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Optimal control problem for Lengyel–Epstein model with obstacles and state constraints

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Abstract. This paper considers the state constrained optimal control problem for Lengyel–Epstein model with obstacles. We prove existence and regularity results for this model by applying the standard methods. We show the relationship between the control problem and its approximation. Moreover, we derive the necessary conditions for the optimal control of our original problem by using the approximate problem.

Keywords: Lengyel–Epstein model, optimal control, necessary conditions, obstacles.

1 Introduction

This paper is concerned with the state constrained optimal control problem for the Lengyel–Epstein model

$$\min L(u,v,w) = \int_0^T \bigl[g\bigl(t,u(t)\bigr) + h\bigl(w(t)\bigr)\bigr]\,\mathrm{d}t \tag{1}$$

subject to

$$\begin{aligned}
&u_t - \Delta u + cu + \frac{4uv}{1+u^2} + \kappa\partial I_{[\sigma_*,\sigma^*]}(u) \ni a - \phi \quad \text{in } Q := \Omega\times(0,T),\\
&v_t - \delta\Delta v - b\theta u + \frac{\theta uv}{1+u^2} = \theta\phi + Bw \quad \text{in } Q,\\
&u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{in } \Omega,\\
&u(x,t) = 0, \quad v(x,t) = 0 \quad \text{on } \Sigma := \partial\Omega\times(0,T)
\end{aligned} \tag{2}$$

and

$$F(u) \subset S, \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N = 1,2,3$) with a smooth boundary $\partial\Omega$, say of class C^2, u and v are the dimensionless concentration for activator and inhibitor, respectively; a, b, c and θ are dimensionless parameters of the chemical system; $\delta > 0$ is proportional to the ratio of the diffusion coefficients of the main species. The obstacle $\partial I_{[\sigma_*,\sigma^*]}(u)$ is the subdifferential of the indicator function $I_{[\sigma_*,\sigma^*]}(u)$ on the closed interval $[\sigma_*,\sigma^*]$; $\kappa > 0$, $\sigma_*, \sigma^* \in \mathbb{R}$ are the given constants. $u_0(x)$, $h_0(x)$ and $\phi(x,t)$ are given functions and Bw is the control term. Here $F(u) \subset S$ is the state constraint, which can be regarded as the description of the physical background of the Lengyel–Epstein model.

Equation (2) without the control term Bw and $\kappa = 0$ is the classical Lengyel–Epstein model (see [4, 6, 10, 11, 13, 15, 16, 20, 26]). It comes from the reaction between chlorine dioxide, iodine and malonic acid (CDIMA reaction), and is one of the most thoroughly studied oscillatory chemical systems both in experiment and in numeric. In [10], the photosensitive CDIMA reaction was investigated by using the Lengyel–Epstein model modified to include the effect of external illumination. Jensen et al. studied the localized structures and front propagation in the Lengyel–Epstein model [13]. Recently, based on Runge–Kutta method, Bastian, Kartawidjaja [4] solved the parallel performance of the Lengyel–Epstein model. More recently, Váquez et al. [10] studied the chaotic behaviors induced by modulated illumination in the Lengyel–Epstein model under Turing considerations. As we all know, in some physical examples, the range of the activator u would not be the whole real numbers $\mathbb{R}$, but often be a bounded closed interval $[\sigma_*,\sigma^*]$. Here we are going to pay attention to this point and give an adequate mathematical treatment to it. Note that $\partial I_{[\sigma_*,\sigma^*]}(u)$ is a multi-valued and maximal monotone graph in $\mathbb{R}$, which can coincide with the subdifferential of $I_{[\sigma_*,\sigma^*]}(u)$. Namely, $I_{[\sigma_*,\sigma^*]}(u)$ is assumed to be $+\infty$ out of a bounded interval.

Throughout this paper we denote $L^2(\Omega)$ by H with the usual norm denoted by $|\cdot|_2$, and $H_0^1(\Omega)$ by V endowed with norm $\|v\|_V = |\nabla v|_2$, which is denoted by $\|\cdot\|_1$. Set $H^1(0,T;H) = \{y \in L^2(0,T;H);\ y_t \in L^2(0,T;H)\}$, $H^{2,1}(Q) = \{y \in L^2(0,T;H^2(\Omega));\ y_t \in L^2(Q)\}$ and $W^{1,2}(0,T;V^*) = \{y \in L^2(0,T;V^*);\ y_t \in L^2(0,T;V^*)\}$. Then we have $V \subset H \subset V^* = H^{-1}$ and denote $\langle\cdot\rangle$ be the scalar product of H and the pairing between V and V^*.

A pair (u,v) is said to be a weak solution of (2) if and only if

$$(u,v) \in \big(C\big([0,T];H\big) \cap L^2(0,T;V) \cap W^{1,2}(0,T;V^*)\big)^2$$

and (u,v) satisfies

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(u,\tilde u\big) + (\nabla u, \nabla \tilde u) + \left(cu + \frac{4uv}{1+u^2} + \kappa\partial I_{[\sigma_*,\sigma^*]}(u),\ \tilde u\right) \\
&\quad \ni (a-\phi,\tilde u) \quad \forall \tilde u \in V, \\
&\frac{\mathrm{d}}{\mathrm{d}t}\big(v,\tilde v\big) + \delta(\nabla v, \nabla \tilde v) + \left(-b\theta u + \frac{\theta uv}{1+u^2},\ \tilde v\right) = \big(Bw,\tilde v\big) \quad \forall\, \tilde v \in V, \\
&u(0) = u_0, \qquad v(0) = v_0
\end{aligned} \tag{4}$$

in the sense of $D'(0,T)$. Let U be a real Hilbert space and $B : U \to H$ be a linear continuous operator. Assume that Z is a Banach space with the dual Z^*, which is strictly convex, and $S \subset Z$ is a closed convex subset with finite codimensionality.

The following items are the assumptions on data:

(H1) $F : L^2(0,T;H) \to Z$ is in the class of C^1.

(H2) $g : [0,T] \times H \to \mathbb{R}^+$ is measurable in t, $g(t,0) \in L^\infty(0,T)$ and for every $\Lambda > 0$, there exists $L_\Lambda > 0$ independent on t such that for $t \in [0,T]$ and $|y|_2 + |z|_2 \leqslant \Lambda$,

$$\bigl|g(t,y) - g(t,z)\bigr| \leqslant L_\Lambda |y - z|_2.$$

(H3) $U \to \bar{\mathbb{R}}$ is lower semicontinuous and convex with the following growth property: there exist $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$h(u) \geqslant c_1 \|u\|_U^2 + c_2 \quad \forall u \in U.$$

Let

$$A_{\mathrm{ad}} = \bigl\{(u,v,w) \in Y \times Y \times L^2(0,T;U)\colon\ (u,v) \text{ is the solution of (2) corresponding to } w, F(u) \subset S\bigr\},$$

where

$$Y = \biggl\{ y \in L^2\bigl(0,T;H^2(\Omega)\bigr) \cap C\bigl([0,T];V\bigr),\ \frac{\mathrm{d}y}{\mathrm{d}t} \in L^2(0,T;H) \biggr\}$$

and $F(u) \subset S$ is the state constraint. In this paper, we consider the following optimal control problem:

$$\text{Minimize (P):}\ L(u,v,w) \text{ over all } (u,v,w) \in A_{\mathrm{ad}}.$$

It is known that for each $w \in L^2(Q)$, $u_0 \in V$ and $v_0 \in V$, system (2) has a unique solution $u, v \in Y$ (see [9]). The first question regarding problem (P) is if there is an admissible solution, namely, if the set A_{ad} is nonempty. Taking into account the arguments in the proof of the main results in [3], we may assume in the sequel that problem (P) admits at least one admissible solution.

In the past decades, much attention has been paid to the optimal control problems governed by nonlinear parabolic system including semilinear equations, variational inequalities and system with phase transitions [5, 7, 8, 12, 14, 18, 21, 22, 23, 27, 28]. In particular, the optimal control for semilinear parabolic system without state constraint was discussed in [14, 21, 25, 29]. Recently, in [23], based on the energy estimates and the compact methods, Ryu and Yagi considered the optimal control problems of adsorbate–induced phase transition model. More recently, a first order optimality condition for non-homogeneous Cauchy–Neumann boundary optimal control problem of non-linear phase-field system was derived in [5]. In [24], the authors studied Pontryagin's maximum principle for optimal control problems (with a state constraint) governed by the 3-dimensional Navier–Stokes equations. In order to overcome the problem associated with the state constraint, the authors first defined a new penalty functional depending on a small parameter ε with which they approximated the original problem with a family of optimal control problems (P^ε) without state constraints. Pontryagin's maximum principle is derived for the approximate problem (P^ε) and the limit as ε goes to 0 yields an optimality condition for the original control problem with a state constraint. These are the steps followed in this article. The main differences between the present work and works mentioned above are as follows. In this paper, the nonlinearity involved in the Lengyel–Epstein model is stronger than that in the 3-dimensional Navier–Stokes equations, which makes the analysis of the optimal control problems in this article more involved. Moreover, because of the obstacle $\partial I_{[\sigma_*,\sigma^*]}(u)$ in the first equation of system (2), we cannot obtain the optimal conditions of problem (2) directly. In this paper, we derive the necessary conditions for problem (P) by showing the relation between approximation problem (P^ε) (problem (P^ε) contains the approximation of $\partial I_{[\sigma_*,\sigma^*]}(u)$) and problem (P).

