbb{N}$, and

$$\mu = \max\biggl\{\frac{d(x_1, x_2)}{\theta}, \frac{d(x_2, x_3)}{\theta^2}, \ldots, \frac{d(x_k, x_{k+1})}{\theta^k}\biggr\}.$$

We shall prove by mathematical induction that

$$d_n \leqslant \mu\theta^n \quad \text{for all } n \in \mathbb{N}. \tag{4}$$

Then, by definition of μ it is obvious that (4) is true for $n = 1, 2, \dots, k$. Now suppose the following k inequalities be the induction hypothesis:

$$d_n \leqslant \mu\theta^n, \quad d_{n+1} \leqslant \mu\theta^{n+1}, \quad \dots, \quad d_{n+k-1} \leqslant \mu\theta^{n+k-1}.$$

As $\{x_n\}$ is a termwise α-sequence, it follows from (3) that

$$\begin{aligned}
d_{n+k} &= d(x_{n+k}, x_{n+k+1}) = d\big(T(x_n, \dots, x_{n+k-1}), T(x_{n+1}, \dots, x_{n+k})\big) \\
&\leqslant \min\big\{\alpha(x_i, x_{i+1})\colon\ n \leqslant i \leqslant n+k-1\big\} \\
&\quad \times d\big(T(x_n, \dots, x_{n+k-1}), T(x_{n+1}, \dots, x_{n+k})\big) \\
&\leqslant \alpha_1 d(x_n, x_{n+1}) + \alpha_2 d(x_{n+1}, x_{n+2}) + \dots + \alpha_k d(x_{n+k-1}, x_{n+k}) \\
&= \alpha_1 d_n + \alpha_2 d_{n+1} + \dots + \alpha_k d_{n+k-1} \\
&\leqslant \alpha_1 \mu\theta^n + \alpha_2 \mu\theta^{n+1} + \dots + \alpha_k \mu\theta^{n+k-1} \\
&\leqslant \mu\theta^n \sum_{i=1}^k \alpha_i = \mu\theta^{n+k}.
\end{aligned}$$

Thus, by the mathematical induction inequality (4) is true for all $n \in \mathbb{N}$.

For $m, n \in \mathbb{N}$ and $m > n$, we have

$$\begin{aligned}
d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\
&= d_n + d_{n+1} + \dots + d_{m-1} \\
&\leqslant \mu\theta^n + \mu\theta^{n+1} + \dots + \mu\theta^{m-1} \leqslant \frac{\mu\theta^n}{1-\theta}.
\end{aligned}$$

Since $\theta = \{\sum_{i=1}^k \alpha_i\}^{1/k} < 1$, it follows from the above inequality that

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence. By completeness of X, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0.$$

We shall show that u is a fixed point of T.

Since T is diagonally α-continuous on X, $\{x_n\}$ is a termwise α-sequence in X and $x_n \to u$ as $n \to \infty$, therefore, we obtain

$$u = \lim_{n\to\infty} x_{n+k} = \lim_{n\to\infty} T(x_n, \dots, x_{n+k-1}) = T(u, \dots, u).$$

Thus, u is a fixed point of T. □

Remark 5. For $k = 1$ in the above theorem, we obtain Theorem 2.1 of Samet et al. [29] with $\alpha_1 = k \in [0, 1)$, $\psi(t) = kt$ for all $t \geqslant 0$. For the existence of fixed point of operator $T : X \to X$, Samet et al. [29] assumed the continuity of T. In the above theorem, we assume that T is diagonally α-continuous on X. Following example illustrate this fact more accurately and it shows that our result is an proper extension of the Prešić's theorem. Also, this example shows that in the above theorem, the fixed point of T may not be unique.

