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Fuzzy fixed points of generalized F_2-Geraghty type fuzzy mappings and complementary results

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Abstract. The aim of this paper is to introduce generalized F_2-Geraghty type fuzzy mappings on a metric space for establishing the existence of fuzzy fixed points of such mappings. As an application of our result, we obtain the existence of common fuzzy fixed point for a generalized F_2-Geraghty type fuzzy hybrid pair. These results unify, generalize and complement various known comparable results in the literature. An example and an application to theoretical computer science are presented to support the theory proved herein. Also, to suggest further research on fuzzy mappings, a Feng–Liu type theorem is proved.

Keywords: fuzzy fixed point, fuzzy mapping, sorting algorithm.

1 Introduction and preliminaries

Banach contraction principle [4] is constructive in nature and is one of the most useful tools in the study of nonlinear equations. Because of its simplicity and usefulness, many authors were motivated to extend and generalize this principle. One of the most interesting generalizations of Banach contraction principle was given by Geraghty [8]. Extensions of Geraghty result for multivalued mappings have also been obtained in different directions [9, 10, 19], which in turn generalize a well-known Nadler's fixed point theorem [15], a multivalued version of Banach contraction principle. Recently, Wardowski [21] introduced the concept of F-contraction and obtained a fixed point result as a generalization of Banach contraction principle; for more details in this direction, we refer to [21, 22].

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On the other hand, mathematical models have been used extensively in real world problems related to engineering, computer sciences, economics, social, natural and medical sciences. Because of various uncertainties arising in real world situations, some times, methods of classical mathematics may not be successfully applied to solve them. In fact, fuzzy set theory has been evolved in mathematics as an important tool (initiated by Zadeh [23]) to solve the issues of uncertainty and ambiguity. Heilpern [12] initiated the concept of fuzzy mappings on a metric space and proved a fixed point theorem as a generalization of Nadler's theorem [15]. Abu-Donia [1] studied the Hausdorff metric between fuzzy subsets via its correspondence between classical sets and obtained common fixed point theorems for fuzzy mappings, see also [13]. For instance, the concept of fixed point of a fuzzy mapping has a great deal in the *Theory of noncooperative N-persons fuzzy games*, see [5]; then the reader interested in fixed point results of fuzzy mappings is referred to [1, 3, 6, 18, 20].

In this paper, we introduce a new class of generalized F_2-Geraghty type fuzzy mappings by combining the concepts of F-contraction and Geraghty type contraction. We establish the existence of fuzzy fixed point of such mappings, employing the concept of Pompeiu–Hausdorff distance between α-level sets of fuzzy mappings, useful in constructing Hausdorff dimensions for fuzzy spaces. As an application, coincidence fuzzy point and common fuzzy fixed point of hybrid pair of a single valued self-mapping and a fuzzy mapping are obtained. These results extend and strengthen various known results in [6, 8, 12, 15, 21, 22]. We also provide an example to illustrate our results and an application to theoretical computer science. Moreover, to suggest further research on fuzzy mappings, a Feng–Liu type theorem is given; in fact, without using the concept of the Pompeiu–Hausdorff distance, Feng and Liu [7] proved an interesting generalization of Nadler's theorem [15].

In the sequel, the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of all real numbers, the set of all nonnegative real numbers and the set of all positive integer numbers, respectively.

The following definitions and results will be considered in establishing our results.

Let X be a space of points with generic element of X denoted by x and $I = [0, 1]$. A fuzzy set A in X is characterized by a membership function $A : X \to I$ such that each element $x \in X$ is associated with a real number $A(x) \in [0, 1]$. Let I^X be a collection of all fuzzy subsets of X.

Let (X, d) be a metric space and A a fuzzy set in X. If $\alpha \in (0, 1]$, then the α-level set A_α of A is defined as

$$A_\alpha = \{x\colon\ A(x) \geqslant \alpha\}.$$

For $\alpha = 0$, we have $A_0 = \overline{\{x \in X\colon\ A(x) > 0\}}$, where $\overline{B}$ denotes the closure of a set B in (X, d). A fuzzy set A is said to be more accurate than fuzzy set B, denoted by $A \subset B$ if and only if $A(x) \leqslant B(x)$ for each x in X. It is obvious that if $0 < \alpha \leqslant \beta \leqslant 1$, then $A_\beta \subseteq A_\alpha$. Corresponding to each $\alpha \in [0, 1]$ and $x \in X$, the fuzzy point x_α of X is a fuzzy set $x_\alpha : X \to [0, 1]$ given by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

For $\alpha = 1$, we have the following indicator function of $\{x\}$:

$$x_1(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$W_\alpha(X) = \{A \in I^X\colon\ A_\alpha \text{ is nonempty and compact}\}.$$

For $A, B \in W_\alpha(X)$ and $\alpha \in [0,1]$, let

$$\begin{aligned} p_\alpha(A,B) &= \inf\{d(x,y),\ x \in A_\alpha,\ y \in B_\alpha\}, \\ D_\alpha(A,B) &= \max\Big\{\sup_{x \in A_\alpha} d(x,B_\alpha),\ \sup_{y \in B_\alpha} d(y,A_\alpha)\Big\}, \\ D(A,B) &= \sup_\alpha D_\alpha(A,B). \end{aligned}$$

Note that p_α is a nondecreasing mapping of α and D_α a metric on $W_\alpha(X)$. Let Y be an arbitrary subset in (X,d). A mapping $R : Y \to W_\alpha(X)$ is called a fuzzy mapping over the set Y, that is, a mapping which associates with each y in Y the fuzzy set $R_y \in W_\alpha(X)$. As a fuzzy set, R_y in X is characterized by a membership function $R_y : X \to [0,1]$, so $R_y(x)$ is a membership of x in R_y. Thus, a fuzzy mapping R over Y is a fuzzy subset of $Y \times X$ having membership function $R_y(x) = R(y,x)$.

In a more general sense than that given in [12], a mapping $R : X \to I^X$ is a fuzzy mapping over X (see [18]). Notice that the α-level set of a fuzzy mapping R over X is given by

$$(R_x)_\alpha = \{y \in X\colon\ R_x(y) \geqslant \alpha\}.$$

Set $K(X) = \{\mu \in I^X\colon \hat{\mu} \in CB(X)\}$, where $CB(X)$ is the set of all closed and bounded subsets of X, $\hat{\mu} = \{x \in X\colon\ \mu(x) = \max_{y \in X} \mu(y)\}$ and $\Lambda : K(X) \to CB(X)$, where $\Lambda(\mu) = \hat{\mu}$. Abu-Donia [1] considers the fuzzy mapping $R : X \to K(X)$ instead of $R : X \to I^X$. Denote the composition $\Lambda \circ R$ by $\hat{R}$. Thus,

$$\hat{R}(x) = \Big\{y \in X\colon\ R_x(y) = \max_{z \in X} R_x(z)\Big\}.$$

For $\alpha \in (0,1]$, $\alpha \leqslant \max_{z \in X} R_x(z)$ implies that $\hat{R}(x) \subseteq (R_x)_\alpha$. If $\alpha > \max_{z \in X} R_x(z) = \omega$ (say), then $(R_x)_\alpha \subseteq \hat{R}(x) = (R_x)_\omega = \{y \in X\colon\ R_x(y) \geqslant \omega\}$. Hence, the approximation $\hat{R}(x)$ of the fuzzy set R_x in the sense of Abu-Donia [1] corresponds to some α-level set.

