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Modeling and adaptive tracking for stochastic nonholonomic constrained mechanical systems*

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Abstract. This paper is devoted to the problem of modeling and trajectory tracking for stochastic nonholonomic dynamic systems in the presence of unknown parameters. Prior to tracking controller design, the rigorous derivation of stochastic nonholonomic dynamic model is given. By reasonably introducing so-called internal state vector, a reduced dynamic model, which is suitable for control design, is proposed. Based on the backstepping technique in vector form, an adaptive tracking controller is then derived, guaranteeing that the mean square of the tracking error converges to an arbitrarily small neighborhood of zero by tuning design parameters. The efficiency of the controller is demonstrated by a mechanics system: a vertical mobile wheel in random vibration environment.

Keywords: stochastic systems, nonholonomic dynamic systems, mechanics model, backstepping, adaptive tracking control.

1 Introduction

Nonholonomic systems have been widely accepted as ones that are subject to nonintegrable constraints and whose behaviors must comply with the constraints [16]. There are extensive examples of nonholonomic systems such as mobile cars [17, 18], trailers [1, 20], space robots [6, 9], and underactuated manipulators [2]. In the field of control theory, tracking problem, which is to make the entire closed-loop system track a given desired trajectory, is one of the fundamental control issue. For the latest two decades, the tracking

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control problem for nonholonomic systems has been an active research direction in the control community because of its widespread applications in the real word, and some control techniques such as backstepping method [10], cascaded design approach [7, 11, 18], recurrent neural network [22], and adaptive fuzzy approach [4] are used. However, it should be noted that the most existing references mainly consider the systems in the deterministic setting.

Since the research on stochastic Hamiltonian systems [23] and stochastic Lagrangian systems [5] has become an important direction, it is naturally expected that nonholonomic systems can work in the random vibration environment. Thanks to the presence of [21], which designed the state feedback controller for a class of nonholonomic systems with stochastic disturbances, the stabilization problem had been extensively studied for stochastic nonholonomic at kinematic level. As a continuous work of [21], [14] investigated the output feedback stabilization problem. Recently, the problem of state feedback stabilization for high order stochastic nonholonomic systems was studied in [24]. Since nonlinear parameters are commonly existing components in many practical control systems, [8] considered the problem of adaptive stabilization by state feedback for stochastic nonholonomic systems with nonlinear parameterization. However, many nonholonomic systems in reality possess significant dynamics, and the system inputs are limited to torques or forces generated by the physical actuators. Therefore, it will be more realistic to research the tracking problem at dynamic level than that at kinematic level. In contrast, the dynamic nonholonomic systems with stochastic disturbances have received less attention. Stochastic source seeking for nonholonomic unicycle was investigated in [13], and the modeling and tracking control for general stochastic dynamic nonholonomic systems are still in urgent need.

In this work, we consider the modeling and dynamic tracking problems for nonholonomic systems in the presence of stochastic disturbances. Prior to controller design, the dynamic model for nonholonomic systems in random vibration environment is given. To achieve the tracking objective, by using the Algebra processing technique, we triumphantly reduce the number of state variables, which provides a motion complying with nonholonomic constraints. Based on the reduced dynamic model, a robust tracking control algorithm, inspired by the designs in [23], is then derived. The contributions of this paper are as follows.

1) Because the emerge of stochastic disturbances is the main difficulty in control design for mechanical systems, it is needed to develop a reasonable stochastic model for nonholonomic systems. In this paper, we propose a class of stochastic dynamic nonholonomic systems to describe the motion of nonholonomic systems subject to random disturbances.
2) An adaptive controller is proposed ensuring that the limit of tracking error can be made arbitrarily small by flexibly choosing design parameters, and so does its derivative.
3) As a practical application, a vertical mobile wheel in random vibration environment is given to demonstrate the reasonability of the assumptions and the effectiveness of the modeling and control strategies.

The rest of this paper is organized as follows. Section 2 contains some preliminaries. Problem formulation is presented in Section 3. Section 4 describes the systematic tracking control scheme and proposes the main result. Section 5 provides the simulation results. Conclusions are offered in Section 6.

Notations. $\mathbb{R}_+$ stands for the set of all nonnegative real numbers, $\mathbb{R}^n$ is the n-dimensional Euclidean space. $\mathbb{R}^{n\times r}$ represents the real $n\times r$ matrix space. For a vector, $|x|$ is the usual Euclidean norm and x^{T} is its transpose. $\|A\|_F$ denotes the Frobenius norm for a matrix A defined as $\|A\|_F = (\mathrm{Tr}\{A^{\mathrm{T}}A\})^{1/2}$, where $\mathrm{Tr}\{\cdot\}$ is the square matrix trace. $\mathbf{E}$ denotes the mathematical expectation. $\mathcal{K}$ denotes the set of all functions $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, strictly increasing, and $\gamma(0) = 0$. Let $\mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$ denote the family of all real-valued functions $V(x,t)$ defined on $\mathbb{R}^n \times \mathbb{R}_+$ such that they are continuously twice differentiable in x and once in t. For simplicity, sometimes the arguments of functions are dropped when no confusion arises.

2 Preliminaries

2.1 Stochastic nonlinear systems

Consider the n-dimensional stochastic differential equation of Itô type

$$\mathrm{d}x(t) = f\bigl(x(t),t\bigr)\,\mathrm{d}t + g\bigl(x(t),t\bigr)\,\mathrm{d}W(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of system, $W(t)$ stands for an r-dimensional independent standard Wiener process (or Brownian motion), and the underlying complete probability space is picked to be the quartet $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions (namely, it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Both functions $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times r}$ are locally Lipschitz in $x \in \mathbb{R}^n$ and piecewise continuous in $t \in \mathbb{R}_+$, that is, for any $k > 0$, there is a constant $C_k \geqslant 0$ such that

$$\bigl|f(x_1,t) - f(x_2,t)\bigr| + \bigl\|g(x_1,t) - g(x_2,t)\bigr\|_F \leqslant C_k\bigl|x_1 - x_2\bigr|$$

for any $t \in \mathbb{R}_+$ and $x_1, x_2 \in U_k = \{\xi\colon |\xi| \leqslant k\}$. In addition, when $x = 0$ almost surely, $f(0,t)$ and $g(0,t)$ are bounded almost surely.

For control design and stability analysis, the definitions and lemma listed below will be used.

Definition 1. (See [23].) For $V(x,t) \in \mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}_+)$, the infinitesimal generator is specified by

$$\mathcal{L}V(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}\,\mathrm{Tr}\bigl\{g^{\mathrm{T}}(x,t)V_{xx}(x,t)g(x,t)\bigr\},$$

where

$$V_t = \frac{\partial V}{\partial t}, \quad V_x = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \dots, \frac{\partial V}{\partial x_n}\right) \quad \text{and} \quad V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{n\times n}.$$

Definition 2. (See [5].) System (1) is said to be pth moment exponentially practically stable provided that there exist positive real numbers λ, d, and a function $\varrho \in \mathcal{K}$ such that

$$\mathbf{E}\bigl|x(t)\bigr|^p \leqslant \varrho\bigl(\bigl|x(t_0)\bigr|\bigr)\mathrm{e}^{-\lambda(t-t_0)} + d, \quad t \geqslant t_0.$$

In particular, when $p = 2$, it is usually said to be exponentially practically stable in mean square.