Example 2. Let $X = [0, 1]$ and d be the usual metric on X, then (X, d) is a complete metric space. For $k = 2$, define $T : X^2 \to X$ and $\alpha : X^2 \to [0, \infty)$ by

$$T(x_1, x_2) = \begin{cases} (x_1 + x_2)/4 & \text{if } x_1, x_2 \in [0, 1/2]; \\ 2/3 & \text{if } x_1, x_2 \in (1/2, 2/3); \\ 1 & \text{otherwise,} \end{cases}$$

$$\alpha(x_1, x_2) = \begin{cases} 1 & \text{if } x_1, x_2 \in [0, 1/2]; \\ 0 & \text{otherwise.} \end{cases}$$

Then:

(a) T satisfies all the conditions of Theorem 3 and has two fixed points;
(b) T is not a Prešić operator, that is, it does not satisfy condition (1) and so the Prešić's theorem (Theorem 2) is not applicable here;
(c) T is not diagonally continuous on X.

Proof. (a) We observe following:

(i) *T is an α-Prešić operator with $\alpha_1 = \alpha_2 = 1/4$.* If any one or any two of x_1, x_2, x_3 are in $(1/2, 1]$, then we have $\min\{\alpha(x_1, x_2), \alpha(x_2, x_3)\} = 0$. Therefore, we have to check the validity of (3) only for the case $x_1, x_2, x_3 \in [0, 1/2]$. Now, for $x_1, x_2, x_3 \in [0, 1/2]$, we have

$$\begin{aligned} &\min\bigl\{\alpha(x_1, x_2), \alpha(x_2, x_3)\bigr\} d\bigl(T(x_1, x_2), T(x_2, x_3)\bigr) \\ &\quad = \frac{|x_1 - x_3|}{4} \leqslant \frac{1}{4}\bigl[|x_1 - x_2| + |x_2 - x_3|\bigr] = \frac{1}{4}\bigl[d(x_1, x_2) + d(x_2, x_3)\bigr]. \end{aligned}$$

Thus, T is an α-Prešić operator with $\alpha_1 = \alpha_2 = 1/4$.

(ii) *T is α-admissible.* Let $x_1, x_2, x_3 \in X$ be such that

$$\min\bigl\{\alpha(x_1, x_2), \alpha(x_2, x_3)\bigr\} \geqslant 1.$$

Then $x_1, x_2, x_3 \in [0, 1/2]$ and so $T(x_1, x_2) = (x_1 + x_2)/4, T(x_2, x_3) = (x_2 + x_3)/4$, that is, $T(x_1, x_2), T(x_2, x_3) \in [0, 1/2]$. Therefore,

$$\alpha\bigl(T(x_1, x_2), T(x_2, x_3)\bigr) = 1.$$

Thus, T is α-admissible.

(iii) *There exist* $x_1, x_2 \in X$ *such that* $\min\{\alpha(x_1, x_2), \alpha(x_2, T(x_1, x_2))\} \geqslant 1$. Indeed, for any $x_1, x_2 \in [0, 1/2]$, we have $\min\{\alpha(x_1, x_2), \alpha(x_2, T(x_1, x_2))\} = 1$.

(iv) T *is diagonally* α*-continuous on* X. Indeed, if $\{x_n\}$ is any termwise α-sequence in X and $x_n \to u$ as $n \to \infty$, then we have $\alpha(x_i, x_{i+1}) \geqslant 1$ for all $i \in \mathbb{N}$, that is, $x_i \in [0, 1/2]$ for all $i \in \mathbb{N}$ and so $u \in [0, 1/2]$. Now

$$T(x_n, x_{n+1}) = \frac{x_n}{2} \to \frac{u}{2} = T(u, u).$$

Thus, T satisfies all the conditions of Theorem 3 and has two fixed points, namely, $\mathrm{Fix}(T) = \{0, 1\}$.

(b) T *is not a Prešić operator.* Taking $x_1 = x_2 = 0$, $x_3 = 1$, we have

$$d\bigl(T(x_1, x_2), T(x_2, x_3)\bigr) = d\bigl(T(0, 0), T(0, 1)\bigr) = d(0, 1) = 1$$

and $d(x_1, x_2) = 0, d(x_2, x_3) = 1$. Therefore, there exist no nonnegative constants α_1, α_2 such that $\alpha_1 + \alpha_2 < 1$ and

$$d\bigl(T(x_1, x_2), T(x_2, x_3)\bigr) \leqslant \alpha_1 d(x_1, x_2) + \alpha_2 d(x_2, x_3).$$

Thus, T is not a Prešić operator.