In order to give the necessary conditions for problem (P), we specify our notion of a strong solution to problem (2).

Definition 1. A weak solution (u, v) is called a strong solution to problem (2) on the time interval $[0, T)$ if it satisfies

$$(u, v) \in Y \times Y.$$

The main purpose of this paper is to derive the necessary optimal conditions for (P) governed by the Lengyel–Epstein model with state constraints and obstacles, which can be stated as follows.

Theorem 1. *Suppose that* (H1), (H2) *and* (H3) *hold and* (u^*, v^*, w^*) *is optimal for problem* (P). *Then there exists a tetrad* $(\mu_0, p, q, \zeta_0) \in \mathbb{R} \times W^{1,2}(0,T;V^*) \cap L^2(0,T;V) \cap C([0,T];H) \times W^{1,2}(0,T;V^*) \cap L^2(0,T;V) \cap C([0,T];H) \times Z^*$ *and a measure* $\eta \in L^\infty(Q)^*$ *such that*

$$\begin{aligned} &-p_t - \Delta p + cp + \frac{v^*(1-(u^*)^2)}{(1+(u^*)^2)^2}(4p + \theta q) - b\theta q + \eta \\ &\quad \in -\big[F'(u^*)\big]^* \zeta_0 - \mu_0 \partial g(t, u^*), \\ &-q_t - \delta \Delta q + \frac{u^*}{1+(u^*)^2}(4p + \theta q) = 0, \\ &p(T) = 0, \qquad q(T) = 0 \end{aligned} \tag{5}$$

and

$$\begin{aligned} &B^* q(t) \in \mu_0 \partial h\big(w^*(t)\big), \\ &\big\langle \zeta_0, s - F(u^*) \big\rangle_{Z^*,Z} \leqslant 0 \quad \forall\, s \in S \end{aligned} \tag{6}$$

with $(\mu_0, \zeta_0) \neq 0$. *Furthermore, if* $[F'(u^*)]^*$ *is injective, then* $(\mu_0, p, q) \neq 0$.

Remark 1.

(i) For the definition of a set to be finite codimensional in Z and for related results, one can refer to [1, 17].

(ii) Some basic examples of the F, g, h are: $F(u) = u(x, T)$, $g(t, u) = \alpha |u|_2^2$ and $h(w) = |w|_2^2$, where $\alpha > 0$, one can see [18] for more details.
pagebreak

(iii) Let $F \equiv I$, $Z = L^2(0,T;H)$, $S = \{u \in L^2(0,T;H)\ \int_0^T |u|_2^2 \,\mathrm{d}t \leqslant r\}$. In this case, (3) is equivalent to $\int_0^T |u|_2^2 \,\mathrm{d}t \leqslant r$.

(iv) Let $Z = \mathbb{R}^N$ and $h_i \in H$ with $1 \leqslant i \leqslant N$, which are linearly independent in H. We define

$$F(u) = \biggl(\int\limits_Q u(x,t)h_1(x,t)\,\mathrm{d}x\,\mathrm{d}t,\ \int\limits_Q u(x,t)h_2(x,t)\,\mathrm{d}x\,\mathrm{d}t,\ \dots,\ \int\limits_Q u(x,t)h_N(x,t)\,\mathrm{d}x\,\mathrm{d}t \biggr)$$

and

$$S = \bigl([a_1,b_1],[a_2,b_2],\dots,[a_N,b_N]\bigr) \subset \mathbb{R}^N, \quad a_i < b_i,\ i = 1,2,\dots,N,$$

then S is a convex and closed subset with finite codimensionality in $\mathbb{R}^N$. Consider a state constraint of the form

$$a_i \leqslant \int\limits_Q u(x,t)h_i(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant b_i, \quad i = 1,2,\dots,N,$$

we have $[F'(u)]^* y = \sum_{i=1}^N y_i h_i$ with $y := (y_1,y_2,\dots,y_N) \in \mathbb{R}^N$. Due to $h_i \in H$ are linearly independent in H, $[F'(u)]^*$ is injective.

(v) The (u^*,v^*,w^*) is optimal for problem (P) if and only if there exists $(u^*,v^*,w^*) \in A_{\mathrm{ad}}$, which satisfies that $L(u^*,v^*,w^*) = \min L(u,v,w)$.

(vi) The relations (5), $(6)_2$ form the adjoint system, (p^*,q^*) is called the adjoint state and it represents a Lagrange multiplier associated with the state constraint. Equation $(6)_1$ expresses the maximum principle.

The rest of this paper is organized as follows. In Section 2, we consider the approximation problem (P^ε) of problem (P). After showing the solvability of (P^ε), we obtain the relationship between the optimal control problem (P) and its approximation problem (P^ε). In Section 3, we derive a priori estimates for the optimal pair $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ of (P^ε) and then use a passage-to-limit procedure with $\varepsilon \searrow 0$ to get the optimality conditions for (P).

2 The approximation problem

This section is to show the existence of the optimal control of the approximation problem corresponding to Lengyel–Epstein model. Firstly, we show some technical lemmas and the existence of problem (2), which is presented below for the sake of completeness and easy reference. Next, we prove the existence of the control optimal problem (P^ε), which is the approximation of problem (P). In order to approximate the $\partial I_{[\sigma_*,\sigma^*]}(\cdot)$, we define a nondecreasing function β^ε [19] on $\mathbb{R}$ by

$$\beta^\varepsilon(r) = \operatorname{sgn}(r) \int\limits_0^{|r|} \min\biggl\{ \frac{1}{\varepsilon}, \frac{[-s-\sigma_*]^+}{\varepsilon^2}, \frac{[s-\sigma^*]^+}{\varepsilon^2} \biggr\}\,\mathrm{d}s \quad \forall r \in \mathbb{R},$$

where $[\cdot]^+$ denotes the positive part. Then $\beta^\varepsilon \in C^1$, $(\beta^\varepsilon)' \in W^{1,\infty}(\mathbb{R})$ and

$$0 \leqslant \big(\beta^\varepsilon\big)'(r) \leqslant \frac{1}{\varepsilon}, \quad \big|\big(\beta^\varepsilon\big)(r)\big| \geqslant \frac{1}{\varepsilon}\big([r-\sigma^*]^+ + [-\sigma_* - r]^+\big) - \frac{1}{2} \quad \forall\, r \in \mathbb{R}. \tag{7}$$

We fix a primitive $\hat{\beta}^\varepsilon$ of β^ε such that

$$\hat{\beta}^\varepsilon(0) = 0 \quad \text{and} \quad \hat{\beta}^\varepsilon(r) \geqslant 0 \quad \forall\, r \in \mathbb{R}. \tag{8}$$

Now, we consider the following approximating system of (2)

$$\begin{aligned} &u_t - \Delta u + cu + \frac{4uv}{1+u^2} + \kappa\beta^\varepsilon(u) = a - \phi \quad \text{in } Q,\\ &v_t - \delta\Delta v - b\theta u + \frac{\theta uv}{1+u^2} = \theta\phi + Bw \quad \text{in } Q,\\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{in } \Omega,\\ &u(x,t) = 0, \quad v(x,t) = 0 \quad \text{on } \Sigma. \end{aligned} \tag{9}$$

Lemma 1. *Suppose that β^ε satisfies* (7)–(8), $(u_0, v_0) \in V \times V$ *and* $\phi \in L^2(0,T;H)$.

(i) *Let $Bw \in L^2(0,T;H)$. Then problem* (9) *admits a unique strong solution $(u,v) \in Y \times Y$, which satisfies the following estimates*:

$$\begin{aligned} &\big|\hat{\beta}^\varepsilon\big|_{L^\infty(0,T;L^1(\Omega))} + \big|\beta^\varepsilon\big|_{L^2(0,T;H)} + |u|_{L^\infty(0,T;V)}\\ &\quad + |u|_{L^2(0,T;H^2(\Omega))} + |u'|_{L^2(0,T;H)} \leqslant C \end{aligned} \tag{10}$$

and

$$|v|_{L^\infty(0,T;V)} + |v|_{L^2(0,T;H^2(\Omega))} + |v'|_{L^2(0,T;H)} \leqslant C, \tag{11}$$

where $C > 0$ is a constant independent of u, v and ε.