Lemma 1. (*See* [5].) *Consider system* (1). *Suppose that there are a function* $V(x,t) \in \mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}_+)$*, positive constants* a_i, a_i', p_i, p_i', c, *and* d_c *such that*

$$\sum_{i=1}^{n} a_i|x_i|^{p_i} \leqslant V(x,t) \leqslant \sum_{i=1}^{n} a_i'|x_i|^{p_i'},$$
$$\mathcal{L}V(x,t) \leqslant -cV(x,t) + d_c.$$

Then there exists a unique strong solution $x(t) = x(t; x_0, t_0)$ *of system* (1) *for each initial state* $x(t_0) = x_0 \in \mathbb{R}^n$ *and system* (1) *is pth moment exponentially practically stable with* $p = \min\{p_1, \ldots, p_n\}$.

2.2 Nonholonomic dynamic systems under random excitation

Let us consider the Lagrangian function of general form

$$L(q,\dot q) = T(q,\dot q) - U(q) = \frac{1}{2}\dot q^{\mathrm{T}} D(q)\dot q - U(q), \tag{2}$$

where $q = (q_1, q_2, \ldots, q_n)^{\mathrm{T}} \in \mathbb{R}^n$ is generalized configuration coordinate for a mechanical system, the inertia matrix $D(q) \in \mathbb{R}^{n\times n}$ is symmetric and positive definite for all q, and $U(q)$ is potential energy. According to Proposition 7.1.1 in [16], under steady and ideal constraints, the following Euler–Lagrange equation can be described as

$$\sum_{s=1}^{n}\biggl(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot q_s} + \frac{\partial T}{\partial q_s} + Q_s\biggr)\delta q_s = 0, \tag{3}$$

where Q_s and δq_s $(s = 1, 2, \ldots, n)$ are generalized forces and virtual displacements, respectively. Normally, the generalized force Q_s can be decomposed as the potential force $-\partial U(q)/\partial q_s$, the dissipative force $\varXi_s(q,\dot q)$, the control input force $\sum_{i=1}^{r} h_{si}(q)\tau_i$ and the random excitation $\sum_{i=1}^{r} \varDelta_{si}(q,\dot q)\xi_i$ with $r \leqslant n$, h_{si} being the (s,i)th element of $H(q)$, $\varDelta_{si}$ being the (s,i)th element of $\varDelta(q,\dot q)$ and random disturbance signal $\xi = (\xi_1, \ldots, \xi_r)^{\mathrm{T}}$. Substituting this decomposition into (3), we obtain

$$\sum_{s=1}^{n}\biggl(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_s} + \frac{\partial L}{\partial q_s} + \varXi_s + \sum_{i=1}^{r}(h_{si}\tau_i + \varDelta_{si}\xi_i)\biggr)\delta q_s = 0. \tag{4}$$

Remark 1. The reason why we assume that the number of random excitation ξ_i equals to the number of control torque τ_i is to reduce the complexity of mathematical formula. Indeed, there is no need to require such a constraint.

The classical nonholonomic constraint is represented as [18]

$$J(q)\dot{q} = 0, \tag{5}$$

where $\operatorname{rank} J(q) = g$ $(g < n)$ for all $q \in \mathbb{R}^n$ and

$$J(q) = \begin{bmatrix} j_{11}(q) & j_{12}(q) & \dots & j_{1n}(q) \\ j_{21}(q) & j_{22}(q) & \dots & j_{2n}(q) \\ \vdots & \vdots & & \vdots \\ j_{g1}(q) & j_{g2}(q) & \dots & j_{gn}(q) \end{bmatrix}.$$

We now rewrite (5) as

$$\sum_{s=1}^{n} \tilde{j}_{\varepsilon+\beta,s}(q)\dot{q}_s = 0, \quad \beta = 1, 2, \dots, g;\ \varepsilon = n - g, \tag{6}$$

with $\tilde{j}_{\varepsilon+\beta,s}(q) = j_{\beta s}(q)$. By virtue of Appell–Chetaev condition (see (4.1.12) in [16]), and in view of (6), there holds

$$\sum_{s=1}^{n} \tilde{j}_{\varepsilon+\beta,s}\delta q_s = 0. \tag{7}$$

When a mechanical system's motion is subject to the nonholonomic constraint (5), multiplying (7) by Lagrangian multipliers λ_β and summing for β, one has

$$\sum_{s=1}^{n} \left(\sum_{\beta=1}^{g} \lambda_\beta \tilde{j}_{\varepsilon+\beta,s} \right) \delta q_s = 0,$$

which, together with (4), leads to

$$\sum_{s=1}^{n} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} + \frac{\partial L}{\partial q_s} + \Xi_s + \sum_{i=1}^{r} (h_{si}\tau_i + \Delta_{si}\xi_i) + \sum_{\beta=1}^{g} \lambda_\beta \tilde{j}_{\varepsilon+\beta,s} \right) \delta q_s = 0. \tag{8}$$

Based on the fact of $\operatorname{rank} J(q) = g$, we can assume the following determinant:

$$\begin{vmatrix} \tilde{j}_{\varepsilon+1,\varepsilon+1} & \tilde{j}_{\varepsilon+1,\varepsilon+2} & \cdots & \tilde{j}_{\varepsilon+1,n} \\ \tilde{j}_{\varepsilon+2,\varepsilon+1} & \tilde{j}_{\varepsilon+2,\varepsilon+2} & \cdots & \tilde{j}_{\varepsilon+2,n} \\ \vdots & \vdots & & \vdots \\ \tilde{j}_{n,\varepsilon+1} & \tilde{j}_{n,\varepsilon+2} & \cdots & \tilde{j}_{nn} \end{vmatrix} \neq 0.$$

Hence, there are Lagrangian multipliers λ_β such that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_{\varepsilon+\gamma}} + \frac{\partial L}{\partial q_{\varepsilon+\gamma}} + \Xi_{\varepsilon+\gamma} + \sum_{i=1}^{r} (h_{\varepsilon+\gamma,i}\tau_i + \Delta_{\varepsilon+\gamma,i}\xi_i) + \sum_{\beta=1}^{g} \lambda_\beta \tilde{j}_{\varepsilon+\beta,\varepsilon+\gamma} = 0, \tag{9}$$

where $\gamma = 1, 2, \dots, g$.

From (8) and (9), it follows that

$$\sum_{\sigma=1}^{\varepsilon}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\sigma}+\frac{\partial L}{\partial q_\sigma}+\Xi_\sigma+\sum_{i=1}^{r}(h_{\sigma i}\tau_i+\Delta_{\sigma i}\xi_i)+\sum_{\beta=1}^{g}\lambda_\beta\tilde{j}_{\varepsilon+\beta,\sigma}\right)\delta q_\sigma=0. \quad (10)$$

By the independence of δq_σ in (10), one has

$$-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\sigma}+\frac{\partial L}{\partial q_\sigma}+\Xi_\sigma+\sum_{i=1}^{r}(h_{\sigma i}\tau_i+\Delta_{\sigma i}\xi_i)+\sum_{\beta=1}^{g}\lambda_\beta\tilde{j}_{\varepsilon+\beta,\sigma}=0, \quad \sigma=1,2,\dots,\varepsilon. \quad (11)$$

Taking (9) and (11) into account, it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=\Xi_\sigma+\sum_{i=1}^{r}(h_{\sigma i}\tau_i+\Delta_{\sigma i}\xi_i)+\sum_{\beta=1}^{g}\lambda_\beta\tilde{j}_{\varepsilon+\beta,s}, \quad s=1,2,\dots,n,$$

or, more compactly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right)-\frac{\partial L}{\partial q}(q,\dot{q})=H(q)\tau+\Delta(q,\dot{q})\xi+\Xi(q,\dot{q})+J^{\mathrm{T}}(q)\lambda, \quad (12)$$

where $\tau=(\tau_1,\tau_2,\dots,\tau_r)^{\mathrm{T}}$, $\Xi=(\Xi_1,\Xi_2,\dots,\Xi_n)^{\mathrm{T}}$, and $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_g)^{\mathrm{T}}$.