(c) T *is not diagonally continuous.* Indeed, consider the sequence $\{x_n\}$, where $x_1 = x_2 = 5/9$, $x_n = (2 - 1/n)/3$, $n = 3, 4, \dots$. Then $x_n \to 2/3$ as $n \to \infty$. Note that $\lim_{n\to\infty} T(x_n, x_{n+1}) = 2/3 \neq 1 = T(2/3, 2/3)$. $\square$

In the next theorem, we replace the hypothesis of α-diagonal continuity of T by another hypothesis.

Theorem 4. *Let* (X, d) *be a complete metric space,* k *a positive integer and* $T : X^k \to X$ *be an* α*-Prešić operator. Suppose the following conditions are satisfied*:

(i) T *is an* α*-admissible operator*;

(ii) *there exist* $x_1, x_2, \dots, x_k \in X$ *such that*

$$\min\bigl\{\alpha(x_i, x_{i+1}), \alpha\bigl(x_k, T(x_1, x_2, \dots, x_k)\bigr)\colon 1 \leqslant i \leqslant k - 1\bigr\} \geqslant 1;$$

(iii) *if* $\{x_n\}$ *is any termwise* α*-sequence in* X *such that* $x_n \to u$ *as* $n \to \infty$*, then* $\alpha(x_n, u) \geqslant 1$ *for all* $n \in \mathbb{N}$ *and* $\alpha(u, u) \geqslant 1$.

Then T *has a fixed point in* X.

Proof. Following the proof of Theorem 3, the sequence $\{x_n\}$ is a termwise α and Cauchy sequence in X, where $\{x_n\}$ is defined as in the proof of Theorem 3. By completeness of X, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. For any $n \in \mathbb{N}$, from triangular inequality we obtain

$$\begin{aligned} d\bigl(u, T(u, \dots, u)\bigr) &\leqslant d(u, x_{n+k}) + d\bigl(x_{n+k}, T(u, \dots, u)\bigr) \\ &= d(u, x_{n+k}) + d\bigl(T(x_n, \dots, x_{n+k-1}), T(u, \dots, u)\bigr) \end{aligned}$$

$$
\begin{aligned}
&\leqslant d(u, x_{n+k}) + d\big(T(x_n, \dots, x_{n+k-1}), T(x_{n+1}, \dots, x_{n+k-1}, u)\big) \\
&\quad + d\big(T(x_{n+1}, \dots, x_{n+k-1}, u), T(x_{n+2}, \dots, x_{n+k-1}, u, u)\big) \\
&\quad + \dots + d\big(T(x_{n+k-1}, u, \dots, u), T(u, \dots, u)\big).
\end{aligned}
$$

As $\{x_n\}$ is a termwise α-sequence, by assumption (iii) we have $\alpha(x_n, u) \geqslant 1$ for all $n \in \mathbb{N}$ and $\alpha(u, u) \geqslant 1$, therefore, from the above inequality we obtain

$$
\begin{aligned}
&d\big(u, T(u, \dots, u)\big) \\
&\quad \leqslant d(u, x_{n+k}) + \min\big\{\alpha(x_i, x_{i+1}), \alpha(x_{n+k-1}, u)\colon n \leqslant i \leqslant n+k-2\big\} \\
&\qquad \times d\big(T(x_n, \dots, x_{n+k-1}), T(x_{n+1}, \dots, x_{n+k-1}, u)\big) \\
&\qquad + \min\big\{\alpha(x_i, x_{i+1}), \alpha(x_{n+k-1}, u), \alpha(u, u)\colon n+1 \leqslant i \leqslant n+k-2\big\} \\
&\qquad \times d\big(T(x_{n+1}, \dots, x_{n+k-1}, u), T(x_{n+2}, \dots, x_{n+k-1}, u, u)\big) \\
&\qquad + \dots + \min\big\{\alpha(x_{n+k-1}, u), \alpha(u, u)\big\} d\big(T(x_{n+k-1}, u, \dots, u), T(u, \dots, u)\big),
\end{aligned}
$$

