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On fixed point results for α-implicit contractions in quasi-metric spaces and consequences*

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Abstract. In this paper, we prove some fixed point results involving α-implicit contractions in quasi-metric spaces. Moreover, we provide some known results on G-metric spaces. An example and an application on a solution of a nonlinear integral equation are also presented.

Keywords: fixed point, implicit contraction, quasi-metric space, G-metric space.

1 Introduction and preliminaries

It is well known that passing from metric spaces to quasi-metric spaces, (i.e. dropping the requirement that the metric function $d : X \times X \to \mathbb{R}$ verifies $d(x, y) = d(y, x)$) carries with it immediate consequences to the general theory. For instance, the topological notions of quasi-metric spaces, such as, limit, continuity, completeness, Cauchyness all should be re-considered under the left and right approaches since the quasi-metric is not symmetric. Furthermore, uniqueness of limit of a sequence should be examined carefully since one can easily consider a sequence which has a left limit and right limit which are not equal to the each other. That's why a few results on fixed points in such spaces are considered.

The definition of a quasi-metric is given as follows:

Definition 1. Let X be a non-empty and let $d : X \times X \to [0, \infty)$ be a function which satisfies:

(d1) $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) \leqslant d(x, z) + d(z, y)$.

Then d is called a quasi-metric and the pair (X, d) is called a quasi-metric space.

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Remark 1. Any metric space is a quasi-metric space, but the converse is not true in general.

Now, we give convergence, completeness and continuity on quasi-metric spaces.

Definition 2. Let (X, d) be a quasi-metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0. \tag{1}$$

Example 1. (See [1].) Let X be a subset of $\mathbb{R}$ containing $[0, 1]$ and define, for all $x, y \in X$,

$$q(x, y) = \begin{cases} x - y & \text{if } x \geqslant y, \\ 1 & \text{otherwise.} \end{cases}$$

Then (X, q) is a quasi-metric space. Notice that $\{q(1/n, 0)\} \to 0$ but $\{q(0, 1/n)\} \to 1$. Therefore, $\{1/n\}$ right-converges to 0 but it does not converge from the left. We also point out that this quasi-metric verifies the following property: if a sequence $\{x_n\}$ has a right-limit x, then it is unique.

Remark 2. A quasi-metric space is Hausdorff, that is, we have the uniqueness of limit of a convergent sequence.

Definition 3. Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is left-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n \geqslant m > N$.

Definition 4. Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is right-Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geqslant n > N$.

Definition 5. Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > N$.

Remark 3. A sequence $\{x_n\}$ in a quasi-metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.

Definition 6. Let (X, d) be a quasi-metric space. We say that:

1. (X, d) is left-complete if and only if each left-Cauchy sequence in X is convergent.
2. (X, d) is right-complete if and only if each right-Cauchy sequence in X is convergent.
3. (X, d) is complete if and only if each Cauchy sequence in X is convergent.

Definition 7. Let (X, d) be a quasi-metric space. The map $f : X \to X$ is continuous if for each sequence $\{x_n\}$ in X converging to $x \in X$, the sequence $\{fx_n\}$ converges to fx, that is,

$$\lim_{n\to\infty} d(fx_n, fx) = \lim_{n\to\infty} d(fx, fx_n) = 0. \tag{2}$$

On the other hand, the study of fixed point for mappings satisfying an implicit relation is initiated and studied by Popa [19] and [20]. It leads to interesting known fixed points results. Following Popa's approach, many authors proved some fixed point, common fixed point and coincidence point results in various ambient spaces, see [3, 6, 9, 21, 23].

In the literature, there are several types of implicit contraction mappings where many nice consequences of fixed point theorems could be derived. First, denote Ψ the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying:

(ψ1) ψ is nondecreasing,
(ψ2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t \in \mathbb{R}^+$, where ψ^n is the nth iterate of ψ.

Remark 4. It is easy to see that if $\psi \in \Psi$, then $\psi(t) < t$ for any $t > 0$.

We introduce the following definition.

Definition 8. Let Γ be the set of all continuous functions $F(t_1, \dots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ such that:

(F1) F is nondecreasing in variable t_1 and nonincreasing in variable t_5;
(F2) There exists $h_1 \in \Psi$ such that for all $u, v \geqslant 0$, $F(u, v, v, u, u+v, 0) \leqslant 0$ implies $u \leqslant h_1(v)$;
(F3) There exists $h_2 \in \Psi$ such that for all $t, s > 0$, $F(t, t, 0, 0, t, s) \leqslant 0$ implies $t \leqslant h_2(s)$.