Definition 1. (See [6].) A fuzzy point x_α in X is called a fuzzy fixed point of fuzzy mapping R if $x_\alpha \subset R_x$, that is, $R_x(x) \geqslant \alpha$ or $x \in (R_x)_\alpha$. Hence, the fixed degree of x in R_x is at least α. If $\{x\} \subset R_x$, then x is a fixed point of the fuzzy mapping R.

Ali and Abbas [2] gave the following definitions.

Definition 2. (See [2].) Let $R : X \to W_\alpha(X)$ be a fuzzy mapping and $g : X \to X$ a self-mapping. A fuzzy point x_α in X is called:

(i) coincidence fuzzy point of hybrid pair (g, R) if $(gx)_\alpha \subset R_x$, that is, $R_x(gx) \geqslant \alpha$ or $gx \in (R_x)_\alpha$ (the fixed degree of gx in R_x is at least α);

(ii) common fuzzy fixed point of the hybrid pair (g, R) if $x_\alpha = (gx)_\alpha \subset R_x$, that is, $x = gx \in (R_x)_\alpha$ (the fixed degree of x and gx in R_x is the same and is at least α).

The sets of all fuzzy fixed points, coincidence fuzzy points and common fuzzy fixed points of the hybrid pair (g, R) are denoted by $F_\alpha^{ix}(R)$, $C_\alpha(R, g)$ and $F_\alpha^{ix}(R, g)$, respectively.

Definition 3. (See [2].) Let $R : X \to W_\alpha(X)$ be a fuzzy mapping and $g : X \to X$ a self-mapping. Then:

(i) the hybrid pair (g, R) is called w-fuzzy compatible if $g(R_x)_\alpha \subseteq (R_{gx})_\alpha$, whenever $x \in C_\alpha(R, g)$;

(ii) a mapping g is called R-fuzzy weakly commuting at some point $x \in X$ if $g^2x \in (R_{gx})_\alpha$.

Lemma 1. (*See* [11].) *Let X be a nonempty set and $g : X \to X$. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one to one.*

Lemma 2. (*See* [12].) *Let (X, d) be a metric space, $x, y \in X$ and $A, B \in W_\alpha(X)$. The following hold*:

1. *If $p_\alpha(x, A) = 0$, then $x_\alpha \subset A$*;
2. $p_\alpha(x, A) \leqslant d(x, y) + p_\alpha(y, A)$;
3. *If $x_\alpha \subset A$, then $p_\alpha(x, B) \leqslant D_\alpha(A, B)$.*

Theorem 1. (*See* [6].) *Let (X, d) be a complete metric space and R a fuzzy mapping from X to $W_\alpha(X)$, where $\alpha \in (0, 1)$. If $D_\alpha(R_x, R_y) \leqslant qd(x, y)$ for each $x, y \in X$, where $q \in (0, 1)$; then there exists $x \in X$ such that x_α is a fuzzy fixed point.*

Lemma 3. (*See* [14].) *Let (X, d) be a complete metric space and R a fuzzy mapping from X to $W_\alpha(X)$ and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset R_{x_0}$.*

Set $S = \{\psi : \mathbb{R}^+ \to [0, 1)\colon\ t_n \to 0 \text{ whenever } \psi(t_n) \to 1\}$. Consider

$$\psi_1(t) = \begin{cases} \mathrm{e}^{-3t} & \text{if } t > 0, \\ 0 & \text{if } t = 0 \end{cases} \quad \text{and} \quad \psi_2(t) = \begin{cases} 1/(t+1) & \text{if } t > 0, \\ 0 & \text{if } t = 0, \end{cases}$$

then $\psi_1, \psi_2 \in S$, and hence, $S \neq \emptyset$.

Geraghty [8] proved the following result.

Theorem 2. *Let (X, d) be a complete metric space and $T : X \to X$. If there exists $\psi \in S$ such that $d(Tx, Ty) \leqslant \psi(d(x, y))d(x, y)$ holds for all $x, y \in X$; then T has a unique fixed point $z \in X$ and for each $x \in X$, the sequence $\{T^n x\}$ converges to z.*

Consider the following conditions for a mapping $F:(0,+\infty)\to\mathbb{R}$:

(C1) F is strictly increasing, that is, for all $\alpha,\beta\in(0,+\infty)$, $\alpha<\beta$ implies $F(\alpha)<F(\beta)$;
(C2) There exists $k\in(0,1)$ such that $\lim_{\alpha\to 0^+}\alpha^k F(\alpha)=0$;
(C3) For every sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to+\infty}\alpha_n=0$ if and only if $\lim_{n\to+\infty}F(\alpha_n)=-\infty$.

Let

$$\digamma_2=\{F_2\colon\ F_2 \text{ satisfies conditions (C1) and (C2)}\},$$
$$\digamma_3=\{F_3\colon\ F_3 \text{ satisfies conditions (C1), (C2) and (C3)}\}.$$

Wardowski [21] introduced the concept of F-contraction as follows.

Definition 4. (See [21].) Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be an F-contraction on X if there exists $\tau>0$ such that

$$d(Tx,Ty)>0 \quad\Longrightarrow\quad \tau+F\bigl(d(Tx,Ty)\bigr)\leqslant F\bigl(d(x,y)\bigr)$$

for all $x,y\in X$, where $F\in\digamma_3$.

From now onwards, we shall call it F_3-contraction instead of F-contraction. Note that F_3-contraction is continuous, see [21].

Wardowski [21] proved the following result as a generalization of Banach contraction principle.

Theorem 3. (*See* [21].) *Let (X,d) be a complete metric space and $T:X\to X$ an F_3-contraction. Then T has a unique fixed point $x^*\in X$ and, for every $x_0\in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.*

Now, we introduce generalized Geraghty type fuzzy mapping over a metric space (X,d).