(ii) *Let $w_n \in L^2(0,T;U)$, $w_n \to u$ weakly in $L^2(0,T;U)$ and $(u,v), (u_n,v_n)$ be the solutions of* (9) *corresponding to w and w_n, respectively. Then on some subsequence of (u_n,v_n), still denoted by itself, we have*

$$\big(u_n,v_n\big) \to (u,v) \quad \textit{weakly in } \big(L^2\big(0,T;H^2(\Omega)\big)\big)^2, \tag{12}$$

$$\big(u_n,v_n\big) \to (u,v) \quad \textit{strongly in } \big(C\big([0,T];H\big) \cap L^2(0,T;V)\big)^2 \tag{13}$$

and

$$\big(u_n',v_n'\big) \to (u',v') \quad \textit{weakly in } \big(L^2(0,T;H)\big)^2. \tag{14}$$

Proof. The existence of weak solution is proved by the standard Galerkin method. Indeed, let

$$A = \begin{bmatrix} -\Delta & 0 \\ 0 & -\delta\Delta \end{bmatrix}.$$

Then A is a linear, self-adjoint operator in H with $D(A)$ dense in H. Therefore, we can define the powers A^s of s, $s \in \mathbb{R}$, and V is nothing other than $D(A^{1/2})$. Thus, there exists an orthonormal family ψ_j ($j \in \mathbb{N}$) of H and a sequence η_j ($j \in \mathbb{N}$) such that

$$0 < \eta_1 \leqslant \eta_2 \leqslant \cdots \leqslant \eta_j \to \infty \quad \text{as } j \to \infty,$$
$$A\psi_j = \eta_j \psi_j.$$

For $n \in \mathbb{N}$, we define the discrete ansatz space by $V_n = \operatorname{span}\{\psi_1, \psi_2, \dots \psi_n\} \subset V$. Set $u_n(t) = u_n(x,t) = \sum_{i=1}^n u_i^n(t)\psi_i(x)$ and require that $u_{n,0}(x) \to u_0$ in H. Performing the Galerkin procedure for system (9),

$$\begin{aligned}
&u_{n,t} - \Delta u_n + cu_n + \frac{4u_n v_n}{1+u_n^2} + \kappa\beta^\varepsilon(u_n) = a - \phi \quad \text{in } Q,\\
&v_{n,t} - \delta\Delta v_n - b\theta u_n + \frac{\theta u_n v_n}{1+u_n^2} = \theta\phi + Bw \quad \text{in } Q,\\
&u_n(x,0) = u_{n,0}(x), \quad v_n(x,0) = v_{n,0}(x) \quad \text{in } \Omega,\\
&u_n = v_n = 0 \quad \text{on } \Sigma.
\end{aligned} \tag{15}$$

According to the ODE theory, there is a unique solution to (15) in the interval $[0, T_n)$, where $T_n \to T$ is a consequence of the following a priori estimates.

Multiplying $(15)_1$ by u_n and $(15)_2$ by v_n and integrating them, respectively, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_n|_2^2 + |\nabla u_n|_2^2 + c|u_n|_2^2 + \int_\Omega \left(\frac{4u_n^2 v_n}{1+u_n^2} + \kappa\hat\beta^\varepsilon(u_n)\right)\mathrm{d}x\\
&\quad \leqslant (a-\phi, u_n)
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v_n|_2^2 + \delta|\nabla v_n|_2^2 + \int_\Omega \left(-b\theta u_n v_n + \frac{\theta u_n v_n^2}{1+u_n^2}\right)\mathrm{d}x\\
&\quad = (\theta\phi + Bw, v_n).
\end{aligned} \tag{17}$$

Here we have use the fact that

$$\hat\beta^\varepsilon(u_n) = \hat\beta^\varepsilon(u_n) - \hat\beta^\varepsilon(0) = \beta^\varepsilon(\xi)u_n \leqslant \beta^\varepsilon(u_n)u_n,$$

where ξ locates between 0 and u_n. Observing that

$$\int_\Omega \frac{4u_n^2 v_n}{1+u_n^2}\,\mathrm{d}x \leqslant \int_\Omega 4|v_n|\,\mathrm{d}x.$$

Therefore, from (16), Young's inequality and Hölder's inequality it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_n|_2^2 + |\nabla u_n|_2^2 + \int_\Omega \kappa\hat\beta^\varepsilon(u_n)\,\mathrm{d}x \leqslant C\big(|u_n|_2^2 + |v_n|_2^2\big) + C. \tag{18}$$

Here and throughout the proof of Lemma 1, we shall denote by C several positive constants independent of u_n, v_n and ε. With similar arguments in the above, we show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v_n|_2^2 + \delta|\nabla v_n|_2^2 \leqslant C\big(|u_n|_2^2 + |v_n|_2^2\big) + C, \tag{19}$$

which, together with (18), implies that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(|u_n|_2^2 + |v_n|_2^2\big) + |\nabla u_n|_2^2 + \delta|\nabla v_n|_2^2 + \kappa\int_\Omega \hat{\beta}^\varepsilon \,\mathrm{d}x \\
&\quad \leqslant C\big(|u_n|_2^2 + |v_n|_2^2\big) + C,
\end{aligned} \tag{20}$$

which, combined with (8) and the Gronwall's inequality, yields

$$\begin{aligned}
&|u_n|_{L^\infty(0,T;H)} + |u_n|_{L^2(0,T;V)} + |v_n|_{L^\infty(0,T;H)} + |v_n|_{L^2(0,T;V)} \\
&\quad + \kappa\big|\hat{\beta}^\varepsilon\big|_{L^1(0,T;L^1(\Omega))} \leqslant C.
\end{aligned} \tag{21}$$

Here we have use the fact that

$$\hat{\beta}^\varepsilon(0) = 0 \quad \text{and} \quad \hat{\beta}^\varepsilon(r) \geqslant 0 \quad \text{for any } r \in \mathbb{R}. \tag{22}$$

On the other hand, testing $(15)_1$ by $-\Delta u_n$ and $(15)_2$ by $-\Delta v_n$, respectively, and integrating the resulting equations over Ω, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla u_n|_2^2 + |\Delta u_n|_2^2 + \kappa\int_\Omega (\beta^\varepsilon)'(u_n)|\nabla u_n|^2 \,\mathrm{d}x \\
&\quad \leqslant |c||\nabla u_n|_2^2 + \left|\int_\Omega \frac{4u_n v_n}{1+u_n^2}\Delta u_n \,\mathrm{d}x\right| + (a-\phi, -\Delta u_n)
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla v_n|_2^2 + \delta|\Delta v_n|_2^2 \\
&\quad \leqslant \left|\int_\Omega \left(b\theta u_n\Delta v_n + \frac{\theta u_n v_n}{1+u_n^2}\Delta v_n\right)\mathrm{d}x\right| + (\theta\phi + Bw, -\Delta v_n).
\end{aligned} \tag{24}$$

Notice that

$$\begin{aligned}
&|c||\nabla u_n|_2^2 + \left|\int_\Omega \frac{4u_n v_n}{1+u_n^2}\Delta u_n \,\mathrm{d}x\right| + (a-\phi, -\Delta u_n) \\
&\quad \leqslant \frac{1}{2}|\Delta u_n|_2^2 + \int_\Omega \frac{16u_n^2 v_n^2}{(1+u_n^2)^2}\,\mathrm{d}x + |c||\nabla u_n|_2^2 + C \\
&\quad \leqslant \frac{1}{2}|\Delta u_n|_2^2 + C|v_n|_2^2 + |c||\nabla u_n|_2^2 + C
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
&\left| \int_\Omega \left(b\theta u_n \Delta v_n + \frac{\theta u_n v_n}{1+u_n^2} \Delta v_n \right) \mathrm{d}x \right| + (\theta\phi + Bw, \, -\Delta v_n) \\
&\quad \leqslant \frac{\delta}{2} |\Delta v_n|_2^2 + \int_\Omega \frac{2\theta^2 u_n^2 v_n^2}{\delta(1+u_n^2)^2} \, \mathrm{d}x + C|u_n|_2^2 + C \\
&\quad \leqslant \frac{\delta}{2} |\Delta v_n|_2^2 + C\big(|u_n|_2^2 + |v_n|_2^2\big) + C. \qquad (26)
\end{aligned}$$