which, together with (3), gives

$$
\begin{aligned}
&d\big(u, T(u, \dots, u)\big) \\
&\quad \leqslant d(u, x_{n+k}) + \alpha_1 d_n + \alpha_2 d_{n+1} + \dots + \alpha_{k-1} d_{n+k-2} + \alpha_k d(x_{n+k-1}, u) \\
&\qquad + \alpha_1 d_{n+1} + \alpha_2 d_{n+2} + \dots + \alpha_{k-2} d_{n+k-2} + \alpha_{k-1} d(x_{n+k-1}, u) \\
&\qquad + \dots + \alpha_1 d(x_{n+k-1}, u).
\end{aligned}
$$

Since $x_n \to u$ as $n \to \infty$, it follows from the above inequality that $d(u, T(u, \dots, u)) = 0$, that is, $T(u, \dots, u) = u$. Thus, u is a fixed point of T. $\square$

Remark 6. The above theorem extends Theorem 2.2 of Samet et al. [29] (for the case $\psi(t) = kt$, $t \geqslant 0$) in product spaces. Note that, for the existence of fixed point in the above theorem, in assumption (iii), we use an additional condition that $\alpha(u, u) \geqslant 1$, which was not assumed by Samet et al. [29] (compare assumption (iii) of the above theorem and assumption (iii) of Theorem 2.2 of Samet et al. [29]). The following example shows that this condition is not superfluous.

Example 3. Let $X = [0, 1]$ and d be the usual metric on X, then (X, d) is a complete metric space. For $k = 2$, define $T : X^2 \to X$ and $\alpha : X^2 \to [0, \infty)$ by

$$
T(x, y) = \begin{cases} 1 & \text{if } x = y = 0; \\ (x + y)/4 & \text{otherwise} \end{cases}
$$

and

$$
\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1],\ y < x; \\ 0 & \text{otherwise.} \end{cases}
$$

Then all the conditions of Theorem 4, except (iii), are satisfied and T has no fixed point in X.

Proof. We observe following:

(i) *T is an α-Prešić operator with $\alpha_1 = \alpha_2 = 1/4$.* If any two of x_1, x_2, x_3 are equal to 0, then we have $\min\{\alpha(x_1, x_2), \alpha(x_2, x_3)\} = 0$. Similarly, if $x_1 = 0$ or $x_2 = 0$ then again we have $\min\{\alpha(x_1, x_2), \alpha(x_2, x_3)\} = 0$. Therefore we have to check the validity of (3) only for the cases $x_1, x_2, x_3 \in (0, 1]$, $x_3 < x_2 < x_1$, and when $x_3 = 0, x_1, x_2 \in (0, 1]$ with $x_2 < x_1$. If $x_1, x_2, x_3 \in (0, 1]$, then

$$\begin{aligned}
&\min\bigl\{\alpha(x_1, x_2), \alpha(x_2, x_3)\bigr\} d\bigl(T(x_1, x_2), T(x_2, x_3)\bigr) \\
&\quad = \frac{|x_1 - x_3|}{4} \leqslant \frac{1}{4}\bigl[|x_1 - x_2| + |x_2 - x_3|\bigr] = \frac{1}{4}\bigl[d(x_1, x_2) + d(x_2, x_3)\bigr].
\end{aligned}$$

If $x_3 = 0, x_1, x_2 \in (0, 1]$ with $x_2 < x_1$, then with a similar calculation one can verify inequality (3).