Note that in Definition 8 and with respect to Popa and Patriciu [22], we did not take the same hypotheses on h_1 and h_2 and we also add the fact that F is nondecreasing in variable t_1.

As in [22], we give the following examples.

Example 2. $F(t_1, \dots, t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a, b, c, d, e \geqslant 0$, $a + b + c + 2d + e < 1$.

Example 3. $F(t_1, \dots, t_6) = t_1 - k \max\{t_2, \dots, t_6\}$, where $k \in [0, 1/2)$.

Some other examples could be derived from [22].

Very recently, Samet et al. [25] introduced the concept of α-admissible maps and suggested a very interesting class of mapping, α-ψ contraction mappings, to investigate the existence and uniqueness of a fixed point.

Definition 9. (See [25].) For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be mappings. We say that the self-mapping T on X is α-admissible if for all $x, y \in X$, we have

$$\alpha(x, y) \geqslant 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geqslant 1. \tag{3}$$

Many papers dealing with above notion have been considered to prove some (common) fixed point results (for example, see [2, 10, 11, 13, 15, 16]).

Now, we introduce the concept of α-implicit contractive mappings in the setting of quasi-metric spaces.

Definition 10. Let (X,d) be a quasi-metric space and $f : X \to X$ be a given mapping. We say that f is an α-implicit contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $F \in \Gamma$ such that

$$F\big(\alpha(x,y)d(fx,fy), d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\big) \leqslant 0 \tag{4}$$

for all $x, y \in X$.

In this paper, we provide some fixed point results involving α-implicit contractions on quasi-metric spaces. As consequences of our obtained results, we also prove some existing fixed point results on G-metric spaces. We also provide an illustrated example and an application on a solution of a nonlinear integral equation.

2 Fixed point theorems

In this section, we shall state and prove our main results.

Theorem 1. *Let (X,d) be a complete quasi-metric space and $f : X \to X$ be an α-implicit contractive mapping. Suppose that*:

(i) *f is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geqslant 1$ and $\alpha(fx_0, x_0) \geqslant 1$*;
(iii) *f is continuous.*

Then there exists a $u \in X$ such that $fu = u$.

Proof. By assumption (ii), there exists a point $x_0 \in X$ such that $\alpha(x_0, fx_0) \geqslant 1$ and $\alpha(fx_0, x_0) \geqslant 1$. We define a sequence $\{x_n\}$ in X by $x_{n+1} = fx_n = f^{n+1}x_0$ for all $n \geqslant 0$. Suppose that $x_{n_0} = x_{n_0+1}$ for some n_0. So the proof is completed since $u = x_{n_0} = x_{n_0+1} = fx_{n_0} = fu$. Consequently, throughout the proof, we assume that

$$x_n \neq x_{n+1} \quad \text{for all } n. \tag{5}$$

Since f is α-admissible and $\alpha(x_0, x_1) = \alpha(x_0, fx_0) \geqslant 1$, so observe that

$$\alpha(fx_0, fx_1) = \alpha(x_1, x_2) \geqslant 1.$$

By repeating the process above, we derive that

$$\alpha(x_n, x_{n+1}) \geqslant 1 \quad \text{for all } n = 0, 1, \dots. \tag{6}$$

Now consider the case where $\alpha(fx_0, x_0) \geqslant 1$. By using the same technique above, we get that

$$\alpha(x_{n+1}, x_n) \geqslant 1 \quad \text{for all } n = 0, 1, \dots. \tag{7}$$

From (4), we have

$$\begin{aligned} &F\big(\alpha(x_{n-1}, x_n)d(fx_{n-1}, fx_n), d(x_{n-1}, x_n), d(x_{n-1}, fx_{n-1}),\\ &\quad d(x_n, fx_n), d(x_{n-1}, fx_n), d(x_n, fx_{n-1})\big) \leqslant 0, \end{aligned}$$

that is,

$$F\big(\alpha(x_{n-1},x_n)d(x_n,x_{n+1}),d(x_{n-1},x_n),d(x_{n-1},x_n),\\ d(x_n,x_{n+1}),d(x_{n-1},x_{n+1}),0\big)\leqslant 0.$$

By (6) and $(d2)$ in the fifth variable, we have using (F1)

$$F\big(d(x_n,x_{n+1}),d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}),\\ d(x_{n-1},x_n)+d(x_n,x_{n+1}),0\big)\leqslant 0. \tag{8}$$