Inserting (25) and (26) into (23) and (24), respectively, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla u_n|_2^2 + |\Delta u_n|_2^2 + \kappa \int_\Omega (\beta^\varepsilon)'(u_n)|\nabla u_n|^2 \, \mathrm{d}x \\
&\quad \leqslant \frac{1}{2}|\Delta u_n|_2^2 + C|v_n|_2^2 + |c||\nabla u_n|_2^2 + C \qquad (27)
\end{aligned}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla v_n|_2^2 + \delta|\Delta v_n|_2^2 \leqslant \frac{\delta}{2}|\Delta v_n|_2^2 + C\big(|u_n|_2^2 + |v_n|_2^2\big) + C, \qquad (28)$$

which, combined with (7), (21) and the Gronwall's inequality, implies that

$$|u_n|_{L^\infty(0,T;V)} + |u_n|_{L^2(0,T;H^2(\Omega))} \leqslant C \qquad (29)$$

and

$$|v_n|_{L^\infty(0,T;V)} + |v_n|_{L^2(0,T;H^2(\Omega))} \leqslant C. \qquad (30)$$

Now, multiplying $(15)_1$ by β^ε, integrating over $[0, T]$ and invoking the Young's inequality, we derive

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big|\hat{\beta}^\varepsilon\big|_{L^1(\Omega)} + \kappa\big|\beta^\varepsilon\big|_2^2 + \int_\Omega \big(\beta^\varepsilon\big)'(u_n)|\nabla u_n|^2 \, \mathrm{d}x \\
&\quad = \int_\Omega \left[\left(a - \phi - cu_n - \frac{4u_n v_n}{1+u_n^2} \right) \beta^\varepsilon(u_n) \right] \mathrm{d}x \\
&\quad \leqslant \frac{\kappa}{2}\big|\beta^\varepsilon\big|_2^2 + C\big(|\phi|_2^2 + |u_n|_2^2 + |v_n|_2^2 + 1\big). \qquad (31)
\end{aligned}$$

Thanks to (7), (21) and the Gronwall's inequality, we derive

$$\big|\hat{\beta}^\varepsilon\big|_{L^\infty(0,T;L^1(\Omega))} + \big|\beta^\varepsilon\big|_{L^2(0,T;H)} \leqslant C. \qquad (32)$$

Finally, multiplying $(15)_1$ and $(15)_2$ by $u_{n,t}$ and $v_{n,t}$, respectively, after some basic calculation, we end up with

$$|u_n'|_{L^2(0,T;H)} \leqslant C \quad \text{and} \quad |v_n'|_{L^2(0,T;H)} \leqslant C. \qquad (33)$$

By (29), (30) and (32)–(33) and applying the rather standard argument, we can conclude that there exist a function (u, v) and a subsequence of (u_n, v_n), still denoted by themselves, such that

$$(u_n, v_n) \to (u, v) \quad \text{weakly in } Y \times Y \tag{34}$$

and (u, v) is the solution of problem (9). The uniqueness of the solution to problem (9) can be got easily, we omit it.

Now, we prove the w-dependence of this lemma. To this end, replacing (u, v) and w by (u_n, v_n) and w_n in (9), respectively, we obtain

$$\begin{aligned}
&u_{n,t} - \Delta u_n + cu_n + \frac{4u_n v_n}{1+u_n^2} + \kappa\beta^\varepsilon(u_n) = a - \phi \quad \text{in } Q,\\
&v_{n,t} - \delta\Delta v_n - b\theta u_n + \frac{\theta u_n v_n}{1+u_n^2} = \theta\phi + Bw_n \quad \text{in } Q,\\
&u_n(x,0) = u_0(x), \quad v_n(x,0) = v_0(x) \quad \text{in } \Omega,\\
&u_n = v_n = 0 \quad \text{on } \Sigma.
\end{aligned} \tag{35}$$

By the above analysis, we have

$$\begin{aligned}
&\big|\hat{\beta}^\varepsilon\big|_{L^\infty(0,T;L^1(\Omega))} + \big|\beta^\varepsilon\big|_{L^2(0,T;H)} + |u_n|_{L^\infty(0,T;V)} + |u_n|_{L^2(0,T;H^2(\Omega))}\\
&\quad + |u_n'|_{L^2(0,T;H)} \leqslant C
\end{aligned} \tag{36}$$

and

$$|v_n|_{L^\infty(0,T;V)} + |v_n|_{L^2(0,T;H^2(\Omega))} + |v_n'|_{L^2(0,T;H)} \leqslant C, \tag{37}$$

where $C > 0$ is a constant independent of n and ε. By (36)–(37) and using Ascoli–Arzela theorem and compactness lemma, we infer that there exists a subsequence of (u_n, v_n), still denoted by itself, such that

$$(u_n, v_n) \to (u, v) \qquad \text{weakly in } \big(L^2\big(0,T;H^2(\Omega)\big)\big)^2, \tag{38}$$

$$(u_n, v_n) \to (u, v) \qquad \text{strongly in } \big(C\big([0,T];H\big) \cap L^2(0,T;V)\big)^2 \tag{39}$$

and

$$(u_n', v_n') \to (u', v') \quad \text{weakly in } \big(L^2(0,T;H)\big)^2 \tag{40}$$

as $n \to \infty$. The proof is completed. □

Lemma 2. *Suppose that β^ε satisfies* (7)–(8)*, let $w_\varepsilon \in L^2(0,T;U)$ with $w_\varepsilon \to w^*$ weakly in $L^2(0,T;U)$ as $\varepsilon \to 0$, $(u_\varepsilon, v_\varepsilon)$ is the solution of* (9) *corresponding to w_ε. Then on some subsequence $(u_{\varepsilon_n}, v_{\varepsilon_n})$ of $(u_\varepsilon, v_\varepsilon)$, there exists a triple $(u, v, \eta) \in Y \times Y \times L^2(0,T;H)$ such that*

$$\eta \in \partial I_{[\sigma_*,\sigma^*]}(u) \quad \textit{a.e. in } L^2(0,T;H), \tag{41}$$

while

$$\big(u_{\varepsilon_n}, v_{\varepsilon_n}\big) \to (u, v) \quad \textit{weakly in } \big(L^2\big(0, T; H^2(\Omega)\big)\big)^2, \tag{42}$$

$$\big(u_{\varepsilon_n}, v_{\varepsilon_n}\big) \to (u, v) \quad \textit{strongly in } \big(C\big([0, T]; H\big) \cap L^2(0, T; V)\big)^2, \tag{43}$$

$$\big(u'_{\varepsilon_n}, v'_{\varepsilon_n}\big) \to (u', v') \quad \textit{weakly in } \big(L^2(0, T; H)\big)^2, \tag{44}$$

$$\beta^{\varepsilon}(u_{\varepsilon_n}) \to \eta \quad \textit{weakly in } L^2(0, T; H) \tag{45}$$

as $\varepsilon_n \to 0$ *and* (u, v, η) *is a solution of* (2) *satisfying the following estimates*

$$|u|_Y^2 + |v|_Y^2 + |\eta|_{L^2(Q_T)}^2 \leqslant C \tag{46}$$

with $C > 0$ *is independent of* ε, n.

Proof. Rewrite (9) as following:

$$\begin{aligned}
&u_{\varepsilon,t} - \Delta u_{\varepsilon} + c u_{\varepsilon} + \frac{4 u_{\varepsilon} v_{\varepsilon}}{1 + u_{\varepsilon}^2} + \kappa \beta^{\varepsilon}(u_{\varepsilon}) = a - \phi \quad \text{in } Q,\\
&v_{\varepsilon,t} - \delta \Delta v_{\varepsilon} - b\theta u_{\varepsilon} + \frac{\theta u_{\varepsilon} v_{\varepsilon}}{1 + u_{\varepsilon}^2} = \theta\phi + B w_{\varepsilon} \quad \text{in } Q,\\
&u_{\varepsilon}(x, 0) = u_0(x), \quad v_{\varepsilon}(x, 0) = v_0(x) \quad \text{in } \Omega,\\
&u_{\varepsilon}(x, t) = v_{\varepsilon}(x, t) = 0 \quad \text{on } \Sigma.
\end{aligned} \tag{47}$$

Employing almost exactly the same arguments as in the proof of Lemma 1, we conclude that the results (42)–(44). Furthermore, by a standard argument in [2], we get $\eta \in \partial I_{[\sigma_*, \sigma^*]}(u)$ a.e. in $L^2(0, T; H)$. This completes the proof. □

Now, we let (u^*, v^*, w^*) be optimal for problem (P). For each $\varepsilon > 0$, assume $(u_{\varepsilon}^*, v_{\varepsilon}^*, w_{\varepsilon}^*)$ is the solution to

$$\begin{aligned}
&u_t - \Delta u + c u + \frac{4uv}{1 + u^2} + \kappa \beta^{\varepsilon}(u) = a - \phi \quad \text{in } Q,\\
&v_t - \delta \Delta v - b\theta u + \frac{\theta uv}{1 + u^2} = \theta\phi + B w^* \quad \text{in } Q,\\
&u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \quad \text{in } \Omega,\\
&u(x, t) = v(x, t) = 0 \quad \text{on } \Sigma.
\end{aligned} \tag{48}$$