(ii) *T is α-admissible.* If $x_1, x_2, x_3 \in X$ and $\min\{\alpha(x_1, x_2), \alpha(x_2, x_3)\} \geqslant 1$, then we have $x_1, x_2, x_3 \in (0, 1]$ with $x_3 < x_2 < x_1$ or $x_3 = 0, x_1, x_2 \in (0, 1]$ with $x_2 < x_1$. If $x_1, x_2, x_3 \in (0, 1]$ with $x_3 < x_2 < x_1$, we have $T(x_1, x_2) = (x_1 + x_2)/4$, $T(x_2, x_3) = (x_2 + x_3)/4$, and so, $T(x_1, x_2), T(x_2, x_3) \in (0, 1]$ and $T(x_2, x_3) < T(x_1, x_2)$. Therefore, $\alpha(T(x_1, x_2), T(x_2, x_3)) = 1$. Similar result holds for the case $x_3 = 0$, $x_1, x_2 \in (0, 1]$ with $x_2 < x_1$. Thus, T is α-admissible.

(iii) *There exist $x_1, x_2 \in X$ such that* $\min\{\alpha(x_1, x_2), \alpha(x_2, T(x_1, x_2))\} \geqslant 1$. Indeed, for any $x_1, x_2 \in (0, 1]$ with $x_2 < x_1 < 3x_2$, we have

$$\min\bigl\{\alpha(x_1, x_2), \alpha\bigl(x_2, T(x_1, x_2)\bigr)\bigr\} = 1.$$

(iv) *$\{x_n\}$ is a termwise α-sequence in X and $x_n \to u \in X$ as $n \to \infty$.* Then we have $\alpha(x_i, x_{i+1}) \geqslant 1$ for all $i \in \mathbb{N}$, and by definition of α, $\{x_n\}$ is a strictly decreasing sequence in $[0, 1]$. Therefore, $u < x_n$ for all $n \in \mathbb{N}$ and so $\alpha(x_n, u) = 1$ for all $n \in \mathbb{N}$. Now, if we take $x_n = 1/n$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a termwise α-sequence and $x_n \to 0$ $(= u)$ as $n \to \infty$. Note that $\alpha(x_n, 0) = \alpha(x_n, u) = 1$ for all $n \in \mathbb{N}$, but $\alpha(0, 0) = \alpha(u, u) = 0 \ngeqslant 1$.

Thus, all the conditions of Theorem 4, except $\alpha(u, u) \geqslant 1$ (in assumption (iii)), are satisfied and T has no fixed point. □

As shown in Example 2, the fixed point of the mapping T satisfying the conditions of Theorem 3 may not be unique. To establish the uniqueness of fixed point, we use the following definition.

Definition 4. Let X be any nonempty set and $\alpha : X^2 \to [0, \infty)$ be a function. Let $A \subseteq X, A \neq \emptyset$, then A is called α-well-ordered, if for all $x, y \in A$, we have $\alpha(x, y) \geqslant 1$ or $\alpha(y, x) \geqslant 1$ or both. Note that, if A is α-well-ordered, then $\alpha(x, x) \geqslant 1$ for all $x \in A$.

Theorem 5. *Suppose all the assumptions of Theorem* 3 (*resp. Theorem* 4) *are satisfied. In addition, suppose that the set of all fixed points of T,* $\mathrm{Fix}(T)$ *is α-well-ordered then* $\mathrm{Fix}(T)$ *is singleton.*

Proof. The existence of fixed point is followed by Theorem 3 (resp. Theorem 4). Now suppose the set of all fixed points of T is α-well-ordered and $u, v \in \mathrm{Fix}(T)$ are distinct, that is, $T(u, \ldots, u) = u$, $T(v, \ldots, v) = v$ and $u \neq v$. As $\mathrm{Fix}(T)$ is α-well-ordered, we have

$$\min\bigl\{\alpha(u,v), \alpha(u,u), \alpha(v,v)\bigr\} \geqslant 1$$

or

$$\min\bigl\{\alpha(v,u), \alpha(u,u), \alpha(v,v)\bigr\} \geqslant 1$$

or both.