Due to (F2), we obtain

$$d(x_n,x_{n+1})\leqslant h_1\big(d(x_{n-1},x_n)\big). \tag{9}$$

If we go on like this, we get

$$d(x_n,x_{n+1})\leqslant h_1^n\big(d(x_0,x_1)\big). \tag{10}$$

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence in the quasi-metric space (X,d). Take $m>n$. By using (d2),

$$\begin{aligned} d(x_n,x_m) &\leqslant d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{m-1},x_m)\\ &\leqslant \big(h_1^n+h_1^{n+1}+\cdots+h_1^{m-1}\big)\big(d(x_0,x_1)\big)\\ &\leqslant \sum_{k=n}^{\infty} h_1^k\big(d(x_0,x_1)\big) \end{aligned} \tag{11}$$

which implies that $d(x_n,x_m)\to 0$ as $n,m\to\infty$ since $h_1\in\Psi$. It follows that $\{x_n\}$ is a right-Cauchy sequence.

Similarly, by (4) we have

$$F\big(\alpha(x_n,x_{n-1})d(fx_n,fx_{n-1}),d(x_n,x_{n-1}),d(fx_{n-1},x_{n-1}),d(fx_n,x_n),\\ d(fx_n,x_{n-1}),d(fx_{n-1},x_n)\big)\leqslant 0,$$

that is, using (7) and $(F1)$, we have

$$F\big(d(x_{n+1},x_n),d(x_n,x_{n-1}),d(x_n,x_{n-1}),d(x_{n+1},x_n),d(x_{n+1},x_{n-1}),0\big)\leqslant 0.$$

Using again (F1) and (d2),

$$F\big(d(x_{n+1},x_n),d(x_n,x_{n-1}),d(x_n,x_{n-1}),d(x_{n+1},x_n),\\ d(x_{n+1},x_n)+d(x_n,x_{n-1}),0\big)\leqslant 0. \tag{12}$$

By (F2), we obtain

$$d(x_{n+1},x_n)\leqslant h_1\big(d(x_n,x_{n-1})\big). \tag{13}$$

If we go on like this, we get

$$d(x_{n+1}, x_n) \leqslant h_1^n\big(d(x_1, x_0)\big). \tag{14}$$

Thus, by using $(d2)$, for $n > m$,

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leqslant \big(h_1^{n-1} + h_1^{n-2} + \cdots + h_1^m\big)\big(d(x_1, x_0)\big) \\ &\leqslant \sum_{k=m}^{\infty} h_1^k\big(d(x_1, x_0)\big) \end{aligned} \tag{15}$$

which implies that $d(x_n, x_m) \to 0$ as $n, m \to \infty$ since $h_1 \in \Psi$. It follows that $\{x_n\}$ is a left-Cauchy sequence.

Thus, $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is quasi-complete, so there exists a point u in X such that $x_n \to u$ as $n \to \infty$, that is, from Definition 2,

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0. \tag{16}$$

We shall prove that $fu = u$.

Since f is continuous, we obtain

$$\lim_{n\to\infty} d(x_{n+1}, fu) = \lim_{n\to\infty} d(fx_n, fu) = 0 \tag{17}$$

and

$$\lim_{n\to\infty} d(fu, x_{n+1}) = \lim_{n\to\infty} d(fu, fx_n) = 0, \tag{18}$$

that is, $\lim_{n\to\infty} x_{n+1} = fu$. Taking Remark 2 into account, that is due the uniqueness of limit, we conclude that $fu = u$, that is, u is a fixed point of f. □

Note that in Theorem 1, the continuity hypothesis of F is not required. But this hypothesis is essential for Theorem 2. In the next result, we drop the continuity hypothesis of f and we replace it by the following:

(H) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geqslant 1$ for all k.

Theorem 2. *Let (X, d) be a complete quasi-metric space and $f : X \to X$ be an α-implicit contractive mapping. Suppose that*:

(i) *f is $\alpha-$admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geqslant 1$ and $\alpha(fx_0, x_0) \geqslant 1$*;
(iii) (H) *is verified.*

Then there exists a $u \in X$ such that $fu = u$.

Proof. Following the proof of Theorem 1, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n$ for all $n \geqslant 0$ is Cauchy and converges to some $u \in X$. From condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geqslant 1$ for all k. We shall show that $fu = u$.