It follows from Lemma 2 that

$$\delta(\varepsilon) := |u_{\varepsilon}^* - u^*|_{L^2(0, T; H)} \to 0. \tag{49}$$

Now, for each $\varepsilon > 0$, the approximating optimal control problems (P^{ε}) is as follows:

$$\text{Minimize } L_{\varepsilon}(w) \text{ over } w \in L^2(0, T; U),$$

where $L_\varepsilon : L^2(0,T;U) \to \mathbb{R}$ by

$$L_\varepsilon(w) = \int_0^T \bigl[g_\varepsilon(t,u_\varepsilon) + h(w)\bigr]\,\mathrm{d}t + \frac{1}{2}|w-w^*|^2_{L^2(0,T;U)} + \frac{1}{2\delta(\varepsilon)}\bigl[d_S\bigl(F(u_\varepsilon)\bigr) + \delta(\varepsilon)\bigr]^2 \tag{50}$$

and $(u_\varepsilon, v_\varepsilon)$ is the solution of (9). Here, $d_S(F(u))$ denotes the distance between $F(u)$ and S,

$$g_\varepsilon(t,y) = \int_{\mathbb{R}^n} g(t, P_n y - \varepsilon \Lambda_n \tau)\rho_n(\tau)\,\mathrm{d}\tau \tag{51}$$

is the approximations of g [1], where $n = [1/\varepsilon]$, ρ_n is a mollifier in $\mathbb{R}^n$ and $P_n : H \to X_n$ is the projection of H on X_n, which is the finite dimensional space generated by $\{e_i\}_{i=1}^n$, $\{e_i\}_{i=1}^\infty$ is an orthonormal basis in H, $\Lambda_n : \mathbb{R}^n \to X_n$ is the operator defined by $\Lambda_n(\tau) = \sum_{i=1}^n \tau_i e_i$ with $\tau = (\tau_1, \tau_2, \dots, \tau_n)$.

In this case, one can transform the original state constrained optimal control problem (P) into non-constrained optimal control problem (P^ε) and use the method [3] to obtain the optimality condition for problem (P) by a passage-to-limit procedure for $\varepsilon \searrow 0$.

First of all, we show the existence of the optimal solutions for (P^ε).

Lemma 3. (P^ε) *has at least one optimal solution.*

Proof. Let $\varepsilon > 0$ be fixed. It is clear that $\inf L_\varepsilon(w) > -\infty$. Let $d_\varepsilon = \inf\{L_\varepsilon(w)\colon w \in L^2(0,T;U)\}$ and w_n be a minimizing sequence such that

$$d_\varepsilon \leqslant L_\varepsilon(w_n) \leqslant d_\varepsilon + \frac{1}{n}, \tag{52}$$

which, together with (H2), (H3) and (50), implies that w_n is bounded in $L^2(0,T;U)$. Without loss of generality, we may assume that $w_n \to \tilde{w}$ weakly in $L^2(0,T;U)$. Let (u_n, v_n) and $(\tilde{u}, \tilde{v})$ be the solutions of (9) corresponding to w_n and $\tilde{w}$, respectively. It follows from Lemma 1 that on some subsequence of (u_n, v_n), still denoted by itself,

$$(u_n, v_n) \to (\tilde{u}, \tilde{v}) \quad \text{weakly in } Y \times Y \quad \text{and strongly in } \bigl(C\bigl([0,T];H\bigr) \cap L^2(0,T;V)\bigr)^2. \tag{53}$$

With the help of (H2), (51) and (53), we also obtain

$$\int_0^T \bigl|g_\varepsilon(t,u_n) - g_\varepsilon\bigl(t,\tilde{u}\bigr)\bigr|_2\,\mathrm{d}t \leqslant C\int_0^T |u_n - \tilde{u}|_2\,\mathrm{d}t \to 0 \quad \text{as } n \to \infty. \tag{54}$$

On the other hand, due to (53) and (H1), we have

$$\lim_{n\to\infty} F(u_n) = F\bigl(\tilde{u}\bigr) \tag{55}$$

and therefore,

$$\lim_{n\to\infty}\frac{1}{2\delta(\varepsilon)}\big[d_S\big(F(u_n)\big)+\delta(\varepsilon)\big]^2=\frac{1}{2\delta(\varepsilon)}\big[d_S\big(F(\tilde u)\big)+\delta(\varepsilon)\big]^2. \tag{56}$$

Finally, (50) and (54)–(56) imply that $(\tilde u,\tilde v,\tilde w)$ is the optimal for problem (P^ε). This concludes the proof of the Lemma 3. □

Lemma 4. *Let w_ε be optimal for problem* (P^ε) *and $(u_\varepsilon, v_\varepsilon)$ be the solution of* (9) *corresponding to w_ε. Then on some subsequence ε_n,*

$$\big(u_{\varepsilon_n},v_{\varepsilon_n}\big)\to(u^*,v^*)\quad\textit{strongly in } \big(C\big([0,T];H\big)\cap L^2(0,T;V)\big)^2, \tag{57}$$

$$w_{\varepsilon_n}\to w^*\qquad\textit{strongly in } L^2(0,T;U). \tag{58}$$

Proof. Since w_ε is solution to (P^ε), we have

$$L_\varepsilon(w_\varepsilon)\leqslant\int_0^T\big[g_\varepsilon(t,u_\varepsilon^*)+h(w^*)\big]\,\mathrm{d}t+\frac{1}{2\delta(\varepsilon)}\big[d_S\big(F(u_\varepsilon^*)\big)+\delta(\varepsilon)\big]^2, \tag{59}$$

which, together with (49), implies that

$$\begin{aligned}\frac{1}{2\delta(\varepsilon)}\big[d_S\big(F(u_\varepsilon^*)\big)+\delta(\varepsilon)\big]^2&\leqslant\frac{1}{2\delta(\varepsilon)}\big[\big|F(u_\varepsilon^*)-F(u^*)\big|_Z+\delta(\varepsilon)\big]^2\\&\leqslant\frac{1}{2\delta(\varepsilon)}\big[C|u_\varepsilon^*-u^*|_{L^2(0,T;H)}+\delta(\varepsilon)\big]^2\\&\leqslant\frac{(1+C)^2}{2}\delta(\varepsilon)\to0\quad\text{as }\varepsilon\to0,\end{aligned} \tag{60}$$

which, combined with (59), implies that

$$\limsup_{\varepsilon\to0}L_\varepsilon(w_\varepsilon)\leqslant\int_0^T\big[g(t,u^*)+h(w^*)\big]\,\mathrm{d}t, \tag{61}$$

which implies that w_ε is bounded in $L^2(0,T;U)$. Without loss of generality, we may assume that $w_\varepsilon\to\tilde w$ weakly in $L^2(0,T;U)$, which, together with Lemma 2, implies that there exists a sequence of ε_n such that

$$(u_{\varepsilon_n},v_{\varepsilon_n})\to(\tilde u,\tilde v)\quad\text{strongly in }\big(C\big([0,T];H\big)\cap L^2(0,T;V)\big)^2. \tag{62}$$

On the other hand, (50) and (61) imply that

$$\lim_{\varepsilon_n\to0}d_S\big(F(u_{\varepsilon_n})\big)=0 \tag{63}$$

and thus,

$$\lim_{\varepsilon_n\to0}d_S\big(F(\tilde u)\big)=0. \tag{64}$$

Thus, we conclude from (50), (62) and (64) that

$$\int_0^T \big[g(t,u^*)+h(w^*)\big]\,\mathrm{d}t = L(w^*) \geqslant \liminf_{\varepsilon_n\to 0} L_{\varepsilon_n}(w_{\varepsilon_n}) \geqslant \int_0^T \big[g(t,\tilde u)+h(\tilde w)\big]\,\mathrm{d}t + \frac{1}{2}|\tilde w - w^*|^2_{L^2(0,T;U)}. \tag{65}$$

Hence, $\tilde u = u^*$, $\tilde v = v^*$, $\tilde w = w^*$ and

$$w_\varepsilon \to w^* \text{ strongly in } L^2(0,T;U), \tag{66}$$

Finally, it follows from Lemma 2 that

$$(u_{\varepsilon_n}, v_{\varepsilon_n}, w_{\varepsilon_n}) \to (u^*, v^*, w^*) \quad \text{strongly in } \big(C\big([0,T];H\big)\cap L^2(0,T;V)\big)^2 \times L^2(0,T;U). \tag{67}$$

This completes the proof. □

3 The optimality condition for (P$^\varepsilon$) and (P)

In the following, we derive the optimality condition for problem (P) by showing the relation between approximation problem (P$^\varepsilon$) and problem (P). We start this section with the necessary conditions for $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ to be optimal for (P$^\varepsilon$).