Since $L(q,\dot{q})=(1/2)\sum_{i=1}^{n}\sum_{j=1}^{n}d_{ij}(q)\dot{q}_i\dot{q}_j-U(q)$ with d_{ij} being the (i,j)th element of $D(q)$, (12) can be rewritten as

$$\begin{aligned}\sum_{j=1}^{n}d_{kj}(q)\ddot{q}_j+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial d_{kj}}{\partial q_i}-\frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j+\frac{\partial U}{\partial q_k}\\ =H_k(q)\tau+\Delta_k(q,\dot{q})\xi+\Xi_k(q,\dot{q})+J_k^{\mathrm{T}}(q)\lambda,\end{aligned} \quad (13)$$

where $k=1,2,\dots,n$, H_k, Ξ_k and J_k^{T} are the kth row vector of H, Ξ and J^{T}, respectively. Noticing $\sum_{i=1}^{n}\sum_{j=1}^{n}(\partial d_{kj}/\partial q_i)\dot{q}_i\dot{q}_j=\sum_{i=1}^{n}\sum_{j=1}^{n}(1/2)(\partial d_{kj}/\partial q_i+\partial d_{ki}/\partial q_j)\dot{q}_i\dot{q}_j$ and defining the Christoffel symbols $\tilde{c}_{ijk}=(1/2)(\partial d_{kj}/\partial q_i+\partial d_{ki}/\partial q_j)-\partial d_{ij}/\partial q_k$, one can transform (13) into

$$\begin{aligned}\sum_{j=1}^{n}d_{kj}(q)\ddot{q}_j+\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{c}_{ijk}(q)\dot{q}_i\dot{q}_j+\frac{\partial U}{\partial q_k}\\ =H_k(q)\tau+\Delta_k(q,\dot{q})\xi+\Xi_k(q,\dot{q})+J_k^{\mathrm{T}}(q)\lambda.\end{aligned} \quad (14)$$

By defining $G(q)=(\partial U/\partial q_1,\partial U/\partial q_2,\dots,\partial U/\partial q_n)^{\mathrm{T}}$ and $c_{kj}=\sum_{i=1}^{n}\tilde{c}_{ijk}(q)\dot{q}_i$ as the (k,j)th element of matrix $C(q,\dot{q})$, equation (14) can be described in general form

$$D(q)\ddot{q}+C(q,\dot{q})\dot{q}+G(q)=J^{\mathrm{T}}(q)\lambda+\Xi(q,\dot{q})+H(q)\tau+\Delta(q,\dot{q})\xi. \quad (15)$$

Based on the arguments stated above, the stochastic dynamic equation of nonholonomic systems when satisfying constraint (5) is obtained as follows:

$$\begin{aligned}&D(q)\ddot{q}+C(q,\dot{q})\dot{q}+G(q)=J^{\mathrm{T}}(q)\lambda+\Xi(q,\dot{q})+H(q)\tau+\Delta(q,\dot{q})\xi,\\ &J(q)\dot{q}=0.\end{aligned} \quad (16)$$

3 Problem formulation

Since rank $J(q) = g$, row vectors of $J(q)$ are independent each other for any q. Hence, we here consider vectors that are independent each other and annihilate row vectors of $J(q)$ and denote them by $s_1(q), \dots, s_{n-g}(q)$. Let $S(q)$ be the full rank matrix made up of these vectors:

$$S(q) = \bigl[s_1(q), \dots, s_{n-g}(q)\bigr]$$

such that

$$S^{\mathrm{T}}(q)J^{\mathrm{T}}(q) = 0. \tag{17}$$

From (5) and (17), it follows that there exists an $(n-g)$-dimensional vector z satisfying

$$\dot{q} = S(q)\dot{z}, \tag{18}$$

where $\dot{z}$ is called the internal state variable vector.

Taking time derivative of (18) results in

$$\ddot{q} = \dot{S}(q)\dot{z} + S(q)\ddot{z}.$$

Thus, the dynamic equation (15), when satisfying the constraint (5), can be reformulated in terms of the internal state variable $\dot{z}$ as

$$D(q)S(q)\ddot{z} + \bar{C}(q,\dot{q})\dot{z} + G(q) = H(q)\tau + \varDelta(q,\dot{q})\xi + J^{\mathrm{T}}(q)\lambda + \varXi(q,\dot{q}), \tag{19}$$

where $\bar{C}(q,\dot{q}) = D(q)\dot{S}(q) + C(q,\dot{q})S(q)$.

Remark 2. Up to now, in view of (19), the equality constraint equation (5) has been embedded into the dynamic equation (19). This implies that the resulting affine nonlinear system is suitable for control purposes and forms the basis for the subsequent developments.

Multiplying $S^{\mathrm{T}}(q)$ on both sides of (19) and noting $S^{\mathrm{T}}(q)J^{\mathrm{T}}(q) = 0$, one has

$$D_1(q)\ddot{z} + C_1(q,\dot{q})\dot{z} + G_1(q) = S^{\mathrm{T}}(q)H(q)\tau + S^{\mathrm{T}}(q)\varDelta(q,\dot{q})\xi + S^{\mathrm{T}}(q)\varXi(q,\dot{q}), \tag{20}$$

where $D_1(q) = S^{\mathrm{T}}(q)D(q)S(q)$, $C_1(q,\dot{q}) = S^{\mathrm{T}}(q)\bar{C}(q,\dot{q})$, and $G_1(q) = S^{\mathrm{T}}(q)G(q)$. By substituting $\mathrm{d}B/\mathrm{d}t$ for ξ in (20), the Stratonovich stochastic differential equation is given by

$$\begin{aligned} \mathrm{d}z &= \dot{z}\,\mathrm{d}t, \\ \mathrm{d}\dot{z} &= \bigl(-D_1^{-1}(q)\bigl(C_1(q,\dot{q})\dot{z} + G_1(q) - S^{\mathrm{T}}(q)\varXi(q,\dot{q})\bigr) \\ &\quad + D_1^{-1}(q)S^{\mathrm{T}}(q)H(q)\tau\bigr)\,\mathrm{d}t + D_1^{-1}(q)S^{\mathrm{T}}(q)\varDelta(q,\dot{q}) \circ \mathrm{d}B, \end{aligned}$$

where B is an r-dimensional independent Wiener process. According to equation (6.1.3) in [19], it can be easily verified that the Wong–Zakai correction term equals to

$$\frac{1}{2}\begin{pmatrix} \varLambda_1\partial\varLambda_1/\partial q + \varLambda_2\partial\varLambda_1/\partial\dot{q} \\ \varLambda_1\partial\varLambda_2/\partial q + \varLambda_2\partial\varLambda_2/\partial\dot{q} \end{pmatrix} = \begin{pmatrix} 0 \\ \varOmega(q,\dot{q}) \end{pmatrix},$$

where $\Lambda_1 = 0$, $\Lambda_2 = D_1^{-1}(q)S^{\mathrm{T}}(q)\Delta(q,\dot q)$ and $\Omega(q,\dot q) = (1/2)(\Lambda_1 \partial\Lambda_2/\partial q + \Lambda_2 \partial\Lambda_2/\partial\dot q)$. Therefore, the equivalent Itô stochastic differential equation is defined as

$$\begin{aligned} \mathrm{d}z &= \dot z\,\mathrm{d}t,\\ \mathrm{d}\dot z &= \bigl(-D_1^{-1}(q)\bigl(C_1(q,\dot q)\dot z + G_1(q) - S^{\mathrm{T}}(q)\Xi(q,\dot q)\bigr) + \Omega(q,\dot q)\\ &\quad + D_1^{-1}(q)S^{\mathrm{T}}(q)H(q)\tau\bigr)\,\mathrm{d}t + D_1^{-1}(q)S^{\mathrm{T}}(q)\Delta(q,\dot q)\,\mathrm{d}B. \end{aligned} \tag{21}$$