Suppose $\min\{\alpha(u,v), \alpha(u,u), \alpha(v,v)\} \geqslant 1$ (proof for the second case is similar). Now using (3) we obtain

$$\begin{aligned} d(u,v) &= d\bigl(T(u,\ldots,u), T(v,\ldots,v)\bigr) \\ &\leqslant d\bigl(T(u,\ldots,u), T(u,\ldots,u,v)\bigr) + d\bigl(T(u,\ldots,u,v), T(u,\ldots,u,v,v)\bigr) \\ &\quad + \cdots + d\bigl(T(u,v,\ldots,v), T(v,\ldots,v)\bigr) \\ &\leqslant \min\bigl\{\alpha(u,u), \alpha(u,v)\bigr\} d\bigl(T(u,\ldots,u), T(u,\ldots,u,v)\bigr) \\ &\quad + \min\bigl\{\alpha(u,v), \alpha(u,u), \alpha(v,v)\bigr\} d\bigl(T(u,\ldots,u,v), T(u,\ldots,u,v,v)\bigr) \\ &\quad + \cdots + \min\bigl\{\alpha(u,v), \alpha(v,v)\bigr\} d\bigl(T(u,v,\ldots,v), T(v,\ldots,v)\bigr) \\ &\leqslant \alpha_k d(u,v) + \alpha_{k-1} d(u,v) + \cdots + \alpha_1 d(u,v) \\ &= \sum_{i=1}^{k} \alpha_i d(u,v) < d(u,v). \end{aligned}$$

This contradiction shows that $\mathrm{Fix}(T)$ is singleton. □

Remark 7. In Example 2, the fixed point of T is not unique and $\mathrm{Fix}(T) = \{0, 1\}$. Note that, $\mathrm{Fix}(T)$ is not α-well-ordered. Indeed, $\alpha(0,1) = \alpha(1,0) = \alpha(1,1) = 0 < 1$.

Corollary 1. (*See* [24, 25].) *Let (X, d) be a complete metric space, k a positive integer and $T : X^k \to X$ be a Prešić operator. Then there exists a unique point $x \in X$ such that $T(x, x, \ldots, x) = x$. Moreover, if $x_1, x_2, \ldots, x_k$ are arbitrary points in X and for $n \in \mathbb{N}$, $x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \ldots, \lim x_n)$.*

Proof. Define $\alpha : X^2 \to [0, \infty)$ by $\alpha(x,y) = 1$ for all $x, y \in X$. Then it is easy to see that all the conditions of Theorem 5 are satisfied, and the result follows from Theorem 5. □

Let a nonempty set X be equipped with a partial order $\preccurlyeq$ such that (X, d) is a metric space, then the triple $(X, \preccurlyeq, d)$ is called an ordered metric space. A sequence $\{x_n\}$ in X is said to be nondecreasing with respect to $\preccurlyeq$ if $x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq \cdots$. Let k be a positive integer and $T : X^k \to X$ be a mapping, then T is said to be nondecreasing with respect to $\preccurlyeq$ if for any finite nondecreasing sequence $\{x_n\}_{n=1}^{k+1}$, we have $T(x_1, x_2, \ldots, x_k) \preccurlyeq T(x_2, x_3, \ldots, x_{k+1})$. Mapping T is said to be an ordered Prešić type contraction if (see also, [17, 18]) there exist nonnegative constants $\alpha_1, \alpha_2, \ldots, \alpha_k$

such that $\sum_{i=1}^{k} \alpha_i < 1$ and

$$d\big(T(x_1, \ldots, x_k), T(x_2, \ldots, x_{k+1})\big) \leqslant \sum_{i=1}^{k} \alpha_i d(x_i, x_{i+1})$$

for every nondecreasing finite sequences $\{x_n\}_{n=1}^{k+1}$ in X.

Following corollary is a fixed point result for an ordered Prešić type contraction.