Lemma 5. *Suppose that β^ε satisfies* (7)–(8) *and* (H1)–(H3) *hold. Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be optimal for problem* (P$^\varepsilon$). *Then there exists a tetrad $(\mu_\varepsilon, p_\varepsilon, q_\varepsilon, \zeta_\varepsilon) \in \mathbb{R}\times W^{1,2}(0,T;V^*)\cap L^2(0,T;V)\cap C([0,T];H)\times W^{1,2}(0,T;V^*)\cap L^2(0,T;V)\cap C([0,T];H)\times Z^*$ such that*

$$\begin{aligned}
&-p_{\varepsilon,t} - \Delta p_\varepsilon + c p_\varepsilon + \frac{v_\varepsilon(1-u_\varepsilon^2)}{(1+u_\varepsilon^2)^2}(4p_\varepsilon + \theta q_\varepsilon) - b\theta q_\varepsilon + \kappa\big(\beta^\varepsilon\big)'(u_\varepsilon)p_\varepsilon \\
&\quad = -\big(F'(u_\varepsilon)\big)^*\zeta_\varepsilon - \mu_\varepsilon \nabla g_\varepsilon(t,u_\varepsilon), \\
&-q_{\varepsilon,t} - \delta\Delta q_\varepsilon + \frac{u_\varepsilon}{1+u_\varepsilon^2}(4p_\varepsilon + \theta q_\varepsilon) = 0, \\
&p_\varepsilon(x,t) = 0, \quad q_\varepsilon(x,t) = 0 \quad \text{in } \Sigma, \\
&p_\varepsilon(T) = q_\varepsilon(T) = 0
\end{aligned} \tag{68}$$

and

$$B^* q_\varepsilon = \mu_\varepsilon\big[\nabla h_\varepsilon(w_\varepsilon) + w_\varepsilon - w^*\big] \quad \text{a.e. } t\in[0,T]. \tag{69}$$

Proof. Let w_ε be optimal for problem (P$^\varepsilon$) and $(u_\varepsilon, v_\varepsilon)$ be the solution of (9) corresponding to w_ε. Set $w_\varepsilon^\chi = w_\varepsilon + \chi w$ for any $w\in L^2(0,T;U)$, $(u_\varepsilon^\chi, v_\varepsilon^\chi)$ is the solution of (9) corresponding to w_ε^χ. Then it is clear that

$$\big(u_\varepsilon^\chi, v_\varepsilon^\chi\big) \to (u_\varepsilon, v_\varepsilon) \quad \text{strongly in } C\big([0,T];H\big)\cap L^2(0,T;V) \text{ as } \chi\to 0. \tag{70}$$

Now, owing to w_ε is the optimal for problem (P$^\varepsilon$), we have $(L_\varepsilon(w_\varepsilon^\chi) - L_\varepsilon(w_\varepsilon))/\chi \geqslant 0$ for all $\chi > 0$. Hence, employing the same arguments as in the proof of [1], we conclude that

$$0 \leqslant \mu_\varepsilon \int_0^T \big[\langle \nabla g_\varepsilon(t, u_\varepsilon), y_\varepsilon \rangle + \langle \nabla h(w_\varepsilon) + w_\varepsilon - w^*, w \rangle_U\big]\,\mathrm{d}t + \big\langle \big(F'(u_\varepsilon)\big)^* \zeta_\varepsilon, y_\varepsilon \big\rangle_{Z^*,Z}, \tag{71}$$

where $(y_\varepsilon, \bar{y}_\varepsilon)$ is the solution of

$$\begin{aligned} &y_{\varepsilon,t} - \Delta y_\varepsilon + c y_\varepsilon + \frac{4v_\varepsilon(1-u_\varepsilon^2)}{(1+u_\varepsilon^2)^2} y_\varepsilon + \frac{4u_\varepsilon}{1+u_\varepsilon^2}\bar{y}_\varepsilon + \kappa(\beta^\varepsilon)'(u_\varepsilon) y_\varepsilon = 0, \\ &\bar{y}_{\varepsilon,t} - \delta\Delta \bar{y}_\varepsilon + \frac{\theta u_\varepsilon}{1+u_\varepsilon^2}\bar{y}_\varepsilon + \frac{\theta v_\varepsilon(1-u_\varepsilon^2)}{(1+u_\varepsilon^2)^2} y_\varepsilon - b\theta y_\varepsilon = Bw, \\ &y_\varepsilon(0) = 0, \qquad \bar{y}_\varepsilon(0) = 0, \end{aligned} \tag{72}$$

$\nabla g_\varepsilon(t, u_\varepsilon)$ denotes the gradient of g_ε to the second variable at u_ε, $\nabla h(w_\varepsilon)$ denotes the gradient of h at w_ε and

$$|\zeta_\varepsilon|_{Z^*} = \begin{cases} \nabla d_S(F(u_\varepsilon)) & \text{if } F(u_\varepsilon) \notin S, \\ 0 & \text{if } F(u_\varepsilon) \in S. \end{cases} \tag{73}$$

Thanks to S is convex and closed, we may also infer that

$$|\zeta_\varepsilon|_{Z^*} = 1 \quad \text{if } F(u_\varepsilon) \notin S. \tag{74}$$

Let

$$\mu_\varepsilon = \frac{\delta(\varepsilon)}{\delta(\varepsilon) + d_S(F(u_\varepsilon))} \tag{75}$$

and $(p_\varepsilon, q_\varepsilon)$ be the solution of (68). Due to [1, Thm. 1.14], the boundary value problem (68) has a unique solution $(p_\varepsilon, q_\varepsilon) \in W^{1,2}(0,T;V^*) \cap L^2(0,T;V) \cap C([0,T];H) \times W^{1,2}(0,T;V^*) \cap L^2(0,T;V) \cap C([0,T];H)$. It follows from (68), (71) and (72) that

$$0 \leqslant \int_0^T -\langle B^* q_\varepsilon, w \rangle + \mu_\varepsilon \langle \nabla h(w_\varepsilon) + w_\varepsilon - w^*,\, w \rangle_U\,\mathrm{d}t, \tag{76}$$

which implies (69). This completes the proof. □

Proof of Theorem 1. By using the properties of β^ε and Lemma 4 that, on a sequence of ε, still denoted by ε,

$$(u_\varepsilon, v_\varepsilon) \to (u, v) \quad \text{weakly in } \big(L^2\big(0,T;H^2(\Omega)\big)\big)^2, \tag{77}$$

$$(u_\varepsilon, v_\varepsilon) \to (u, v) \quad \text{strongly in } \big(C\big([0,T];H\big) \cap L^2(0,T;V)\big)^2, \tag{78}$$

$$(u'_\varepsilon, v'_\varepsilon) \to (u', v') \quad \text{weakly in } \big(L^2(0,T;H)\big)^2 \tag{79}$$

and

$$\beta^\varepsilon(u_{\varepsilon_n}) \to \eta \quad \text{weakly in } L^2(0,T;H). \tag{80}$$

On the other hand, by the same argument in [1], we obtain that on a subsequence, still denoted by ε,

$$(p_\varepsilon, q_\varepsilon) \to (p, q) \quad \text{weakly in } \big(L^2(0,T;V)\big)^2 \text{ and weakly star in } \big(C\big([0,T];H\big)\big)^2, \tag{81}$$

$$(p_\varepsilon, q_\varepsilon) \to (p, q) \quad \text{strongly in } \big(L^2\big([0,T];H\big)\big)^2 \tag{82}$$

and

$$(p'_\varepsilon, q'_\varepsilon) \to (p', q') \quad \text{weakly in } \big(L^2(0,T;V^*)\big)^2. \tag{83}$$

Now, we will prove that

$$\big(\beta^\varepsilon\big)'(u_\varepsilon)p_\varepsilon \to \eta \quad \text{weakly star in } \big(L^\infty(Q_T)\big)^*. \tag{84}$$