Remark 3. It was pointed in [25, p. 109] that compared with Itô stochastic differential equation, Stratonovich stochastic differential equation has a more closer relationship with physical system. So, it is more realistic to transform the kinematic differential equation into Stratonovich stochastic differential equation when dealing with the practical dynamic system under random excitation. On the other hand, since Itô stochastic calculus possesses a series of good properties, Itô stochastic differential equation is frequently used to handle stochastic differential equation (including the existence and uniqueness of solution, stochastic stability, stochastic control and so on). In order to make full use of these mathematic researches, the Stratonovich stochastic differential equation derived from practical problem is often converted into an equivalent Itô stochastic differential equation in control design.

To facilitate the upcoming control designs, we rearrange some terms in (21) as follows:

$$C_1(q,\dot q)\dot z + G_1(q) - S^{\mathrm{T}}(q)\Xi(q,\dot q) = \omega(q,\dot q) + \varpi(q,\dot q),$$

where ω is a known function and ϖ is an unknown function. Furthermore, suppose the power spectral density of white noise ξ equals to $\Sigma/(2\pi)$, which implies $\mathrm{d}B = \Sigma\,\mathrm{d}W$, where $\Sigma \in \mathbb{R}^{r\times r}$ is an unknown nonnegative matrix and W is an r-dimensional independent standard Wiener process. Under this, system (21) can be rewritten as

$$\begin{aligned} \mathrm{d}z &= \dot z\,\mathrm{d}t,\\ \mathrm{d}\dot z &= \bigl(-D_1^{-1}(q)\bigl(\omega(q,\dot q) + \varpi(q,\dot q)\bigr) + D_1^{-1}(q)S^{\mathrm{T}}(q)H(q)\tau + \Omega(q,\dot q)\bigr)\,\mathrm{d}t\\ &\quad + D_1^{-1}(q)S^{\mathrm{T}}(q)\Delta(q,\dot q)\Sigma\,\mathrm{d}W. \end{aligned} \tag{22}$$

By appropriately selecting a set of $(n-g)$-vector of variables $z(q)$ and $\dot z(q)$, the *control objective can be specified as: given a desired z_d and $\dot z_d$, seek a control law τ such that, for any $(q(0),\dot q(0)) \in \Omega_{nh}$, z and $\dot z$ converge to a manifold: $\Omega_{nhd} = \{(q,\dot q) \mid z(q) = z_d,\ \dot q = S(q)\dot z_d\}$ as closely as possible, while keeping all other signals in closed-loop system bounded in probability*. To this end, the following assumptions are needed in this paper.

Assumption 1. The known matrices $S(q)$, $D(q)$, $C(q,\dot q)$, $G(q)$, $H(q)$, and unknown matrices $\Xi(q,\dot q)$ and $\Delta(q,\dot q)$ are functions of both variables z and $\dot z$ only.

Assumption 2. There are unknown positive parameters ϑ_i $(i = 1, 2, 3, 4)$ and known smooth nonnegative functions $\phi_1(z, \dot{z})$, $\phi_2(z, \dot{z})$, $\phi_3(z, \dot{z})$, and $\psi(z, \dot{z}, z_d, \dot{z}_d)$ such that

$$\begin{aligned}
\left|\varpi(q, \dot{q})\right|^2 &\leqslant \phi_1(z, \dot{z})\vartheta_1, \\
\left|\Omega(q, \dot{q})\right|^2 &\leqslant \phi_2(z, \dot{z})\vartheta_2, \\
\left\|\Delta(q, \dot{q})\right\|_F^2 &\leqslant \phi_3(z, \dot{z})\vartheta_3, \\
\left\|\Delta(q, \dot{q}) - \Delta(q_d, \dot{q}_d)\right\|_F^2 &\leqslant \psi(z, \dot{z}, z_d, \dot{z}_d)\left(|z - z_d|^2 + |\dot{z} - \dot{z}_d|^2\right)\vartheta_4.
\end{aligned}$$

Assumption 3. $r \geqslant n - g$ and the matrix $S^{\mathrm{T}}(q)H(q)$ is of row full rank.

Remark 4. There are three points to be emphasized:

1) The hypothesis that system matrices $S(q)$, $D(q)$, $C(q, \dot{q})$, $G(q)$, and $H(q)$ only rely on z and $\dot{z}$ holds for most nonholonomic systems, such as wheeled mobile robot [7, 18] and vertical wheel rolling [4].
2) Assumption 2 is reasonable. The first three inequations in Assumption 2 are the direct results of Lemma 2.1 in [12]. For the last inequation in Assumption 2, applying the identity $f(x) - f(x_0) = (\int_0^1 (\mathrm{d}f/\mathrm{d}\lambda)|_{\lambda=\alpha(x-x_0)+x_0}\,\mathrm{d}\alpha)(x - x_0) = \bar{f}(x, x_0)(x - x_0)$ to the elements of $\Delta(q, \dot{q})$, a continuous function $\bar{\psi}$ can be found in such a way that $\|\Delta(q, \dot{q}) - \Delta(q_d, \dot{q}_d)\|_F^2 \leqslant \bar{\psi}(z, \dot{z}, z_d, \dot{z}_d)(|z - z_d|^2 + |\dot{z} - \dot{z}_d|^2)$. In view of Lemma 2.1 in [12], the last inequation holds readily.
3) Assumption 3 guarantees all $n - g$ degrees of freedom can be (independently) actuated. This assumption also holds for a large class of nonholonomic mechanical systems such as the aforementioned wheeled mobile, a knife-edge and Bottema platform. In these systems, the internal state $\dot{z}(q)$ and variable $z(q)$ possess practical physical meanings.