Definition 5. Let (X,d) be a metric space. A fuzzy mapping $R:X\to W_\alpha(X)$ is said to be generalized F_i-Geraghty type fuzzy mapping if there exist $\tau>0$ and $\psi\in S$ such that

$$\begin{aligned}&D_\alpha(R_x,R_y)>0\\ &\quad\Longrightarrow\quad \tau+F_i\bigl(D_\alpha(R_x,R_y)\bigr)\leqslant F_i\bigl(\psi\bigl(M_\alpha^R(x,y)\bigr)N_\alpha^R(x,y)\bigr)\end{aligned}\qquad(1)$$

holds for all $x,y\in X$, where $F_i\in\digamma_i$ for $i\in\{2,3\}$ and

$$M_\alpha^R(x,y)=\max\biggl\{d(x,y),p_\alpha(x,R_x),p_\alpha(y,R_y),\frac{p_\alpha(x,R_y)+p_\alpha(y,R_x)}{2}\biggr\},$$
$$N_\alpha^R(x,y)=\max\bigl\{d(x,y),p_\alpha(x,R_x),p_\alpha(y,R_y)\bigr\}.$$

Definition 6. Let (X,d) be a metric space, $R : X \to W_\alpha(X)$ a fuzzy mapping and $g : X \to X$. A pair (g,R) is called generalized F_i-Geraghty type fuzzy hybrid pair if there exist $\tau > 0$ and $\psi \in S$ such that

$$D_\alpha(R_x, R_y) > 0 \implies \tau + F_i\bigl(D_\alpha(R_x, R_y)\bigr) \leqslant F_i\bigl(\psi\bigl(M_\alpha^{g,R}(x,y)\bigr)N_\alpha^{g,R}(x,y)\bigr) \qquad (2)$$

holds for all $x, y \in X$, where $F_i \in \mathcal{F}_i$ for $i \in \{2,3\}$ and

$$M_\alpha^{g,R}(x,y) = \max\biggl\{d(gx,gy), p_\alpha(gx,R_x), p_\alpha(gy,R_y), \frac{p_\alpha(gx,R_y)+p_\alpha(gy,R_x)}{2}\biggr\},$$
$$N_\alpha^{g,R}(x,y) = \max\bigl\{d(gx,gy), p_\alpha(gx,R_x), p_\alpha(gy,R_y)\bigr\}.$$

2 Fuzzy fixed points of F_2-Geraghty type fuzzy mappings

First, we prove a fuzzy fixed point result for generalized F_2-Geraghty type fuzzy mappings on a complete metric space.

Theorem 4. *Let (X,d) be a complete metric space and $R : X \to W_\alpha(X)$ a generalized F_2-Geraghty type fuzzy mapping. Then there exists a point $x \in X$ such that $x_\alpha \subset R_x$, that is, $F_\alpha^{ix}(R)$ is nonempty.*

Proof. Let u_0 be a given point in X. As $R_{u_0} \in W_\alpha(X)$, we can choose $u_1 \in (R_{u_0})_\alpha$ such that $d(u_0,u_1) = p_\alpha(u_0, R_{u_0})$. If $u_0 = u_1$, then $u_0 = u_1 \in (R_{u_0})_\alpha$ and the result follows trivially. Suppose that $u_0 \neq u_1$; since $R_{u_1} \in W_\alpha(X)$, there exists $u_2 \in (R_{u_1})_\alpha$ such that

$$d(u_1,u_2) = p_\alpha(u_1, R_{u_1}) \leqslant D_\alpha(R_{u_0}, R_{u_1}).$$

If $u_1 = u_2$, then $u_1 = u_2 \in (R_{u_1})_\alpha$ and the proof is finished. Suppose $u_1 \neq u_2$, then $D_\alpha(R_{u_0}, R_{u_1}) > 0$. This further implies

$$\begin{aligned}
&F_2\bigl(d(u_1,u_2)\bigr)\\
&\quad\leqslant F_2\bigl(D_\alpha(R_{u_0}, R_{u_1})\bigr) \leqslant F_2\bigl(\psi\bigl(M_\alpha^R(u_0,u_1)\bigr)N_\alpha^R(u_0,u_1)\bigr) - \tau\\
&\quad\leqslant F_2\biggl(\psi\biggl(\max\biggl\{d(u_0,u_1), p_\alpha(u_0,R_{u_0}), p_\alpha(u_1,R_{u_1}), \frac{p_\alpha(u_0,R_{u_1})+p_\alpha(u_1,R_{u_0})}{2}\biggr\}\biggr)\\
&\qquad\times \max\bigl\{d(u_0,u_1), p_\alpha(u_0,R_{u_0}), p_\alpha(u_1,R_{u_1})\bigr\}\biggr) - \tau\\
&\quad\leqslant F_2\biggl(\psi\biggl(\max\biggl\{d(u_0,u_1), d(u_0,u_1), d(u_1,u_2), \frac{d(u_0,u_2)+d(u_1,u_1)}{2}\biggr\}\biggr)\\
&\qquad\times \max\bigl\{d(u_0,u_1), d(u_0,u_1), d(u_1,u_2)\bigr\}\biggr) - \tau\\
&\quad\leqslant F_2\bigl(\psi\bigl(\max\{d(u_0,u_1), d(u_1,u_2)\}\bigr)\max\bigl\{d(u_0,u_1), d(u_1,u_2)\bigr\}\bigr) - \tau.
\end{aligned}$$

Suppose that $d(u_1, u_2) \nless d(u_0, u_1)$. Then we obtain

$$F_2\bigl(d(u_1, u_2)\bigr) \leqslant F_2\bigl(\psi\bigl(d(u_1, u_2)\bigr)d(u_1, u_2)\bigr) - \tau.$$

Thus, since $\psi \in S$, we have

$$F_2\bigl(d(u_1, u_2)\bigr) \leqslant F_2\bigl(d(u_1, u_2)\bigr) - \tau,$$

which implies $\tau \leqslant 0$, a contradiction. Hence, $d(u_1, u_2) < d(u_0, u_1)$ and so

$$F_2\bigl(d(u_1, u_2)\bigr) \leqslant F_2\bigl(\psi\bigl(d(u_0, u_1)\bigr)d(u_0, u_1)\bigr) - \tau.$$

Continuing this way, we can obtain a sequence $\{u_n\}$ in X such that $u_n \in (R_{u_{n-1}})_\alpha$, $u_{n+1} \in (R_{u_n})_\alpha$ and also

$$d(u_n, u_{n+1}) = p_\alpha(u_n, R_{u_n}) \leqslant D_\alpha(R_{u_{n-1}}, R_{u_n}).$$

Now $u_n = u_{n+1}$ gives that $u_n = u_{n+1} \in (R_{u_n})_\alpha$; hence, the result follows. Suppose that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$, then $D_\alpha(R_{u_{n-1}}, R_{u_n}) > 0$. Thus, we have

$$\begin{aligned}
&F_2\bigl(d(u_n, u_{n+1})\bigr) \\
&\quad\leqslant F_2\bigl(D_\alpha(R_{u_{n-1}}, R_{u_n})\bigr) \leqslant F_2\bigl(\psi\bigl(M_\alpha^R(u_{n-1}, u_n)\bigr)N_\alpha^R(u_{n-1}, u_n)\bigr) - \tau \\
&\quad\leqslant F_2\Biggl(\psi\Biggl(\max\Biggl\{d(u_{n-1}, u_n), p_\alpha(u_{n-1}, R_{u_{n-1}}), p_\alpha(u_n, R_{u_n}), \\
&\qquad \frac{p_\alpha(u_{n-1}, R_{u_n}) + p_\alpha(u_n, R_{u_{n-1}})}{2}\Biggr\}\Biggr) \\
&\qquad \times \max\bigl\{d(u_{n-1}, u_n), p_\alpha(u_{n-1}, R_{u_{n-1}}), p_\alpha(u_n, R_{u_n})\bigr\}\Biggr) - \tau \\
&\quad\leqslant F_2\Biggl(\psi\Biggl(\max\Biggl\{d(u_{n-1}, u_n), d(u_{n-1}, u_n), d(u_n, u_{n+1}), \\
&\qquad \frac{d(u_{n-1}, u_{n+1}) + d(u_n, u_n)}{2}\Biggr\}\Biggr) \\
&\qquad \times \max\bigl\{d(u_{n-1}, u_n), d(u_{n-1}, u_n), d(u_n, u_{n+1})\bigr\}\Biggr) - \tau \\
&\quad\leqslant F_2\bigl(\psi\bigl(\max\bigl\{d(u_{n-1}, u_n), d(u_n, u_{n+1})\bigr\}\bigr) \max\bigl\{d(u_{n-1}, u_n), d(u_n, u_{n+1})\bigr\}\bigr) - \tau.
\end{aligned}$$