By (4), we have successively

$$F\bigl(\alpha(x_{n(k)-1}, u)d(fx_{n(k)-1}, fu), d(x_{n(k)-1}, u), d(x_{n(k)-1}, fx_{n(k)-1}), d(u, fu), d(x_{n(k)-1}, fu), d(u, fx_{n(k)-1})\bigr) \leqslant 0.$$

Using (F1) and $\alpha(x_{n(k)-1}, u) \geqslant 1$, we get

$$F\bigl(d(x_{n(k)}, fu), d(x_{n(k)-1}, u), d(x_{n(k)-1}, x_{n(k)}), d(u, fu), d(x_{n(k)-1}, fu), d(u, x_{n(k)})\bigr) \leqslant 0.$$

Letting k tend to infinity and using continuity of F, we have

$$F\bigl(d(u, fu), 0, 0, d(u, fu), d(u, fu), 0\bigr) \leqslant 0.$$

By (F2), it follows that $d(u, fu) \leqslant 0$ which implies $u = fu$. □

For the uniqueness, we need an additional condition:

(U) For all $x, y \in \mathrm{Fix}(f)$, we have $\alpha(x, y) \geqslant 1$, where $\mathrm{Fix}(f)$ denotes the set of fixed points of f.

Theorem 3. *Adding condition* (U) *to the hypotheses of Theorem* 1 (*resp. Theorem* 2)*, we obtain that u is the unique fixed point of f.*

Proof. We argue by contradiction, that is, there exist $u, v \in X$ such that $u = fu$ and $v = fv$ with $u \neq v$. By (4), we get

$$F\bigl(\alpha(u, v)d(fu, fv), d(u, v), d(u, fu), d(v, fv), d(u, fv), d(v, u)\bigr) \leqslant 0,$$

i.e.,

$$F\bigl(\alpha(u, v)d(u, v), d(u, v), 0, 0, d(u, v), d(v, u)\bigr) \leqslant 0.$$

Due to the fact that $\alpha(u, v) \geqslant 1$, so by (F1), we get

$$F\bigl(d(u, v), d(u, v), 0, 0, d(u, v), d(v, u)\bigr) \leqslant 0.$$

Since F satisfies property (F3), so

$$d(u, v) \leqslant h_2\bigl(d(v, u)\bigr). \tag{19}$$

Analogously, we obtain

$$d(v, u) \leqslant h_2\bigl(d(u, v)\bigr). \tag{20}$$

Combining (19) to (20), we get

$$d(u, v) \leqslant h_2\bigl(d(v, u)\bigr) \leqslant h_2^2\bigl(d(u, v) < d(u, v)\bigr). \tag{21}$$

It is a contradiction. Hence $u = v$. □

In the sequel, we present the following corollaries as consequences of Theorem 1 (resp. Theorem 2).

Corollary 1. *Let (X,d) be a complete quasi-metric space and $f : X \to X$ be such that*

$$\alpha(x,y)d(fx,fy) \leqslant ad(x,y) + bd(x,fx) + cd(y,fy) + dd(x,fy) + ed(y,fx) \quad (22)$$

for all $x,y \in X$, where $a,b,c,d,e \geqslant 0$ and $a+b+d+2d+e<1$. Suppose that:

(i) *f is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geqslant 1$ and $\alpha(fx_0,x_0) \geqslant 1$*;
(iii) *f is continuous or* (H) *is verified.*

Then there exists a $u \in X$ such that $fu = u$.

Proof. It suffices to take F in Theorem 1 (resp. Theorem 2) as given in Example 3, that is, $F(t_1,\dots,t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a,b,c,d,e \geqslant 0$ and $a+b+c+2d+e<1$. □

Corollary 2. *Let (X,d) be a complete quasi-metric space and $f : X \to X$ be such that*

$$\alpha(x,y)d(fx,fy) \leqslant k\max\bigl\{d(x,y),d(x,fx),d(y,fy),d(x,fy),d(y,fx)\bigr\} \quad (23)$$

for all $x,y \in X$, where $k \in [0,1/2)$. Suppose that:

(i) *f is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geqslant 1$ and $\alpha(fx_0,x_0) \geqslant 1$*;
(iii) *f is continuous or (H) is verified.*

Then there exists a $u \in X$ such that $fu = u$.

Proof. It suffices to take F in Theorem 1 (resp. Theorem 2) as given in Example 3, that is, $F(t_1,\dots,t_6) = t_1 - k\max\{t_2,\dots,t_6\}$, where $k \in [0,1/2)$. □

We present the following example illustrating Corollary 2.