4 Control design and stability analysis

4.1 Adaptive tracking control via state feedback

Define tracking error $e_1(t) = z(t) - z_d(t)$. To develop a helpful tracking error system, similar to [3], we introduce the filtered tracking error $e_2(t) = \dot{e}_1(t) + k_1 e_1(t)$, where k_1 is a positive design parameter. From (22), the following error dynamic equations are obtained:

$$\begin{aligned}
\mathrm{d}e_1 &= (e_2 - k_1 e_1)\,\mathrm{d}t, \\
\mathrm{d}e_2 &= \big(-D_1^{-1}(q)\big(\omega(q, \dot{q}) + \varpi(q, \dot{q})\big) + \Omega(q, \dot{q}) - \ddot{z}_d - k_1^2 e_1 + k_1 e_2 \\
&\quad + D_1^{-1}(q)S^{\mathrm{T}}(q)H(q)\tau\big)\,\mathrm{d}t + D_1^{-1}(q)S^{\mathrm{T}}(q)\Delta(q, \dot{q})\Sigma\,\mathrm{d}W.
\end{aligned} \tag{23}$$

Step 1. Pick the Lyapunov function candidate

$$U_1 = \frac{1}{4}\left(e_1^{\mathrm{T}} e_1\right)^2.$$

Its infinitesimal generator along the solution of system (23) satisfies

$$\mathcal{L}U_1 = e_1^{\mathrm{T}} e_1 e_1^{\mathrm{T}} (e_2 - k_1 e_1) = -k_1 \big(e_1^{\mathrm{T}} e_1\big)^2 + e_1^{\mathrm{T}} e_1 e_1^{\mathrm{T}} e_2. \tag{24}$$

Step 2. Define $\theta = \max\{\vartheta_1, \vartheta_2, \|\varSigma\|_F^4 \vartheta_4^2, \|\varSigma\|_F^2 \vartheta_4, \|\varSigma\|_F^4 \vartheta_3^2\}$ and consider the Lyapunov function candidate

$$U_2 = \frac{1}{4}\big(e_1^{\mathrm{T}} e_1\big)^2 + \frac{1}{4}\big(e_2^{\mathrm{T}} e_2\big)^2 + \frac{\lambda}{2}\tilde{\theta}^2,$$

where $\lambda > 0$ is a design parameter, $\tilde{\theta} = \hat{\theta} - \theta$, and $\hat{\theta}$ is the estimate of θ.

By taking (23) and (24) into account, it yields that

$$\begin{aligned} \mathcal{L}U_2 = {} & -k_1\big(e_1^{\mathrm{T}} e_1\big)^2 + e_2^{\mathrm{T}} e_2 e_2^{\mathrm{T}} \big(-D_1^{-1}(q)\big(\omega(q,\dot{q}) + \varpi(q,\dot{q})\big) + \varOmega(q,\dot{q})\big) \\ & + e_1^{\mathrm{T}} e_1 e_1^{\mathrm{T}} e_2 - e_2^{\mathrm{T}} e_2 e_2^{\mathrm{T}} \ddot{z}_d - k_1^2 e_2^{\mathrm{T}} e_2 e_2^{\mathrm{T}} e_1 + k_1 e_2^{\mathrm{T}} e_2 e_2^{\mathrm{T}} e_2 \\ & + \frac{1}{2}\operatorname{Tr}\big\{\varSigma^{\mathrm{T}} \varDelta^{\mathrm{T}}(q,\dot{q}) S(q) D_1^{-T}(q)\big(2e_2 e_2^{\mathrm{T}} + e_2^{\mathrm{T}} e_2 I\big) D_1^{-1}(q) S^{\mathrm{T}}(q) \varDelta(q,\dot{q})\varSigma\big\} \\ & + \lambda\tilde{\theta}\dot{\hat{\theta}} + e_2^{\mathrm{T}} e_2 e_2^{\mathrm{T}} D_1^{-1}(q) S^{\mathrm{T}}(q) H(q)\tau. \end{aligned} \tag{25}$$

By virtue of Young's inequality, it can be directly deduced that

$$e_1^{\mathrm{T}} e_1 e_1^{\mathrm{T}} e_2 \leqslant |e_1|^3 |e_2| \leqslant \frac{k_1}{4}\big(e_1^{\mathrm{T}} e_1\big)^2 + \frac{27}{4k_1^3}\big(e_2^{\mathrm{T}} e_2\big)^2. \tag{26}$$

Using Assumption 2 and noting $|A| \leqslant \|A\|_F$ for a vector or matrix A, one has

$$\begin{aligned} \big|-D_1^{-1}(q)\varpi(q,\dot{q}) + \varOmega(q,\dot{q})\big|^2 & \leqslant 2\big\|D_1^{-1}\big\|_F^2 \|\varpi\|_F^2 + 2\|\varOmega\|_F^2 \\ & \leqslant 2\big\|D_1^{-1}\big\|_F^2 \phi_1(z,\dot{z})\vartheta_1 + 2\phi_2(z,\dot{z})\vartheta_2 \\ & \leqslant \varPhi_1(z,\dot{z})\theta, \end{aligned} \tag{27}$$

where $\varPhi_1 = 2(\|D_1^{-1}\|_F^2 \phi_1(z,\dot{z}) + \phi_2(z,\dot{z}))$. Accordingly, in view of (27) and Young's inequality, it follows that

$$\begin{aligned} & e_2^{\mathrm{T}} e_2 e_2^{\mathrm{T}} \big(-D_1^{-1}\varpi + \varOmega\big) \\ & \quad \leqslant |e_2|^3 \big|-D_1^{-1}\varpi + \varOmega\big| \leqslant |e_2|^3 \big(\varPhi_1(z,\dot{z})\theta\big)^{1/2} = |e_2|\big(|e_2|(\varPhi_1\theta)^{1/4}\big)^2 \\ & \quad \leqslant \frac{e_2^{\mathrm{T}} e_2}{4} + \big(e_2^{\mathrm{T}} e_2\big)^2 \varPhi_1\theta \leqslant \frac{\beta_1 (e_2^{\mathrm{T}} e_2)^2}{8} + \big(e_2^{\mathrm{T}} e_2\big)^2 \varPhi_1\theta + \frac{1}{2\beta_1} \end{aligned} \tag{28}$$

with a design parameter $\beta_1 > 0$.

Next, we pay attention to the Hessian term on the right-hand side of (25). Prior to this purpose, we establish the following inequality:

$$\begin{aligned}
&\left\|D_1^{-1}(q)S^{\mathrm{T}}(q)\Delta(q,\dot q)\Sigma\right\|_F^2\\
&\quad\leqslant \left\|D_1^{-1}\right\|_F^2\|S\|_F^2\left\|\Delta(q,\dot q)-\Delta(q_d,\dot q_d)+\Delta(q_d,\dot q_d)\right\|_F^2\|\Sigma\|_F^2\\
&\quad\leqslant \left\|D_1^{-1}\right\|_F^2\|S\|_F^2\|\Sigma\|_F^2\left(2\left\|\Delta(q,\dot q)-\Delta(q_d,\dot q_d)\right\|_F^2+2\left\|\Delta(q_d,\dot q_d)\right\|_F^2\right)\\
&\quad\leqslant \left\|D_1^{-1}\right\|_F^2\|S\|_F^2\|\Sigma\|_F^2\left(2\psi(z,\dot z,z_d,\dot z_d)\left(|e_1|^2+|e_2-k_1e_1|^2\right)\vartheta_4+2\phi_3(z_d,\dot z_d)\vartheta_3\right)\\
&\quad\leqslant \left\|D_1^{-1}\right\|_F^2\|S\|_F^2\left(2\left(2k_1^2+1\right)\psi|e_1|^2\|\Sigma\|_F^2\vartheta_4\right.\\
&\qquad\left.+4\psi|e_2|^2\|\Sigma\|_F^2\vartheta_4+2\phi_3(z_d,\dot z_d)\|\Sigma\|_F^2\vartheta_3\right),
\end{aligned}$$