We claim that

$$d(u_n, u_{n+1}) < d(u_{n-1}, u_n) \tag{3}$$

holds for each $n \in \mathbb{N}$. If not, then from (3) we get

$$\begin{aligned}
F_2\bigl(d(u_{n_0}, u_{n_0+1})\bigr) &\leqslant F_2\bigl(\psi\bigl(d(u_{n_0}, u_{n_0+1})\bigr)d(u_{n_0}, u_{n_0+1})\bigr) - \tau \\
&\leqslant F_2\bigl(d(u_{n_0}, u_{n_0+1})\bigr) - \tau
\end{aligned}$$

for some $n_0 \in \mathbb{N}$, which leads to the contradiction $\tau \leqslant 0$. Therefore, we write

$$F_2(\mu_{n+1}) \leqslant F_2\big(\psi(\mu_n)\mu_n\big) - \tau$$

where $\mu_n = d(u_n, u_{n+1})$ and, since $\tau > 0$, then we have $F_2(\mu_{n+1}) < F_2(\psi(\mu_n)\mu_n)$. Also, since F_2 is strictly increasing and $\psi \in S$, it follows that

$$\mu_{n+1} < \psi(\mu_n)\mu_n \leqslant \mu_n, \tag{4}$$

that is, $\{\mu_{n+1}\}$ is a decreasing sequence of nonnegative real numbers, which is bounded below by 0; hence, $\lim_{n\to+\infty} \mu_{n+1} = \lambda \geqslant 0$ for some $\lambda \in \mathbb{R}^+$. Suppose that $\lambda > 0$. On taking limit as n tends to $+\infty$ on both sides of (4), we have

$$\lambda \leqslant \lim_{n\to+\infty} \psi(\mu_n)\lambda \leqslant \lambda,$$

which implies that

$$1 \leqslant \lim_{n\to+\infty} \psi(\mu_n) \leqslant 1,$$

that is, $\lim_{n\to+\infty}\psi(\mu_n)=1$. Then by definition of $\psi\in S$, it follows that $\lim_{n\to+\infty}\mu_n=0$, a contradiction, and hence,

$$\lim_{n\to+\infty} \mu_n = 0. \tag{5}$$

From (C2) there exists $k \in (0,1)$ such that

$$\lim_{n\to+\infty} \mu_n^k F_2(\mu_n) = 0. \tag{6}$$

Now, we have

$$\begin{aligned} F_2(\mu_n) &\leqslant F_2\big(\psi(\mu_{n-1})\mu_{n-1}\big) - \tau \leqslant F_2(\mu_{n-1}) - \tau \leqslant \cdots \\ &\leqslant F_2\big(\psi(\mu_0)\mu_0\big) - n\tau. \end{aligned}$$

It follows that

$$\begin{aligned} &\mu_n^k F_2(\mu_n) - \mu_n^k F_2\big(\psi(\mu_0)\mu_0\big) \\ &\quad \leqslant \mu_n^k\big(F_2\big(\psi(\mu_0)\mu_0\big) - n\tau\big) - \mu_n^k F_2\big(\psi(\mu_0)\mu_0\big) \leqslant -n\tau\mu_n^k \leqslant 0. \end{aligned} \tag{7}$$

On taking limit as n tends to $+\infty$ on both sides of (7) and using (5) and (6), we have $\lim_{n\to+\infty} n\mu_n^k = 0$. Consequently, there exists $n_1 \in \mathbb{N}$ such that $n\mu_n^k < 1$ for all $n \geqslant n_1$, and hence, we have

$$\mu_n < \frac{1}{n^{1/k}} \quad \text{for all } n \geqslant n_1.$$

For $m \in \mathbb{N}$ with $m > n$, we write

$$\begin{aligned} d(u_n, u_m) &\leqslant d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + \cdots + d(u_{m-2}, u_{m-1}) + d(u_{m-1}, u_m) \\ &\leqslant \frac{1}{n^{1/k}} + \frac{1}{(n+1)^{1/k}} + \cdots + \frac{1}{(m-2)^{1/k}} + \frac{1}{(m-1)^{1/k}} \\ &\leqslant \sum_{j=n}^{m-1} \frac{1}{j^{1/k}} \leqslant \sum_{j=n}^{+\infty} \frac{1}{j^{1/k}}. \end{aligned}$$

Using the convergence of the series $\sum_{j=n}^{+\infty} j^{-1/k}$, we get that $\{u_n\}$ is a Cauchy sequence in X. Next, since (X,d) is complete, we have $\lim_{n\to+\infty} d(u_n, z) = 0$ for some $z \in X$. Now, we show that $z_\alpha \subset R_z$; note that

$$\lim_{n\to+\infty} p_\alpha(u_n, R_z) = \lim_{n\to+\infty} d\big(u_n, (R_z)_\alpha\big) = p_\alpha(z, R_z).$$

Clearly, if $p_\alpha(z, R_z) = 0$, then by Lemma 2 we have $z_\alpha \subset R_z$, that is, $z \in (Rz)_\alpha$. On the contrary if $p_\alpha(z, R_z) \neq 0$, then there exist $\varepsilon_0 > 0$ and $N \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ with $n \geqslant N$, one has $p_\alpha(u_n, R_z) > \varepsilon_0 > 0$. Thus,

$$\begin{aligned}
&F_2\big(p_\alpha(u_{n+1}, R_z)\big) \\
&\quad\leqslant F_2\big(D_\alpha(R_{u_n}, R_z)\big) \leqslant F_2\big(\psi\big(M_\alpha^R(u_n, z)\big)N_\alpha^R(u_n, z)\big) - \tau \\
&\quad\leqslant F_2\bigg(\psi\bigg(\max\bigg\{d(u_n, z), d(u_n, u_{n+1}), p_\alpha(z, R_z), \frac{p_\alpha(u_n, R_z) + p_\alpha(z, R_{u_n})}{2}\bigg\}\bigg) \\
&\qquad\times \max\big\{d(u_n, z), d(u_n, u_{n+1}), p_\alpha(z, R_z)\big\}\bigg) - \tau.
\end{aligned}$$

Then, by definition of F_2, it follows that

$$\begin{aligned}
&p_\alpha(u_{n+1}, R_z) \\
&\quad\leqslant \psi\bigg(\max\bigg\{d(u_n, z), d(u_n, u_{n+1}), p_\alpha(z, R_z), \frac{p_\alpha(u_n, R_z) + p_\alpha(z, R_{u_n})}{2}\bigg\}\bigg) \\
&\qquad\times \max\big\{d(u_n, z), d(u_n, u_{n+1}), p_\alpha(z, R_z)\big\}.
\end{aligned}$$

On taking limit as n tends to $+\infty$ in the previous inequality, we have

$$p_\alpha(z, R_z) \leqslant \lim_{n\to+\infty} \psi\big(M_\alpha^R(u_n, R_z)\big)p_\alpha(z, R_z),$$

which implies that

$$\lim_{n\to+\infty} M_\alpha^R(u_n, R_z) = p_\alpha(z, R_z) = 0$$

and by Lemma 2, we have $z_\alpha \subset R_z$. □

Next, we give a corollary.