Example 4. Let $X = [0,\infty)$ endowed with the quasi-metric

$$d(x,y) = |x| \quad \text{if } x \neq y \quad \text{and} \quad d(x,y) = 0 \quad \text{if } x = y.$$

It is clear that (X,d) is a complete quasi-metric space. Consider the mapping $T : X \to X$ defined by

$$Tx = \begin{cases} x^2 - 3x + 2 & \text{if } x > 2 \\ x/3 & \text{if } x \in [0,2]. \end{cases}$$

At first, we observe that the Banach contraction principle for $d_0(x,y) = |x-y|$ cannot be applied in this case since we have

$$d_0(T0,T4) = 6 > 4 = d_0(0,4).$$

Now, we define the mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

If $x,y \in [0,1]$ and $x \neq y$, we have

$$\begin{aligned} \alpha(x,y)d(Tx,Ty) &= d(Tx,Ty) \leqslant |Tx| = \frac{x}{3} = \frac{1}{3}d(x,y) \\ &\leqslant k\max\bigl\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\bigr\}, \end{aligned} \tag{24}$$

where $k = 1/3$. Similarly, it is obvious that (24) holds in the cases ($x,y \in [0,1]$ with $x = y$) and (x or y is not in $[0,1]$). Now, we shall prove that the hypothesis (H) is satisfied. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. Then by definition of α, we get

$$(x_n, x_{n+1}) \in [0,1] \times [0,1] \quad \text{for all } n.$$

Assume that $x > 1$. Then $x_n \neq x$ for all n. Since $x_n \to x \in X$, so $d(x,x_n) = |x| \to 0$, which is a contradiction. Thus, $x \in [0,1]$. We get that

$$(x_n, x) \in [0,1] \times [0,1] \quad \text{for all } n,$$

that is, $\alpha(x_n, x) = 1$, i.e., (H) is verified. Take $x_0 = 1$. We have

$$\alpha(x_0, Tx_0) = \alpha\left(1, \frac{1}{3}\right) = 1 \quad \text{and} \quad \alpha(Tx_0, x_0) = \alpha\left(\frac{1}{3}, 1\right) = 1.$$

The mapping T is α-admissible. In fact, let $x,y \in X$ such that $\alpha(x,y) \geqslant 1$, so $x,y \in [0,1]$. Then

$$\alpha(Tx,Ty) = \alpha\left(\frac{x}{3}, \frac{y}{3}\right) = 1.$$

All hypotheses of Corollary 2 hold and the mapping T has a fixed point in X. Note that in this case, we have two fixed points of T which are $u = 0$ and $v = 2 + \sqrt{2}$.

3 Consequences

In this section, we give some consequences of our main results.

3.1 Standard fixed point theorems

We start with the following corollary.

Corollary 3. *Let (X,d) be a complete quasi-metric space and $f : (X,d) \to (X,d)$ be agiven mapping. Suppose that*

$$F\bigl(d(fx,fy), d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\bigr) \leqslant 0 \tag{25}$$

for all $x,y \in X$, where $F \in \varGamma$. Then f has a unique fixed point.

Proof. It suffices to take $\alpha(x, y) = 1$ for all $x, y \in X$ in Theorem 2. Notice that the hypothesis (U) is satisfied, so we apply Theorem 3. □

The following corollary is a Ćirić contraction type [8].

Corollary 4. *Let (X, d) be a complete quasi-metric space and $f : (X, d) \to (X, d)$ be a given mapping such that*

$$d(fx, fy) \leqslant k \max\bigl\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\bigr\} \tag{26}$$

for all $x, y \in X$, where $k \in [0, 1/2)$. Then f has a unique fixed point.

Proof. It suffices to take F as given in Example 3, that is, $F(t_1, \dots, t_6) = t_1 - k \times \max\{t_2, \dots, t_6\}$, where $k \in [0, 1/2)$. Then, we apply Corollary 3. □

3.2 Fixed point theorems on metric spaces endowed with a partial order

Definition 11. Let $(X, \preccurlyeq)$ be a partially ordered set and $f : X \to X$ be a given mapping. We say that f is nondecreasing with respect to $\preccurlyeq$ if

$$x, y \in X, \quad x \preccurlyeq y \quad \Longrightarrow \quad fx \preccurlyeq fy.$$

Definition 12. Let $(X, \preccurlyeq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be nondecreasing with respect to $\preccurlyeq$ if $x_n \preccurlyeq x_{n+1}$ for all n.