which, together with Young's inequality, leads to

$$\begin{aligned}
&\frac12\operatorname{Tr}\left\{\Sigma^{\mathrm{T}}\Delta^{\mathrm{T}}SD_1^{-T}\left(2e_2e_2^{\mathrm{T}}+e_2^{\mathrm{T}}e_2I\right)D_1^{-1}S^{\mathrm{T}}\Delta\Sigma\right\}\\
&\quad\leqslant\frac32|e_2|^2\left\|D_1^{-1}S^{\mathrm{T}}\Delta\Sigma\right\|_F^2\\
&\quad\leqslant\frac32|e_2|^2\left(\left\|D_1^{-1}\right\|_F^2\|S\|_F^2\left(2\left(2k_1^2+1\right)\psi|e_1|^2\|\Sigma\|_F^2\vartheta_4\right.\right.\\
&\qquad\left.\left.+4\psi|e_2|^2\|\Sigma\|_F^2\vartheta_4+2\phi_3(z_d,\dot z_d)\|\Sigma\|_F^2\vartheta_3\right)\right)\\
&\quad\leqslant\frac{k_1}{4}|e_1|^4+\frac{9}{k_1}\left(2k_1^2+1\right)^2|e_2|^4\left\|D_1^{-1}\right\|_F^4\|S\|_F^4\psi^2\|\Sigma\|_F^4\vartheta_4^2\\
&\qquad+6|e_2|^4\left\|D_1^{-1}\right\|_F^2\|S\|_F^2\psi\|\Sigma\|_F^2\vartheta_4\\
&\qquad+9\beta_2|e_2|^4\phi_3^2(z_d,\dot z_d)\left\|D_1^{-1}\right\|_F^4\|S\|_F^4\|\Sigma\|_F^4\vartheta_3^2+\frac{1}{4\beta_2}\\
&\quad\leqslant\frac{k_1}{4}\left(e_1^{\mathrm{T}}e_1\right)^2+\Phi_2(e_1,e_2,z_d,\dot z_d)\theta\left(e_2^{\mathrm{T}}e_2\right)^2+\frac{1}{4\beta_2},
\end{aligned}\tag{29}$$

where $\beta_2>0$ is a design parameter and

$$\begin{aligned}
\Phi_2&=\frac{9}{k_1}\left(2k_1^2+1\right)^2\left\|D_1^{-1}\right\|_F^4\|S\|_F^4\psi^2+6\left\|D_1^{-1}\right\|_F^2\|S\|_F^2\psi\\
&\quad+9\beta_2\phi_3^2(z_d,\dot z_d)\left\|D_1^{-1}\right\|_F^4\|S\|_F^4.
\end{aligned}$$

Substituting (26), (28) and (29) back into (25) results in

$$\begin{aligned}
\mathcal{L}U_2&\leqslant-\frac{k_1}{2}\left(e_1^{\mathrm{T}}e_1\right)^2+e_2^{\mathrm{T}}e_2e_2^{\mathrm{T}}\Bigg(D_1^{-1}S^{\mathrm{T}}H\tau-D_1^{-1}\omega-\ddot z_d-k_1^2e_1\\
&\quad+\left(\frac{27}{4k_1^3}+\frac{\beta_1}{8}+k_1\right)e_2+(\Phi_1+\Phi_2)\hat\theta e_2\Bigg)+\frac{1}{2\beta_1}+\frac{1}{4\beta_2}\\
&\quad+\tilde\theta\left(\lambda\dot{\hat\theta}-(\Phi_1+\Phi_2)\left(e_2^{\mathrm{T}}e_2\right)^2\right).
\end{aligned}\tag{30}$$

By choosing the control law and adaptive law as

$$\tau = \left(D_1^{-1}S^{\mathrm{T}}H\right)^{\#}\Biggl(D_1^{-1}\omega + \ddot{z}_d + k_1^2 e_1 - \left(\frac{k_2}{2} + \frac{27}{4k_1^3} + \frac{\beta_1}{8} + k_1\right)e_2 - (\varPhi_1 + \varPhi_2)\hat{\theta}e_2\Biggr), \tag{31}$$

$$\dot{\hat{\theta}} = \frac{1}{\lambda}(\varPhi_1 + \varPhi_2)\left(e_2^{\mathrm{T}}e_2\right)^2 - \varepsilon\hat{\theta}, \tag{32}$$

where # is any right inverse and k_2, ε are positive design parameters. From (30)–(32),

$$\mathcal{L}U_2 \leqslant -\frac{k_1}{2}\left(e_1^{\mathrm{T}}e_1\right)^2 - \frac{k_2}{2}\left(e_2^{\mathrm{T}}e_2\right)^2 - \varepsilon\lambda\tilde{\theta}\hat{\theta} + \frac{1}{2\beta_1} + \frac{1}{4\beta_2}$$

follows readily. By completing the squares, it is not hard to show that

$$\mathcal{L}U_2 \leqslant -cU_2 + d_c, \tag{33}$$

where $c = \min\{2k_1, 2k_2, \varepsilon\}$ and $d_c = 1/(2\beta_1) + 1/(4\beta_2) + \varepsilon\lambda\theta^2/2$.

Thus, from the above analysis, the resulting closed-loop error system are obtained as

$$\begin{aligned} \mathrm{d}e_1 &= (e_2 - k_1 e_1)\mathrm{d}t, \\ \mathrm{d}e_2 &= \Biggl(-D_1^{-1}(q)\varpi(q,\dot{q}) + \varOmega(q,\dot{q}) - \left(\frac{k_2}{2} + \frac{27}{4k_1} + \frac{\beta_1}{8}\right)e_2 \\ &\quad - (\varPhi_1 + \varPhi_2)\hat{\theta}e_2\Biggr)\mathrm{d}t + D_1^{-1}(q)S^{\mathrm{T}}(q)\varDelta(q,\dot{q})\varSigma\,\mathrm{d}W, \\ \dot{\hat{\theta}} &= \frac{1}{\lambda}(\varPhi_1 + \varPhi_2)\left(e_2^{\mathrm{T}}e_2\right)^2 - \varepsilon\hat{\theta}. \end{aligned} \tag{34}$$

Based on closed-loop system (34), stability analysis will be given in the following subsection.

Remark 5. From (22) and Assumption 2, it can be easily seen that the uncertainties are caused by unknown parameters ϑ_i ($1 \leqslant i \leqslant 4$) and $\varSigma$. Adaptive law (32) is given to estimate θ but not unknown parameters ϑ_i and $\varSigma$. Namely, the adaptive law (32) is designed "directly" instead of "indirectly".

4.2 Stability analysis

We are now in a position to present the main result of this paper.

Theorem 1. *Under Assumptions* 1–3, *for any initial values* $e_1(t_0)$, $e_2(t_0) \in \mathbb{R}^{n-g}$, $\tilde{\theta}(t_0) = \hat{\theta}(t_0) - \theta$, *the tracking error closed-loop system* (34) *possesses a unique solution on* $[t_0, +\infty)$ *and is exponentially practically stable in mean square. Moreover, the mean* *square of tracking errors $e_1(t)$ and $\dot e_1(t)$ meet*

$$\lim_{t\to+\infty}\mathbf{E}\big|e_1(t)\big|^2 \leqslant \left(\frac{4d_c}{c}\right)^{1/2},$$
$$\lim_{t\to+\infty}\mathbf{E}\big|\dot e_1(t)\big|^2 \leqslant 4\big(1+k_1^2\big)\left(\frac{4d_c}{c}\right)^2, \tag{35}$$

where the right-hand sides can be made small enough by choosing design parameters appropriately.