In fact, let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth, bounded and monotone approximation of the signum function such that $\psi(0) = 0$ (see [1, Lemma 3.5]). Now, multiplying $(68)_1$ by $\psi(p_\varepsilon)$ and integrating the resulting equations over $[0, T]$, we get

$$\begin{aligned}
&\int\limits_Q \kappa\big(\beta^\varepsilon\big)'(u_\varepsilon)\psi(p_\varepsilon)p_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \int\limits_Q \bigg(-\big(F'(u_\varepsilon)\big)^*\zeta_\varepsilon - \mu_\varepsilon\nabla g_\varepsilon(t,u_\varepsilon) + p_{\varepsilon,t} + \Delta p_\varepsilon - cp_\varepsilon \\
&\qquad\quad - \frac{v_\varepsilon(1-u_\varepsilon^2)}{(1+u_\varepsilon^2)^2}(4p_\varepsilon - \theta q_\varepsilon) + b\theta q_\varepsilon\bigg)\psi(p_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant \int\limits_0^T \big|\big(F'(u_\varepsilon)\big)^*\zeta_\varepsilon\big|_2^2 + \big|\mu_\varepsilon\nabla g_\varepsilon(t,u_\varepsilon)\big|_2^2 + \gamma(c,\theta,b)\big(|p_\varepsilon|_2^2 + |q_\varepsilon|_2^2 + |p'_\varepsilon|_{V^*}^2 + |v_\varepsilon|_2^2\big)\,\mathrm{d}t \\
&\quad\leqslant C_1,
\end{aligned} \tag{85}$$

where $C_1 > 0$ is independent of ε and $\gamma(c, \theta, b)$ is positive constant depending on c, θ and b. Here we have use the fact that

$$\int\limits_Q \Delta p_\varepsilon \psi(p_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \leqslant C_2 + \int\limits_Q \big|p_\varepsilon\Delta\psi(p_\varepsilon)\big|\,\mathrm{d}x\,\mathrm{d}t \leqslant C_3 + C_4|p_\varepsilon|_2^2.$$

and $\psi \in L^2(0,T;V)$ (see [1, Lemma 5.3]). Here and throughout the proof of Theorem 1, we shall denote by C_i ($i \in \mathbb{N}$) several positive constants independent of ε. Therefore, (85) implies that

$$\kappa\int\limits_Q \big|\big(\beta^\varepsilon\big)'(u_\varepsilon)p_\varepsilon\big|\,\mathrm{d}x\,\mathrm{d}t \leqslant C_5.$$

Hence, by the above inequality, we infer that there exists $\eta \in (L^\infty(Q_T))^*$ such that

$$\big(\beta^\varepsilon\big)'(u_\varepsilon)p_\varepsilon \to \eta \quad \text{weakly star in } \big(L^\infty(Q_T)\big)^*.$$

Thus, (84) holds.

On the other hand, it follows from (74) and (75) that

$$1 \leqslant \mu_\varepsilon + |\zeta_\varepsilon|_{Z^*} \leqslant 2 \quad \text{for any } \varepsilon > 0. \tag{86}$$

Therefore, there exist two generalized subsequences of μ_ε and ζ_ε such that

$$\mu_\varepsilon \to \mu_0 \quad \text{as } \varepsilon \to 0 \quad \text{and} \quad \zeta_\varepsilon \to \zeta_0 \quad \text{weakly star in } Z^* \text{ as } \varepsilon \to 0. \tag{87}$$

Here we use the fact that μ_ε and ζ_ε are bounded on $\mathbb{R}$ and Z^*, respectively. Using Lemma 4, we may pass to the limit in (69) and derive $(6)_1$.

On the other hand, thanks to (H2) and (82), we may also infer from [1, Prop. 1.11] that $\nabla g_\varepsilon(t, u_\varepsilon)$ weak star upper semicontinuous, which implies that

$$\nabla g_\varepsilon(t, u_\varepsilon) \to \rho(t) \quad \text{weakly in } L^2(0,T;H) \text{ as } \varepsilon \to 0, \tag{88}$$

where $\rho(t) \in \partial g(t, u^*)$ a.e. $t \in (0,T)$.

Similarly, due to (H1) [1, Prop. 1.12], we have

$$\big[F'(u_\varepsilon)\big]^*\zeta_\varepsilon \to \big[F'(u^*)\big]^*\zeta_0 \quad \text{weakly in } L^2(0,T;H) \text{ as } \varepsilon \to 0. \tag{89}$$

In the following we will prove

$$\frac{u_\varepsilon}{1+u_\varepsilon^2}p_\varepsilon \to \frac{u^*}{1+(u^*)^2}p \quad \text{weakly in } L^2(0,T;H) \text{ as } \varepsilon \to 0 \tag{90}$$

and

$$\frac{v_\varepsilon(1-u_\varepsilon^2)}{(1+u_\varepsilon^2)^2}p_\varepsilon \to \frac{v^*(1-(u^*)^2)}{(1+(u^*)^2)^2}p \quad \text{weakly in } L^2(0,T;V^*) \text{ as } \varepsilon \to 0. \tag{91}$$

Indeed, let $\varphi \in L^2(0,T;H)$, then we derive

$$\begin{aligned}
&\left(\int_0^T \left|\left\langle \frac{u_\varepsilon}{1+u_\varepsilon^2}p_\varepsilon - \frac{u^*}{1+(u^*)^2}p, \varphi\right\rangle\right| \mathrm{d}t\right)^2 \\
&\quad = \left(\int_0^T \left|\left\langle \frac{u_\varepsilon p_\varepsilon(1+(u^*)^2) - u^* p(1+u_\varepsilon^2)}{(1+u_\varepsilon^2)(1+(u^*)^2)}, \varphi\right\rangle\right| \mathrm{d}t\right)^2 \\
&\quad \leqslant \left(\int_0^T \left|\frac{u_\varepsilon p_\varepsilon(1+(u^*)^2) - u^* p(1+u_\varepsilon^2)}{(1+u_\varepsilon^2)(1+(u^*)^2)}\right|_2 |\varphi|_2 \,\mathrm{d}t\right)^2 \\
&\quad \leqslant \int_0^T \left|\frac{u_\varepsilon p_\varepsilon(1+(u^*)^2) - u^* p(1+u_\varepsilon^2)}{(1+u_\varepsilon^2)(1+(u^*)^2)}\right|_2^2 \mathrm{d}t \int_0^T |\varphi|_2^2 \,\mathrm{d}t.
\end{aligned} \tag{92}$$

On the other hand,

$$
\begin{aligned}
&\left|\frac{u_\varepsilon p_\varepsilon(1+(u^*)^2) - u^* p(1+u_\varepsilon^2)}{(1+u_\varepsilon^2)(1+(u^*)^2)}\right|_2^2 \\
&\quad = \int\limits_\Omega \left|\frac{u_\varepsilon p_\varepsilon(1+(u^*)^2) - u^* p(1+u_\varepsilon^2)}{(1+u_\varepsilon^2)(1+(u^*)^2)}\right|^2 \mathrm{d}x \\
&\quad \leqslant 4 \int\limits_\Omega \frac{(u_\varepsilon p_\varepsilon - u^* p_\varepsilon)^2 + (u^* p_\varepsilon - u^* p)^2 + u_\varepsilon^2 (u^*)^2 [(p_\varepsilon u^* - p u^*)^2 + (p u^* - p u_\varepsilon)^2]}{(1+u_\varepsilon^2)^2 (1+(u^*)^2)^2} \mathrm{d}x \\
&\quad := J_{1,\varepsilon}(t) + J_{2,\varepsilon}(t) + J_{3,\varepsilon}(t) + J_{4,\varepsilon}(t),
\end{aligned} \tag{93}
$$

where

$$
J_{1,\varepsilon}(t) = 4 \int\limits_\Omega \frac{(u_\varepsilon p_\varepsilon - u^* p_\varepsilon)^2}{(1+u_\varepsilon^2)^2 (1+(u^*)^2)^2} \mathrm{d}x, \tag{94}
$$

$$
J_{2,\varepsilon}(t) = 4 \int\limits_\Omega \frac{(u^* p_\varepsilon - u^* p)^2}{(1+u_\varepsilon^2)^2 (1+(u^*)^2)^2} \mathrm{d}x, \tag{95}
$$

$$
J_{3,\varepsilon}(t) = 4 \int\limits_\Omega \frac{u_\varepsilon^2 (u^*)^2 (p_\varepsilon u^* - p u^*)^2}{(1+u_\varepsilon^2)^2 (1+(u^*)^2)^2} \mathrm{d}x \tag{96}
$$

and

$$
J_{4,\varepsilon}(t) = 4 \int\limits_\Omega \frac{u_\varepsilon^2 (u^*)^2 (p u^* - p u_\varepsilon)^2}{(1+u_\varepsilon^2)^2 (1+(u^*)^2)^2} \mathrm{d}x. \tag{97}
$$