Corollary 1. *Let (X,d) be a complete metric space and $R : X \to W_\alpha(X)$ a fuzzy mapping. Suppose that there exist $\tau > 0$ and $\psi \in S$ such that*

$$\tau + F_2\big(D_\alpha(R_x, R_y)\big) \leqslant F_2\big(\psi\big(M_\alpha^R(x, y)\big)d(x, y)\big)$$

for all $x, y \in X$, where $F_2 \in \digamma_2$. Then $F_\alpha^{ix}(R)$ is nonempty.

Using the same techniques as in the proof of Theorem 4, one can prove the following results.

Theorem 5. *Let (X,d) be a complete metric space and $R : X \to W_\alpha(X)$ a fuzzy mapping. Suppose that there exist $\tau > 0$ and $\psi \in S$ such that*

$$\tau + F_2\big(D_\alpha(R_x,R_y)\big) \leqslant F_2\big(\psi\big(N_\alpha^R(x,y)\big)N_\alpha^R(x,y)\big)$$

for all $x,y \in X$, where $F_2 \in \mathcal{F}_2$. Then $F_\alpha^{ix}(R)$ is nonempty.

Theorem 6. *Let (X,d) be a complete metric space and $R : X \to W_\alpha(X)$ a fuzzy mapping. Suppose that there exist $\tau > 0$ and $\psi \in S$ such that*

$$\tau + F_2\big(D_\alpha(R_x,R_y)\big) \leqslant F_2\big(\psi\big(d(x,y)\big)d(x,y)\big) \tag{8}$$

for all $x,y \in X$, where $F_2 \in \mathcal{F}_2$. Then $F_\alpha^{ix}(R)$ is nonempty.

Remark 1. Let (X,d) be a complete metric space and $R : X \to W_\alpha(X)$ a fuzzy mapping. Let $\psi \in S$ and $F_2(\alpha) = \ln(\alpha)$; clearly $F_2 \in \mathcal{F}_2$. Then (8) becomes

$$\tau + \ln\big(D_\alpha(R_x,R_y)\big) \leqslant \ln\big(\psi\big(d(x,y)\big)d(x,y)\big),$$

which further implies that

$$D_\alpha(R_x,R_y) \leqslant \mathrm{e}^{-\tau}\psi\big(d(x,y)\big)d(x,y)$$

for all $x,y \in X$. Hence, Theorem 6 generalizes Theorem 2.

Now, we give an illustrative example.

Example 1. Let $X = \{0,1,2\}$ and $d : X \times X \to \mathbb{R}^+$ be the metric defined by

$$d(0,0) = d(1,1) = d(2,2) = 0, \qquad d(0,2) = d(2,0) = 6,$$
$$d(0,1) = d(0,1) = 10, \qquad d(1,2) = d(2,1) = 16.$$

Let $\alpha \in (0,1/3)$, $F_2(\alpha) = \ln(\alpha)$, $\psi(t) = \mathrm{e}^{-t/50}$ if $t > 0$ and $\psi(0) = 0$. Clearly, $\psi \in S$ and $F_2 \in \mathcal{F}_2$. Define a fuzzy mapping $R : X \to W_\alpha(X)$ by

$$(R_0)(x) = \begin{cases} 2\alpha & \text{if } x = 0, \\ 0 & \text{if } x = 1, \\ \alpha/4 & \text{if } x = 2, \end{cases} \qquad (R_1)(x) = \begin{cases} 3\alpha & \text{if } x = 0, \\ \alpha/5 & \text{if } x = 1, \\ 0 & \text{if } x = 2, \end{cases}$$

$$(R_2)(x) = \begin{cases} \alpha/3 & \text{if } x = 0, \\ \alpha & \text{if } x = 1, \\ \alpha/4 & \text{if } x = 2. \end{cases}$$

Then $(R_0)_\alpha = (R_1)_\alpha = \{0\}$, $(R_2)_\alpha = \{1\}$. Note that, for $x,y \in \{0,1\}$, we have $D_\alpha(R_x,R_y) = 0$. On the other hand, we get

$$D_\alpha(R_1,R_2) = D_\alpha(R_0,R_2) = d(0,1) = 10.$$

For $x = 0$ and $y = 2$, we write

$$\begin{aligned} M_\alpha^R(0,2) &= \max\left\{d(0,2), p_\alpha(0,R_0), p_\alpha(2,R_2), \frac{p_\alpha(0,R_2)+p_\alpha(2,R_0)}{2}\right\} \\ &= \max\left\{d(0,2), d(0,0), d(2,1), \frac{d(0,1)+d(2,0)}{2}\right\} \\ &= \max\{6,0,16,8\} = 16, \\ N_\alpha^R(0,2) &= \max\{6,0,16\} = 16. \end{aligned}$$

Also, for $x = 1$ and $y = 2$, we get

$$\begin{aligned} M_\alpha^R(1,2) &= \max\left\{d(1,2), p_\alpha(1,R_1), p_\alpha(2,R_2), \frac{p_\alpha(1,R_2)+p_\alpha(2,R_1)}{2}\right\} \\ &= \max\left\{d(1,2), d(1,0), d(2,1), \frac{d(1,1)+d(2,0)}{2}\right\} \\ &= \max\{16,10,16,3\} = 16, \\ N_\alpha^R(1,2) &= \max\{16,10,16\} = 16. \end{aligned}$$

Finally, for $x \in \{0,1\}$, we write

$$\begin{aligned} &\ln\bigl(\psi\bigl(M_\alpha^R(x,2)\bigr)N_\alpha^R(x,2)\bigr) - \ln\bigl(D_\alpha(R_x,R_2)\bigr) \\ &\quad = \ln\bigl(16\mathrm{e}^{-16/50}\bigr) - \ln(10) \approx 0.150004. \end{aligned}$$

Hence, there exists $\tau = 0.15$ such that the condition

$$\tau + \ln\bigl(D_\alpha(R_x,R_y)\bigr) \leqslant \ln\bigl(\psi\bigl(M_\alpha^R(x,y)\bigr)N_\alpha^R(x,y)\bigr)$$

holds true for all $x, y \in X$ with $D_\alpha(R_x,R_y) > 0$. Thus, all the conditions of Theorem 4 are satisfied. Moreover, $x = 0$ is the fuzzy fixed point of the fuzzy mapping R. Indeed, for $x = 0$, we have $x_\alpha \subset R_x$ as $(R_0)(0) \geqslant \alpha$, that is, $0 \in (R_0)_\alpha$.