Proof. For the simplicity of presentation, denote $\Pi = (e_1^{\mathrm T}, e_2^{\mathrm T}, \sqrt{2\lambda}\tilde\theta)^{\mathrm T}$. Since $e_1, e_2 \in \mathbb{R}^l$, $l = n - g$, let $e_i = (e_{i1}, e_{i2}, \dots, e_{il})^{\mathrm T}$, $i = 1, 2$. By considering the definition of U_2 and using Lemma 4 in [15], it can be proved that

$$\frac14\left(\sum_{j=1}^{l}\big(e_{1j}^4 + e_{2j}^4\big) + (\sqrt{2\lambda}\tilde\theta)^2\right)$$
$$\leqslant U_2(\Pi) \leqslant \frac14\left(l\sum_{j=1}^{l}\big(e_{1j}^4 + e_{2j}^4\big) + (\sqrt{2\lambda}\tilde\theta)^2\right). \tag{36}$$

Since the functions of the error system (34) satisfy the local Lipschitz condition, in view of (33), (36) and Lemma 1, there exists a unique strong solution to the closed-loop system (34) on $[t_0, +\infty)$ for initial values $e_1(t_0)$, $e_2(t_0)$, $\tilde\theta(t_0) = \hat\theta(t_0) - \theta$, and error system (34) is exponentially practically stable in mean square. Furthermore, applying the same arguments as made in the proof of Lemma 1 in [5] to $U_2(\Pi) = (1/4)(|e_1|^4 + |e_2|^4 + |\sqrt{2\lambda}\tilde\theta|^2)$, one can obtain

$$\mathbf{E}\big|e_1(t)\big|^2 \leqslant \mathrm{e}^{-c(t-t_0)/2}\big(\big|e_1(t_0)\big|^4 + \big|e_2(t_0)\big|^4 + 2\lambda\tilde\theta^2(t_0)\big)^{1/2} + \left(\frac{4d_c}{c}\right)^{1/2},$$
$$\mathbf{E}\big|e_2(t)\big|^2 \leqslant \mathrm{e}^{-c(t-t_0)/2}\big(\big|e_1(t_0)\big|^4 + \big|e_2(t_0)\big|^4 + 2\lambda\tilde\theta^2(t_0)\big)^{1/2} + \left(\frac{4d_c}{c}\right)^{1/2}. \tag{37}$$

Noting $|\dot e_1|^2 \leqslant (|e_2| + k_1|e_1|)^2 \leqslant 2(1 + k_1^2)(|e_2|^2 + |e_1|^2)$, (35) follows from (37). Considering the definitions of c and d_c specified by (33), it is easy to see that the right-hand sides of (35) can be made small enough by picking β_1, β_2 large enough and γ small enough because they are independent of the parameters k_1, k_2, and ε. The proof of Theorem 1 is completed. □

5 Application to mechanical systems

In the preceding parts, we have given the rigorous proof of the dynamic model (16) and the adaptive control algorithm for (31)–(32). In this section, the above methods are to be used to model and tracking control a vertical wheel in random vibration environment.

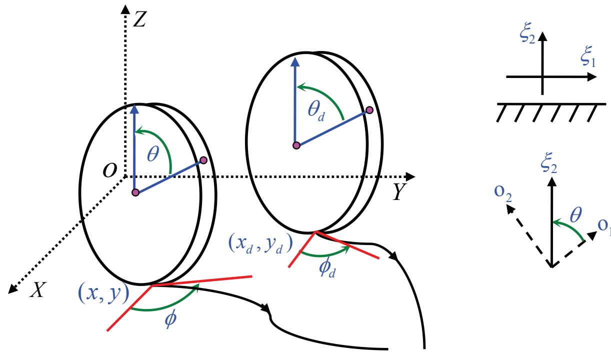

Figure 1. A vertical wheel rolling on a random vibrating smooth plane.

Table 1. Parameters in vertical mobile wheel.

Parameters	Description	Unit
m	mass of wheel	kg
r	radius of wheel	m
I_θ	moment of inertia around vertical direction	$\mathrm{kg}\cdot\mathrm{m}^2$
I_ϕ	moment of inertia around x-axis	$\mathrm{kg}\cdot\mathrm{m}^2$

Consider the control system of a vertical wheel rolling on a smooth plane surface (see Fig. 1) with constant parameters in Table. 1. As in [23] and [5], we assume that there is no air resistance and the random vibration in question is described by the random accelerations of the point (x, y). Let ξ_1, ξ_2 denote the random accelerations of the point (x, y) in x-axis and vertical directions. Let x and y denote the coordinates of the point of contact of the vertical wheel on the plane. Let ϕ be the heading angle of the vertical wheel (measured from the x-axis), and θ stands for the rotational angle of the wheel due to rolling (measured from the vertical direction). Choose the generalized configuration coordinate $q = (x, y, \theta, \phi)^{\mathrm{T}}$. Suppose the configuration q and the velocity $\dot{q}$ can be measured by some measuring equipments.

Step 1. Modeling for the mobile wheel.

Since the Lagrangian of the system is $L = (1/2)(m(\dot{x}^2 + \dot{y}^2) + I_\theta\dot{\theta}^2 + I_\phi\dot{\phi}^2)$ and the (k, j)th element of matrix $C(q, \dot{q})$ is defined as $c_{kj} = \sum_{i=1}^{4} \tilde{c}_{ijk}(q)\dot{q}_i$ with the Christoffel symbols $\tilde{c}_{ijk} = (1/2)(\partial d_{kj}/\partial q_i + \partial d_{ki}/\partial q_j) - \partial d_{ij}/\partial q_k$, then

$$D(q) = \mathrm{diag}(m, m, I_\theta, I_\phi), \qquad C(q, \dot{q}) = 0, \qquad G(q) = 0,$$

where 0 denotes zero matrix with appropriate dimension.

Based on the equivalence principle of mechanics, from equilibration of velocities in the wheel sides, the following kinematic equations are obtained:

$$\dot{x} = r\dot{\theta}\cos\phi, \qquad \dot{y} = r\dot{\theta}\sin\phi.$$

Therefore, the nonholonomic constraints of the system is given as

$$J(q)\dot{q} = 0$$

with

$$g = 2 \quad \text{and} \quad J(q) = \begin{bmatrix} 1 & 0 & -r\cos\phi & 0 \\ 0 & 1 & -r\sin\phi & 0 \end{bmatrix}.$$

Since the contact surface is smooth, ξ_1 does nothing for the wheel. Decomposing ξ_2 along the direction of $\mathbf{o}_2$ (see Fig. 1), the stochastic force F in the direction of θ is introduced as $F = I_\theta \sin\theta \xi_2$. Let τ_1 denote the control torque about the rolling axis of the wheel and τ_2 be the control torque about the vertical axis through the point of the contact. Thus, the matrices $E(q)$ and $\Delta(q)$ are defined respectively by

$$H(q) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}, \qquad \Delta(q) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & I_\theta \sin\theta & 0 \end{bmatrix}^{\mathrm{T}}.$$

Up to now, the dynamic model of the mechanical systems is

$$D(q)\ddot{q} = H(q)\tau + J^{\mathrm{T}}(q)\lambda + \Delta(q)\xi$$

with $\lambda = (\lambda_1, \lambda_2)^{\mathrm{T}}$ and $\xi = (\xi_1, \xi_2)^{\mathrm{T}}$.

Step 2. Control design and simulation.