Moreover, due to the Hölder's inequality and Young's inequality, we have

$$
\begin{aligned}
J_{1,\varepsilon}(t) &= 4 \int\limits_\Omega \frac{(u_\varepsilon p_\varepsilon - u^* p_\varepsilon)^2}{(1+u_\varepsilon^2)^2 (1+(u^*)^2)^2} \mathrm{d}x \leqslant 4 \int\limits_\Omega \frac{|u_\varepsilon - u^*|(|u_\varepsilon| + |u^*|) p_\varepsilon^2}{(1+u_\varepsilon^2)^2 (1+(u^*)^2)^2} \mathrm{d}x \\
&= 4 \int\limits_\Omega \left(\frac{|u_\varepsilon - u^*||u_\varepsilon| p_\varepsilon^2}{(1+u_\varepsilon^2)^2 (1+(u^*)^2)^2} + \frac{|u_\varepsilon - u^*||u^*| p_\varepsilon^2}{(1+u_\varepsilon^2)^2 (1+(u^*)^2)^2} \right) \mathrm{d}x \\
&\leqslant 4 \int\limits_\Omega \left(\frac{|u_\varepsilon - u^*||u_\varepsilon| p_\varepsilon^2}{1+u_\varepsilon^2} + \frac{|u_\varepsilon - u^*||u^*| p_\varepsilon^2}{1+(u^*)^2} \right) \mathrm{d}x \\
&\leqslant 2 \int\limits_\Omega \left(|u_\varepsilon - u^*| p_\varepsilon^2 + |u_\varepsilon - u^*| p_\varepsilon^2 \right) \mathrm{d}x = 4 \int\limits_\Omega |u_\varepsilon - u^*| p_\varepsilon^2 \, \mathrm{d}x \\
&\leqslant 4 |u_\varepsilon - u^*|_2 |p_\varepsilon|_{L^4(\Omega)}^2 \leqslant 4 |u_\varepsilon - u^*|_2 |p_\varepsilon|_{L^6(\Omega)}^2 |\Omega|^{1/6},
\end{aligned} \tag{98}
$$

thus, by the Hölder's inequality and Sobolev embedding theorem, we have

$$\begin{aligned}\int_0^T J_{1,\varepsilon}(t)\,\mathrm{d}t &= 4\int_0^T\int_\Omega \frac{(u_\varepsilon p_\varepsilon - u^* p_\varepsilon)^2}{(1+u_\varepsilon^2)^2(1+(u^*)^2)^2}\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant 4\int_0^T |u_\varepsilon - u^*|_2 |p_\varepsilon|_{L^6(\Omega)}^2 |\Omega|^{1/6}\,\mathrm{d}t \\ &\leqslant 4|\Omega|^{1/6}\max_{0\leqslant t\leqslant T}\bigl|u_\varepsilon(t)-u^*(t)\bigr|_2 \int_0^T |p_\varepsilon|_{L^6(\Omega)}^2\,\mathrm{d}t, \\ &\leqslant 4|\Omega|^{1/6}\max_{0\leqslant t\leqslant T}\bigl|u_\varepsilon(t)-u^*(t)\bigr|_2 \int_0^T |p_\varepsilon|_V^2\,\mathrm{d}t, \end{aligned} \tag{99}$$

which, together with (78) and (81), implies that

$$\int_0^T J_{1,\varepsilon}(t)\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t \to 0 \quad \text{as } \varepsilon \to 0.$$

Similarly, by the Young's inequality and (82), we have

$$\begin{aligned}\int_0^T J_{2,\varepsilon}(t)\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t &= 4\int_0^T\int_\Omega \frac{(u^*)^2(p_\varepsilon - p)^2}{(1+u_\varepsilon^2)^2(1+(u^*)^2)^2}\,\mathrm{d}x\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t \\ &\leqslant 4\int_0^T\int_\Omega \frac{(u^*)^2(p_\varepsilon - p)^2}{(1+(u^*)^2)^2}\,\mathrm{d}x\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t \\ &\leqslant 2\int_0^T\int_\Omega (p_\varepsilon - p)^2\,\mathrm{d}x\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t \to 0 \quad \text{as } \varepsilon \to 0, \end{aligned} \tag{100}$$

$$\begin{aligned}\int_0^T J_{3,\varepsilon}(t)\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t &= 4\int_0^T\int_\Omega \frac{u_\varepsilon^2(u^*)^4(p_\varepsilon - p)^2}{(1+u_\varepsilon^2)^2(1+(u^*)^2)^2}\,\mathrm{d}x\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t \\ &= 4\int_0^T\int_\Omega \frac{u_\varepsilon^2(u^*)^4(p_\varepsilon - p)^2}{(1+2u_\varepsilon^2+u_\varepsilon^4)(1+2(u^*)^2+(u^*)^4)}\,\mathrm{d}x\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t \\ &\leqslant 2\int_0^T\int_\Omega (p_\varepsilon - p)^2\,\mathrm{d}x\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t \to 0 \quad \text{as } \varepsilon \to 0, \end{aligned} \tag{101}$$

$$\begin{aligned}
&\int_0^T J_{4,\varepsilon}(t)\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t \\
&\quad = 4\int_0^T\int_\Omega \frac{u_\varepsilon^2(u^*)^2p^2(u^*-u_\varepsilon)^2}{(1+u_\varepsilon^2)^2(1+(u^*)^2)^2}\,\mathrm{d}x\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t \\
&\quad \leqslant 4\int_0^T\int_\Omega \frac{p^2(u^*-u_\varepsilon)^2}{(1+u_\varepsilon^2)(1+(u^*)^2)}\,\mathrm{d}x\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t \\
&\quad \leqslant 4\int_0^T\int_\Omega \frac{p^2|u^*-u_\varepsilon|(|u^*|+|u_\varepsilon|)}{(1+u_\varepsilon^2)(1+(u^*)^2)}\,\mathrm{d}x\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t \\
&\quad = 4\int_0^T\int_\Omega \left(\frac{|u_\varepsilon-u^*||u_\varepsilon|p_\varepsilon^2}{1+u_\varepsilon^2} + \frac{|u_\varepsilon-u^*||u^*|p_\varepsilon^2}{1+(u^*)^2}\right)\mathrm{d}x\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t \\
&\quad \leqslant 2\int_0^T\int_\Omega \bigl(|u_\varepsilon-u^*|p_\varepsilon^2 + |u_\varepsilon-u^*|p_\varepsilon^2\bigr)\,\mathrm{d}x\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t \\
&\quad = 4\int_0^T\int_\Omega |u_\varepsilon-u^*|p_\varepsilon^2\,\mathrm{d}x\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t \leqslant 4\int_0^T |u_\varepsilon-u^*|_2|p_\varepsilon|^2_{L^4(\Omega)}\,\mathrm{d}t \int_0^T |\varphi|_2^2\,\mathrm{d}t \\
&\quad \leqslant 4|\Omega|^{1/6}\max_{0\leqslant t\leqslant T}\bigl|u_\varepsilon(t)-u^*(t)\bigr|_2 \int_0^T |p_\varepsilon|^2_{L^6(\Omega)}\,\mathrm{d}t, \\
&\quad \leqslant 4|\Omega|^{1/6}\max_{0\leqslant t\leqslant T}\bigl|u_\varepsilon(t)-u^*(t)\bigr|_2 \int_0^T |p_\varepsilon|^2_V\,\mathrm{d}t \to 0 \quad \text{as } \varepsilon\to 0.
\end{aligned} \tag{102}$$

Here we have use the fact $(u^*)^2 \leqslant 1+(u^*)^2$ and $u_\varepsilon^2 \leqslant 1+u_\varepsilon^2$. With similar arguments we can get (91).

With the help of (81)–(84) and (87)–(91), we can pass to the limit in (68) to derive that $(p,q)\in(W^{1,2}(0,T;V^*)\cap L^2(0,T;V)\cap C([0,T];H))^2$ and satisfies (5). On the other hand, observe that $\zeta_\varepsilon\in\partial d_S(F(u_\varepsilon))$, we derive

$$\bigl\langle \zeta_\varepsilon, w-F(u_\varepsilon)\bigr\rangle_{Z^*,Z} \leqslant 0 \quad \forall\, w\in S. \tag{103}$$

Since $u_\varepsilon\to u^*$ strongly in $L^2(0,T;H)$, by (H_1), we yield that $F(u_\varepsilon)\to F(u^*)$ strongly in Z. Letting $\varepsilon\to 0$ in (103) we have $(6)_2$.

Finally, we are in a position to prove that $(\mu_0,\zeta_0)\neq 0$. To this end, we suppose that $\mu_0=0$. It follows from (74) and (75) that

$$0<\delta\leqslant|\zeta_\varepsilon|_{Z^*} \quad \text{for some } \delta>0. \tag{104}$$

On the other hand, by (103), we have

$$\langle \zeta_\varepsilon, w - F(u^*) \rangle_{Z^*,Z} \leqslant \langle \zeta_\varepsilon, F(u_\varepsilon) - F(u^*) \rangle_{Z^*,Z} \to 0 \quad \text{uniformly in } w \in S. \tag{105}$$

Since $S \subset Z$ is a closed convex subset with finite co-dimensionality, so does $S–F(u^*)$, which, together with (104) and (105), implies that $(\mu_0, \zeta_0) \neq 0$ ([17]).

Assume $[F'(u^*)]^*$ is injective and $(\mu_0, p, q) = 0$, and thanks to (5), we derive $(F'(u^*))^* \zeta_0 = 0$, which yields $\zeta_0 = 0$ and $(\mu_0, \zeta_0) = 0$. This is a contradiction with $(\mu_0, \zeta_0) \neq 0$. Thus, if $[F'(u^*)]^*$ is injective, then $(\mu_0, p, q) \neq 0$. We complete the proof. □

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