Remark 2. Note that in Example 1 above, we get

$$D_\alpha(R_0,R_2) = d(0,1) = 10 \quad \text{and} \quad d(0,2) = 6.$$

Consequently, for any choice of $q \in (0,1)$ and $\psi \in S$, we have

$$D_\alpha(R_0,R_2) \nleqslant q d(0,2)$$

and

$$D_\alpha(R_0,R_2) \nleqslant \psi(d(0,2))d(0,2).$$

Hence, Theorems 1 and 2 do not hold in this case. Thus, Theorem 4 is a proper generalization of the results in [6, 8, 12, 15].

Also we note that Theorem 4 generalizes the main results in [18, 20, 21, 22].

Remark 3. Let $T : X \to CC(X)$ (set of all compact subsets of X) be a multivalued mapping and, for all $z \in X$, define the fuzzy mapping $R : X \to W_\alpha(X)$ by

$$R_x(z) = \begin{cases} \alpha & \text{if } z \in Tx, \\ 0 & \text{otherwise} \end{cases}$$

for each $x \in X$. Note that

$$(R_x)_\alpha = \{z\colon R_x(z) \geqslant \alpha\} = Tx.$$

In view of Remark 3, by using the usual Pompeiu–Hausdorff distance of sets, say H, one can easily prove the following result.

Theorem 7. *Let (X, d) be a complete metric space and $T : X \to CC(X)$ a multivalued mapping. Assume that there exist $\tau > 0$ and $\psi \in S$ such that*

$$H(Tx, Ty) > 0 \quad \Longrightarrow \quad \tau + F_2\big(H(Tx, Ty)\big) \leqslant F_2\big(\psi\big(M(x, y)\big)N(x, y)\big),$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\},$$
$$N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$$

and $F_2 \in \digamma_2$. Then T has a fixed point; that is, there exists $z \in X$ such that $z \in Tz$.

3 Complementary results and application

3.1 Coincidence and common fixed points

As an immediate consequence of Theorem 4, we obtain the following common fuzzy fixed point result for F_2-Geraghty type fuzzy hybrid pair (g, R). Denote $(R(X))_\alpha := \bigcup_{x \in X}(R_x)_\alpha$.

Theorem 8. *Let (X, d) be a complete metric space and (g, R) a generalized F_2-Geraghty type fuzzy hybrid pair. Then $C_\alpha(R, g) \neq \emptyset$, provided that $(R(X))_\alpha \subseteq g(X)$ for each α. Moreover, R and g have a common fuzzy fixed point if any of the following conditions holds*:

(i) *R and g are w-fuzzy compatible,* $\lim_{n\to+\infty} g^n x = u$ *for some $x \in C_\alpha(R, g)$ and $u \in X$, and g is continuous at u*;

(ii) *g is R-fuzzy weakly commuting for some $x \in C_\alpha(R, g)$, and $g^2x = gx$*;

(iii) *g is continuous at x for some $x \in C_\alpha(R, g)$, and, for some $u \in X$, we have* $\lim_{n\to+\infty} g^n u = x$.

Proof. By Lemma 1, there exists $E \subseteq X$ such that $g : E \to X$ is one to one and $g(E) = g(X)$. Define a mapping $\mathcal{A} : g(E) \to W_\alpha(X)$ by

$$\mathcal{A}gx = R_x \quad \text{for all } gx \in g(E).$$

As g is one to one on E, so $\mathcal{A}$ is well defined. Therefore, (2) becomes

$$\begin{aligned}
&\tau + F_2\big(D_\alpha(\mathcal{A}gx, \mathcal{A}gy)\big) \\
&\quad = F_2\big(D_\alpha(R_x, R_y)\big) \leqslant F_2\big(\psi\big(M_\alpha^{g,R}(x,y) N_\alpha^{g,R}(x,y)\big)\big) \\
&\quad = F_2\bigg(\psi\bigg(\max\bigg\{d(gx, gy), p_\alpha(gx, R_x), p_\alpha(gy, R_y), \\
&\qquad \frac{p_\alpha(gx, R_y) + p_\alpha(gy, R_x)}{2}\bigg\}\bigg) \\
&\qquad \times \max\big\{d(gx, gy), p_\alpha(gx, R_x), p_\alpha(gy, R_y)\big\}\bigg) \\
&\quad = F_2\bigg(\psi\bigg(\max\bigg\{d(gx, gy), p_\alpha(gx, \mathcal{A}gx), p_\alpha(gy, \mathcal{A}gy), \\
&\qquad \frac{p_\alpha(gx, \mathcal{A}gy) + p_\alpha(gy, \mathcal{A}gx)}{2}\bigg\}\bigg) \\
&\qquad \times \max\big\{d(gx, gy), p_\alpha(gx, \mathcal{A}gx), p_\alpha(gy, \mathcal{A}gy)\big\}\bigg) \\
&\quad = F_2\big(\psi\big(M_\alpha^{\mathcal{A}}(x,y) N_\alpha^{\mathcal{A}}(x,y)\big)\big)
\end{aligned}$$

for all $gx, gy \in g(E)$ such that $D_\alpha(\mathcal{A}gx, \mathcal{A}gy) > 0$. Thus, $\mathcal{A}$ satisfies (1) and all the other conditions of Theorem 4. By an application of Theorem 4 to mapping $\mathcal{A}$, it follows that $\mathcal{A}$ has a fuzzy fixed point $u \in g(E)$.

Now we show that R and g have a coincidence fuzzy point. Since $\mathcal{A}$ has a fuzzy fixed point $u_\alpha \subset \mathcal{A}u$, then we have $u \in (\mathcal{A}u)_\alpha$. Since $(R(X))_\alpha \subseteq g(X)$, there exists $u_1 \in X$ such that $gu_1 = u$, and hence,

$$gu_1 \in (\mathcal{A}gu_1)_\alpha = (R_{u_1})_\alpha,$$

that is, $u_1 \in X$ is a coincidence fuzzy point of R and g. Now, we distinguish the following three cases:

If (i) holds; then for some $x \in C_\alpha(R, g)$, we have $\lim_{n\to+\infty} g^n x = u$, where $u \in X$. Since g is continuous at u, so u is a fixed point of g, that is $u = gu$. As R and g are w-fuzzy compatible and $x \in C_\alpha(R, g)$, then $g(gx) \in (R_{gx})_\alpha$, that is, $gx \in C_\alpha(R, g)$. By iterating this process, we deduce that $g^n x \in C_\alpha(R, g)$ for all $n \geqslant 1$, and hence,

$$g^n x \in R\big(g^{n-1}x\big)_\alpha$$

for all $n \geqslant 1$. We show that $gu \in (R_u)_\alpha$. Note that $p_\alpha(g^n x, R_u) = d(g^n x, (R_u)_\alpha)$, then the continuity of d ensures that

$$\lim_{n\to+\infty} p_\alpha\big(g^n x, R_u\big) = p_\alpha(u, R_u).$$

If $p_\alpha(u, R_u) = 0$, then by Lemma 2 we have $u = gu \in (R_u)_\alpha$ and so u is a fuzzy fixed point. If $p_\alpha(u, R_u) > 0$, then there exist $\varepsilon_0 > 0$ and $N \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$

with $n \geqslant N$, we have $p_\alpha(g^n x, R_u) > \varepsilon_0 > 0$. This implies that

$$\begin{aligned}
&F_2\bigl(p_\alpha\bigl(g^n x, R_u\bigr)\bigr)\\
&\quad\leqslant F_2\bigl(D_\alpha\bigl(R\bigl(g^{n-1}x\bigr), R_u\bigr)\bigr) - \tau\\
&\quad\leqslant F_2\biggl(\psi\biggl(\max\biggl\{d\bigl(g^n x, gu\bigr), p_\alpha\bigl(g^n x, Rg^{n-1}x\bigr), p_\alpha(gu, R_u),\\
&\qquad\frac{p_\alpha(gu, Rg^{n-1}x) + p_\alpha(g^n x, R_u)}{2}\biggr\}\biggr)\\
&\qquad\times \max\bigl\{d\bigl(g^n x, gu\bigr), p_\alpha\bigl(g^n x, Rg^{n-1}x\bigr), p_\alpha(gu, R_u)\bigr\}\biggr) - \tau\\
&\quad\leqslant F_2\biggl(\psi\biggl(\max\biggl\{d\bigl(g^n x, gu\bigr), d\bigl(g^n x, g^n x\bigr), d(gu, R_u),\\
&\qquad\frac{d(gu, g^n x) + p_\alpha(g^n x, R_u)}{2}\biggr\}\biggr)\\
&\qquad\times \max\bigl\{d\bigl(g^n x, gu\bigr), d\bigl(g^n x, g^n x\bigr), p_\alpha(gu, R_u)\bigr\}\biggr) - \tau
\end{aligned}$$

for all $n \geqslant N$. Since F_2 is strictly increasing, then we get

$$\begin{aligned}
p_\alpha\bigl(g^n x, R_u\bigr) \leqslant{}& \psi\biggl(\max\biggl\{d\bigl(g^n x, gu\bigr), 0, d(gu, R_u), \frac{d(gu, g^n x) + p_\alpha(g^n x, R_u)}{2}\biggr\}\biggr)\\
&\times \max\bigl\{d\bigl(g^n x, gu\bigr), 0, p_\alpha(gu, R_u)\bigr\}.
\end{aligned}$$

On taking limit as n tends to $+\infty$, we have

$$p_\alpha(gu, R_u) \leqslant \lim_{n\to+\infty} \psi\bigl(M_\alpha^{g,R}\bigl(g^{n-1}x, u\bigr)\bigr) p_\alpha(gu, R_u),$$

which implies

$$1 \leqslant \lim_{n\to+\infty} \psi\bigl(M_\alpha^{g,R}\bigl(g^{n-1}x, u\bigr)\bigr) \leqslant 1.$$

Since $\psi \in S$, then

$$\lim_{n\to+\infty} M_\alpha^{g,R}\bigl(g^{n-1}x, u\bigr) = p_\alpha(gu, R_u) = 0,$$

which is a contradiction. Thus, $u = gu \in (R_u)_\alpha$, and hence, u_α is a common fuzzy fixed point of R and g.

Suppose that (ii) holds; that is, for some $x \in C_\alpha(R, g)$, g is R-fuzzy weakly commuting and $g^2 x = gx$, then

$$gx = g^2 x \in (R_{gx})_\alpha.$$

Hence, $(gx)_\alpha$ is a common fuzzy fixed point of R and g.

Finally, suppose that (iii) holds; that is, for some $x \in C_\alpha(R, g)$ and for some $u \in X$, $\lim_{n\to+\infty} g^n u = x$. By the continuity of g at x, we get $x = gx \in (R_x)_\alpha$, then the result. □

3.2 Feng–Liu type fixed point theorem

We give a theorem inspired by [7], to suggest a direction for further research. Let (X,d) be a complete metric space and $R: X \to W_\alpha(X)$ be a fuzzy mapping; for a positive constant $r \in (0,1]$ and each $x \in X$, define the set

$$J_r^x := \{y \in (R_x)_\alpha\colon\ r\, d(x,y) \leqslant p_\alpha(x,R_x)\}.$$

We recall that a function $f: X \to \mathbb{R}$ is lower semicontinuous, if for each sequence $\{x_n\} \subset X$ and $x \in X$, we have

$$\lim_{n\to+\infty} x_n = x \quad\Longrightarrow\quad fx \leqslant \liminf_{n\to+\infty} fx_n.$$

Theorem 9. *Let (X,d) be a complete metric space and $R: X \to W_\alpha(X)$ be a fuzzy mapping. Suppose that there exists $q \in (0,r)$, with $r \in (0,1]$, such that, for any $x \in X$, there is $y \in J_r^x$ satisfying the condition $p_\alpha(y,R_y) \leqslant q\, d(x,y)$. Then $F_\alpha^{ix}(R)$ is nonempty, provided that the function $p_\alpha(y,R_y)$ is lower semicontinuous.*

Proof. Since $(R_x)_\alpha$ is a nonempty compact set for any $x \in X$, then J_r^x is a nonempty set for any $r \in (0,1]$. Now, for a fixed point $u_0 \in X$, there exists $u_1 \in J_r^{u_0}$ such that

$$p_\alpha(u_1,R_{u_1}) \leqslant qd(u_0,u_1).$$

If u_1 is not a fuzzy fixed point of R, we choose $u_2 \in J_r^{u_1}$ such that

$$p_\alpha(u_2,R_{u_2}) \leqslant qd(u_1,u_2).$$

Again, if u_2 is not a fuzzy fixed point of R (and so on), by iterating this procedure, we can get an iterative sequence $\{u_n\}$, where $u_{n+1} \in J_r^{u_n}$ and

$$p_\alpha(u_{n+1},R_{u_{n+1}}) \leqslant qd(u_n,u_{n+1}) \quad \text{for all } n \in \mathbb{N}\cup\{0\}. \tag{9}$$

On the other hand, $u_{n+1} \in J_r^{u_n}$ implies

$$rd(u_n,u_{n+1}) \leqslant p_\alpha(u_n,R_{u_n}) \quad \text{for all } n \in \mathbb{N}\cup\{0\}. \tag{10}$$

The next step of the proof is to show that the sequence $\{u_n\}$ is a Cauchy sequence. Using (9) and (10), we get

$$d(u_{n+1},u_{n+2}) \leqslant \frac{q}{r} d(u_n,u_{n+1}) \quad \text{for all } n \in \mathbb{N}\cup\{0\}.$$

This implies

$$d(u_n,u_{n+1}) \leqslant k^n d(u_0,u_1) \quad \text{for all } n \in \mathbb{N},$$

where $k = q/r < 1$.