By defining the so-called internal state variable vector $z(q) = (\theta, \phi)^{\mathrm{T}}$, we obtain

$$\dot{q} = S(q)\dot{z}, \quad \text{where } S(q) = \begin{bmatrix} r\cos\phi & r\sin\phi & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}.$$

After lengthy but simple calculations, the corresponding dynamic model (20) can be expressed concretely as

$$\begin{bmatrix} mr^2 + I_\theta & 0 \\ 0 & I_\phi \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} + \begin{bmatrix} 0 & I_\theta \sin\theta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}.$$

Here, the control objective is specified as: for the given desired reference signals $\theta_d(t) = \sin t$, $\phi_d(t) = \cos t$, design a state feedback adaptive control law such that the tracking error $e_1 = (\theta - \theta_d, \phi - \phi_d)^{\mathrm{T}}$ and its time derivative $\dot{e}_1(t)$ converge to zero as much as possible. Since Δ only depends q, the Wong–Zakai correction term equals zero, i.e., $\Omega(q, \dot{q}) = 0$. It is easy to check that Assumptions 1–3 hold with $\vartheta_1 = \vartheta_2 = 0$, $\vartheta_3 = \vartheta_4 = I_\theta^2$, $\phi_1 = \phi_2 = 0$, $\phi_3(z, \dot{z}) = \sin^2\theta$ and $\psi(z, \dot{z}, z_d, \dot{z}_d) = (\sin\theta - \sin\theta_d)^2$. According to Section 4, the following adaptive tracking algorithm is obtain:

$$\tau = \left(D_1^{-1} S^{\mathrm{T}} H\right)^{-1} \left(\ddot{z}_d + k_1^2 e_1 - \left(\frac{k_2}{2} + \frac{27}{4k_1^3} + k_1 \right) e_2 - \Phi_2 \hat{\theta} e_2 \right),$$

$$\dot{\hat{\theta}} = \frac{1}{\lambda} \Phi_2 \left(e_2^{\mathrm{T}} e_2\right)^2 - \varepsilon\hat{\theta},$$

where

$$\left(D_1^{-1} S^{\mathrm{T}} H\right)^{-1} = \begin{bmatrix} (mr^2 + I_\theta)/r & 0 \\ 0 & I_\phi \end{bmatrix}$$

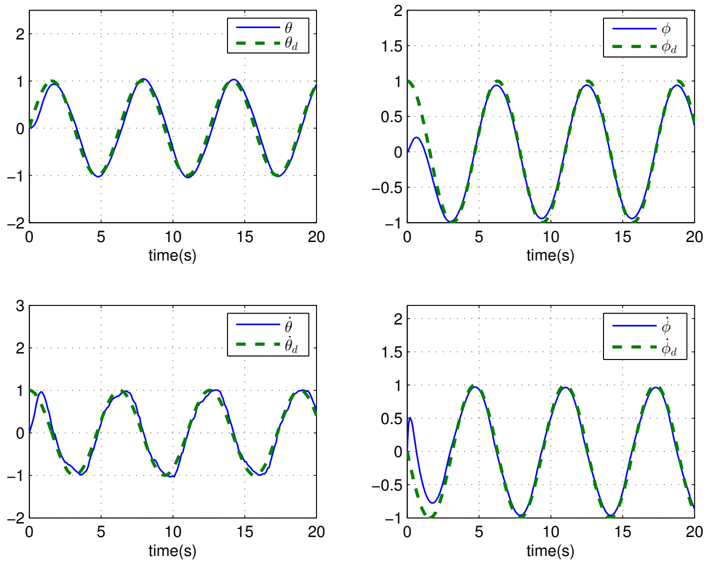

Figure 2. Time evolutions of system states and desired trajectories.

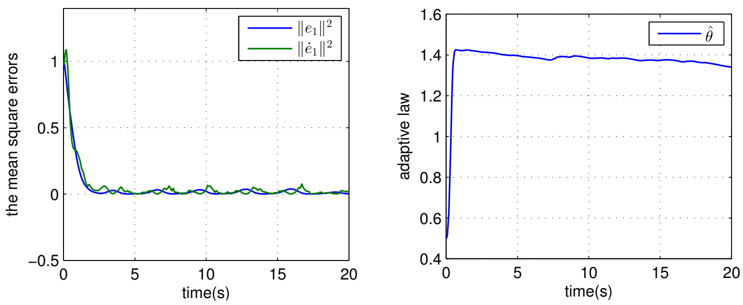

Figure 3. Mean square of tracking errors and estimation of parameter.

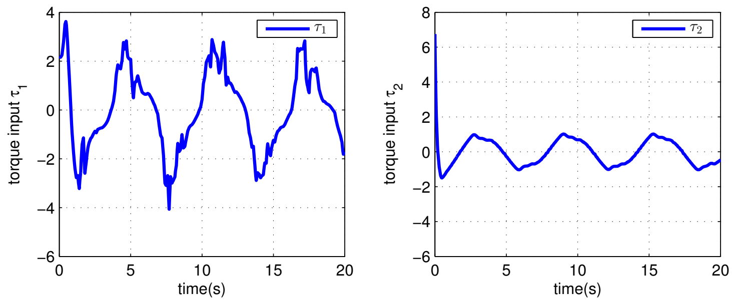

Figure 4. Control torques of vertical mobile wheel.

and

$$\Phi_2 = (r^2+2)^2\left(\frac{1}{(mr^2+I_\theta)^2}+\frac{1}{I_\theta^2}\right)^2\left(\frac{9}{k_1}(2k_1^2+1)^2(\sin\theta-\sin\theta_d)^4+9\beta_2\sin^4\theta_d\right)$$
$$+6(r^2+2)\left(\frac{1}{(mr^2+I_\theta)^2}+\frac{1}{I_\theta^2}\right)(\sin\theta-\sin\theta_d)^2.$$

For the convenience of simulation, the system parameters are taken as $m = 4$ kg, $I_\theta = 1$ kg$\cdot$m^2, $I_\phi = 1$ kg$\cdot$m^2, $r = 0.5$ m, the power spectral density of white noise ξ is chosen as $\Sigma = \left[\begin{smallmatrix}0.5 & 0\\ 0 & 0.5\end{smallmatrix}\right]$, the initial conditions are taken as $\theta(0) = \phi(0) = \dot\theta(0) = \dot\phi(0) = 0$, and the design parameters are picked as $k_1 = 1$, $k_2 = 2$, $\beta_1 = 8$, $\beta_2 = 10$, $\varepsilon = 0.01$, $\lambda = 1$. Simulation results are provided in Figs. 2–4. From the plots in Figs. 2–4, it is shown that the control performance is satisfactory.

6 Conclusions

In this paper, the issue of modeling and adaptive tracking control has been addressed for a class of nonholonomic mechanical systems under stochastic disturbances. The stochastic nonholonomic dynamic model has been formulated. Based on a reduced dynamic model, an adaptive tracking control strategy is constructed, which drives the mean square of tracking error converges to an arbitrarily small neighborhood of zero. A vertical mobile wheel under stochastic disturbances is provided to test the efficiency of the controller.

Compared with traditional nonholonomic systems, the difficulties to handle stochastic nonholonomic dynamics basically focus on three aspects:

1) Since the effect of random disturbances is the major challenge in control design for nonholonomic systems, it is necessary to construct a reasonable stochastic model.
2) For stochastic nonholonomic dynamic systems, prior to tracking controller development, assumptions needed for controller design should be made milder and reasonable.
3) Since the considered model is stochastic nonlinear systems, conventional quadratic Lyapunov function is not suitable for control design, hence, how to choose an appropriate 4th Lyapunov function is another difficulty.

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