Corollary 2. *Let $(X, \preccurlyeq, d)$ be an ordered complete metric space. Let k be a positive integer, $T : X^k \to X$ be a nondecreasing mapping with respect to $\preccurlyeq$. Suppose that the following conditions hold*:

(A) *T is an ordered Prešić type contraction*;

(B) *there exist $x_1, x_2, \ldots, x_k \in X$ such that $\{x_n\}_{n=1}^{k}$ is a nondecreasing sequence and $x_k \preccurlyeq T(x_1, \ldots, x_k)$*;

(C) *if a nondecreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$.*

Then T has a fixed point in X. In addition, if $\mathrm{Fix}(T)$ *is well-ordered with respect to $\preccurlyeq$, that is, for all $x, y \in \mathrm{Fix}(T)$, either $x \preccurlyeq y$ or $y \preccurlyeq x$. Then the fixed point of T is unique.*

Proof. Define $\alpha : X^2 \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preccurlyeq y; \\ 0 & \text{otherwise.} \end{cases}$$

Then, with this function α, the fact that T is nondecreasing implies that T is an α-admissible operator and condition (A) implies that T is an α-Prešić operator. Conditions (B) and (C) imply respectively conditions (ii) and (iii) of Theorem 4. Therefore, the existence of fixed point follows from Theorem 4. The well-orderedness of $\mathrm{Fix}(T)$ with respect to $\preccurlyeq$ implies the α-well-orderedness of $\mathrm{Fix}(T)$. So, uniqueness of the fixed point follows from Theorem 5. $\square$

Acknowledgment. The authors are thankful to the reviewers and the editor for their valuable comments and suggestions on this paper. The first author is also thankful to Professor Mahesh Dube and Professor Stojan Radenović for encouragement and support.

References

1. T. Abdeljawad, Meir–Keeler α-contractive fixed and commonfixed point theorems, *Fixed Point Theory Appl.*, **2013**, Article ID 19, 2013.
2. R.P. Agarwal, P. Kumam, W. Sintunavarat, PPF dependent fixed point theorems for an α_c-admissible non-self mapping in the Razumikhin class, *Fixed Point Theory Appl.*, **2013**, Article ID 280, 2013.
3. M.U. Ali, T. Kamran, N. Shahzad, Best proximity point for α-ψ-proximal contractive multimaps, *Abstr. Appl. Anal.*, **2014**, Article ID 181598, 2014.

4. M.U. Ali, Q. Kiran, N. Shahzad, Fixed point theorems for multi-valued mappings involving α-function, *Abstr. Appl. Anal.*, **2014**, Article ID 409467, 2014.
5. H. Alikhani, Sh. Rezapour, N. Shahzad, Fixed points of a new type contractive mappings and multifunctions, *Filomat*, **27**(7):1315–1319, 2013.
6. P. Amiri, S. Rezapour, N. Shahzad, Fixed points of generalized α-ψ-contractions, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, **108**(2):519–526, 2014.
7. J.H. Asl, S. Rezapour, N. Shahzad, On fixed points of α-ψ-contractive multifunctions, *Fixed Point Theory Appl.*, **2012**, Article ID 212, 2012.
8. Y.Z. Chen, A Prešić type contractive condition and its applications, *Nonlinear Anal., Theory Methods Appl.*, **71**(12):2012–2017, 2009.
9. L.B. Ćirić, S.B. Prešić, On Prešić type generalisation of Banach contraction principle, *Acta Math. Inform. Univ. Ostrav.*, **LXXVI**(2):143–147, 2007.
10. R. George, K.P. Reshma, R. Rajagopalan, A generalised fixed point theorem of Prešić type in cone metric spaces and application to Markov process, *Fixed Point Theory Appl.*, **2011**, Article ID 85, 2011.
11. D. Gopal, M. Abbas, C. Vetro, Some new fixed point theorems in Menger PM-spaces with application to Volterra type integral equation, *Applied Math. Comput.*, **232**:955–967, 2014.
12. D. Gopal, C. Vetro, Some new fixed point theorems in fuzzy metric spaces, *Iran. J. Fuzzy Syst.*, **11**:95–107, 2014.
13. M.E. Gordji, S. Pirbavafa, M. Ramezani, C. Park, D.Y. Shin, Prešić–Kannan–Rus fixed point theorem on partially order metric spaces, *Fixed Point Theory*, **15**(2):463–474, 2014.
14. E. Karapınar, P. Kumam, P. Salimi, On α-ψ-Meir–Keeler contractive mappings, *Fixed Point Theory Appl.*, **2013**, Article ID 94, 2013.
15. M.S. Khan, M. Berzig, B. Samet, Some convergence results for iterative sequences of Prešić type and applications, *Adv. Difference Equ.*, **2012**, Article ID 38, 2012.
16. W. Long, S. Khaleghizadeh, P. Salimi, S. Radenović, S. Shukla, Some new fixed point results in partial ordered metric spaces via admissible mappings, *Fixed Point Theory Appl.*, **2014**, Article ID 117, 2014.
17. N.V. Luong, N.X. Thuan, Some fixed point theorems of Prešić–Ćirić type, *Acta Univ. Apulensis, Math. Inform.*, **30**:237–249, 2012.
18. S.K. Malhotra, S. Shukla, R. Sen, A generalization of Banach contraction principle in ordered cone metric spaces, *J. Adv. Math. Stud.*, **5**(2):59–67, 2012.
19. B. Mohammadi, S. Rezapour, N. Shahzad, Some results on fixed points of α-ψ-Ciric generalized multifunctions, *Fixed Point Theory Appl.*, **2013**, Article ID 24, 2013.
20. J.J. Nieto, R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order*, **22**:223–239, 2005.
21. J.J. Nieto, R. Rodríguez-López, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, *Acta. Math. Sin., Engl. Ser.*, **23**(12):2205–2212, 2007.
22. M. Păcurar, A multi-step iterative method for approximating common fixed points of Prešić–Rus type operators on metric spaces, *Stud. Univ. Babeş-Bolyai, Math.*, **LV**(1):149–162, 2010.