For $m \in \mathbb{N}$ with $m > n$, we write

$$\begin{aligned}
d(u_n, u_m) &\leqslant d(u_n, u_{n+1}) + d(u_{n+1}, u_m) \\
&\leqslant d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + d(u_{n+2}, u_m) \\
&\leqslant d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + \cdots + d(u_{m-2}, u_{m-1}) + d(u_{m-1}, u_m) \\
&\leqslant k^n d(u_0, u_1) + k^{n+1} d(u_0, u_1) + \cdots + k^{m-2} d(u_0, u_1) + k^{m-1} d(u_0, u_1) \\
&= k^n d(u_0, u_1)\bigl[1 + k + k^2 + \cdots + k^{m-n-1}\bigr] \\
&\leqslant \frac{k^n}{1-k}\, d(u_0, u_1).
\end{aligned}$$

Consequently, since

$$\lim_{n\to+\infty} \frac{k^n}{1-k}\, d(u_0, u_1) = 0,$$

we deduce that $\{x_n\}$ is a Cauchy sequence and so, by completeness of the space (X, d), $\lim_{n\to+\infty} d(u_n, z) = 0$ for some $z \in X$. Now we claim that z is a fuzzy fixed point of R. Again, by (9) and (10), we write

$$p_\alpha(u_{n+1}, R_{u_{n+1}}) \leqslant k p_\alpha(u_n, R_{u_n}) \quad \text{for all } n \in \mathbb{N} \cup \{0\},$$

which implies

$$p_\alpha(u_n, R_{u_n}) \leqslant k^n p_\alpha(u_0, R_{u_0}) \quad \text{for all } n \in \mathbb{N},$$

where $k = q/r < 1$. Consequently, by the semicontinuity of function $p_\alpha(y, R_y)$, we get

$$0 \leqslant p_\alpha(z, R_z) \leqslant \lim_{n\to+\infty} p_\alpha(u_n, R_{u_n}) = 0$$

and so

$$p_\alpha(z, R_z) = 0.$$

By Lemma 2, we get that $z_\alpha \subset R_z$. □

3.3 Application to the domain of words

By adapting some ideas in the recent literature [16, 17], we apply Theorem 4 to solve a typical problem in theoretical computer science. Precisely, denote by Σ a nonempty alphabet and by Σ^∞ the set of all finite and infinite sequences over Σ. Also, we denote the empty sequence by $\emptyset$ and assume that $\emptyset \in \Sigma^\infty$. Moreover, on Σ^∞, we consider the prefix order $\sqsubseteq$ given by:

$$x \sqsubseteq y \quad \text{if and only if } x \text{ is a prefix of } y.$$

Now, for any sequence $x \neq \emptyset$ in Σ^∞, let $l(x) \in [1, +\infty]$ be the length of x and assume that $l(\emptyset) = 0$. Then, if $x \in \Sigma^\infty$ has length $n < +\infty$, we have $x := x_1 x_2 \cdots x_n$, otherwise $x := x_1 x_2 \cdots$ in the case of infinite sequence.

Next, if $x, y \in \Sigma^\infty$, then $x \sqcap y$ is the common prefix of x and y. Clearly, $x = y$ if and only if $x \sqsubseteq y$ and $y \sqsubseteq x$ and $l(x) = l(y)$.

Consider the Baire metric $d_\sqsubseteq : \Sigma^\infty \times \Sigma^\infty \to [0, +\infty)$ given by

$$d_\sqsubseteq(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-l(x \sqcap y)} & \text{otherwise.} \end{cases}$$

so that the metric space $(\Sigma^\infty, d_\sqsubseteq)$ is complete. Finally, we refer to the average case time complexity analysis of the Quicksort divide-and-conquer sorting algorithm in [17]. Precisely, we consider the following recurrence relation:

$$\begin{aligned} &T(1) = 0, \\ &T(n) = \frac{2(n-1)}{n} + \frac{n+1}{n} T(n-1), \quad n \in \mathbb{N} \setminus \{1\}. \end{aligned} \tag{11}$$

For $\Sigma = \mathbb{R}^+$, which is the set of all nonnegative real numbers, we introduce the functional $\phi : \Sigma^\infty \to \Sigma^\infty$ that associates $\phi(x) := (\phi(x))_1 (\phi(x))_2 \cdots$ to $x := x_1 x_2 \cdots$ and is given by:

$$\begin{aligned} &\big(\phi(x)\big)_1 = 0, \\ &\big(\phi(x)\big)_n = \frac{2(n-1)}{n} + \frac{n+1}{n} x_{n-1}. \end{aligned}$$

It follows that $l((\phi(x))) = l(x) + 1$ for all $x \in \Sigma^\infty$ and $l((\phi(x))) = +\infty$, whenever $l(x) = +\infty$.

We will show that the functional ϕ has a fixed point by an application of Theorem 4. Let $R : \Sigma^\infty \to W_\alpha(\Sigma^\infty)$ be the fuzzy mapping given by

$$R_x = \big(\phi(x)\big)_\alpha \quad \text{for all } x \in \Sigma^\infty$$

and distinguish the following two cases:

Case 1. If $x = y$, then we write

$$D_\sqsubseteq\big(\big(\phi(x)\big)_\alpha, \big(\phi(x)\big)_\alpha\big) = 0 = d_\sqsubseteq(x, x).$$

Case 2. If $x \neq y$, then we write

$$\begin{aligned} D_\sqsubseteq\big(\big(\phi(x)\big)_\alpha, \big(\phi(y)\big)_\alpha\big) &= d_\sqsubseteq\big(\big(\phi(x)\big)_\alpha, \big(\phi(y)\big)_\alpha\big) = 2^{-(l((\phi(x))_\alpha \sqcap_\phi (\phi(y))_\alpha))} \\ &\leqslant 2^{-(l(\phi(x \sqcap y)))} = 2^{-(l(x \sqcap y)+1)} \\ &= \frac{1}{2} 2^{-l(x \sqcap y)} = \frac{1}{2} d_\sqsubseteq(x, y) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} d_\sqsubseteq(x, y). \end{aligned}$$

It is immediate to conclude that all the conditions of Theorem 4 are satisfied with $\psi(t) = 2^{-1/2}$, $F_2(t) = \ln(t)$ and $\mathrm{e}^{-\tau} = 2^{-1/2}$. Consequently, the fuzzy mapping R has a fuzzy fixed point $z = z_1 z_2 \cdots$, that is, $z \in (R_z)_\alpha$. Also, in view of the definition of R, z is a fixed point of ϕ, and hence, z solves the recurrence relation (11); we have

$$
\begin{aligned}
z_1 &= 0, \\
z_n &= \frac{2(n-1)}{n} + \frac{n+1}{n} z_{n-1}, \quad n \in \mathbb{N} \setminus \{1\}.
\end{aligned}
$$

Conclusion

In this paper, we introduced the generalized F_2-Geraghty type fuzzy mappings and extend the Geraghty type fixed point theorems [8] to fuzzy mappings. These results generalize, unify and extend comparable results in [6, 8, 12, 15, 21, 22]. As an application of Theorem 4, the existence of coincidence fuzzy points and common fuzzy fixed points of a hybrid pair of a single-valued self-mapping and a fuzzy mapping is obtained. A Feng–Liu type fixed point theorem and an application to the domain of words conclude the paper.

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