23. M. Păcurar, Common fixed points for almost Prešić type operators, *Carpathian J. Math.*, **28**(1):117–126, 2012.
24. S.B. Prešić, Sur la convergence des suites, *C. R. Acad. Sci. Paris*, **260**:3828–3830, 1965.
25. S.B. Prešić, Sur une classe d'inéquations aux différences finite et sur la convergence de certaines suites, *Acad. Serbe Sci., Publ. Inst. Math.*, **5**(19):75–78, 1965.
26. A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some application to matrix equations, *Proc. Am. Math. Soc.*, **132**:1435–1443, 2004.
27. V. La Rosa, P. Vetro, Common fixed points for α-ψ-φ-contractions in generalized metric spaces, *Nonlinear Anal. Model. Control*, **19**(1):43–54, 2014.
28. P. Salimi, C. Vetro, P. Vetro, Some new fixed point results in non-Archimedean fuzzy metric spaces, *Nonlinear Anal. Model. Control*, **18**(3):344–358, 2013.
29. B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Anal., Theory Methods Appl.*, **75**(4):2154–2165, 2012.
30. N. Shahzad, S. Shukla, Set-valued G-Prešić operators on metric spaces endowed with a graph and fixed point theorems, *Fixed Point Theory Appl.*, **2015**, Article ID 24, 2015.
31. S. Shukla, Set-valued Prešić–Ćirić type contraction in 0-complete partial metric spaces, *Mat. Vesn.*, **66**(2):178–189, 2014.
32. S. Shukla, Some stability results and Assad–Kirk type fixed point theorems for set-valued Prešić type mappings, *J. Nonlinear Convex Anal.*, **16**(3):509–520, 2015.
33. S. Shukla, M. Abbas, Fixed point results of cyclic contractions in product spaces, *Carpathian J. Math.*, **31**(1):119–126, 2014.
34. S. Shukla, S. Radenović, Some generalizations of Prešić type mappings and applications, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Ser. Nouă, Mat.*, 2014 (accepted), doi:10.1515/aicu-2015-0026.
35. S. Shukla, R. Sen, Set-valued Prešić–Reich type mappings in metric spaces, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, **108**(2):431–440, 2012.
36. S. Shukla, N. Shahzad, G-Prešić operators on metric spaces endowed with a graph and fixed point theorems, *Fixed Point Theory Appl.*, **2014**, Article ID 127, 2014.