Definition 13. Let $(X, \preccurlyeq)$ be a partially ordered set and d be a quasi-metric on X. We say that $(X, \preccurlyeq, d)$ is regular if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preccurlyeq x$ for all k.

We state the following result.

Corollary 5. *Let $(X, \preccurlyeq)$ be a partially ordered set and d be a quasi-metric on X such that (X, d) is complete. Let $f : X \to X$ be a nondecreasing mapping with respect to $\preccurlyeq$. Suppose that there exists a function $F \in \Gamma$ such that*

$$F\bigl(d(fx, fy), d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\bigr) \leqslant 0, \tag{27}$$

for all $x, y \in X$ with $x \succcurlyeq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq fx_0$ or $fx_0 \preccurlyeq x_0$*;
(ii) *f is continuous or $(X, \preccurlyeq, d)$ is regular.*

Then f has a fixed point. Moreover, if $\mathrm{Fix}(f)$ *is well-ordered, we have uniqueness of the fixed point.*

Proof. Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preccurlyeq y \text{ or } x \succcurlyeq y, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, f is an α-implicit contractive mapping, that is,

$$F\big(\alpha(x, y)d(fx, fy), d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\big) \leqslant 0$$

for all $x, y \in X$. From condition (i), we have $\alpha(x_0, fx_0) \geqslant 1$ and $\alpha(fx_0, x_0) \geqslant 1$. Moreover, for all $x, y \in X$, from the monotone property of f, we have

$$\begin{aligned} \alpha(x, y) \geqslant 1 \quad &\Longrightarrow \quad x \succcurlyeq y \quad \text{or} \quad x \preccurlyeq y \\ &\Longrightarrow \quad fx \succcurlyeq fy \quad \text{or} \quad fx \preccurlyeq fy \\ &\Longrightarrow \quad \alpha(fx, fy) \geqslant 1. \end{aligned}$$

Thus, f is α-admissible. Now, if f is continuous, the existence of a fixed point follows from Theorem 1. Suppose now that $(X, \preccurlyeq, d)$ is regular. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. From the regularity hypothesis, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preccurlyeq x$ for all k. This implies from the definition of α that $\alpha(x_{n(k)}, x) \geqslant 1$ for all k. In this case, the existence of a fixed point follows from Theorem 2. To show the uniqueness, let $x, y \in X$. By hypothesis, there exists $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$, which implies from the definition of α that $\alpha(x, z) \geqslant 1$ and $\alpha(y, z) \geqslant 1$. Thus, we deduce the uniqueness of the fixed point by Theorem 3. □

3.3 Fixed point theorems in the context of *G*-metric spaces

Before all, we need the following definitions and concepts.

Definition 14. (See [17].) Let X be a non-empty set, $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$;
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leqslant G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);
(G5) $G(x, y, z) \leqslant G(x, a, a) + G(a, y, z)$ (rectangle inequality) for all $x, y, z, a \in X$.

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 15. (See [17]). A G-metric space (X, G) is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

In their initial paper, Mustafa and Sims [17] also defined the basic topological concepts in G-metric spaces as follows:

Definition 16. (See [17].) Let (X,G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to $x \in X$ if

$$\lim_{n,m\to+\infty} G(x,x_n,x_m) = 0,$$

that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \varepsilon$ for all $n,m \geqslant N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to+\infty} x_n = x$.

Proposition 1. (*See* [17].) *Let (X,G) be a G-metric space. The following are equivalent*:

(i) *$\{x_n\}$ is G-convergent to x*;
(ii) *$G(x_n,x_n,x) \to 0$ as $n \to +\infty$*;
(iii) *$G(x_n,x,x) \to 0$ as $n \to +\infty$.*

Definition 17. (See [17].) Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $m,n,l \geqslant N$, that is, $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to +\infty$.

Proposition 2. (*See* [17].) *Let (X,G) be a G-metric space. Then the followings are equivalent*:

(i) *the sequence $\{x_n\}$ is G-Cauchy*;
(ii) *for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$ for all $m,n \geqslant N$.*

Definition 18. (See [17].) A G-metric space (X,G) is called G-complete if every G-Cauchy sequence is G-convergent in (X,G).

Notice that any G-metric space (X,G) induces a metric d_G on X defined by

$$d_G(x,y) = G(x,y,y) + G(y,x,x) \quad \text{for all } x,y \in X. \tag{28}$$

Furthermore, (X,G) is G-complete if and only if (X,d_G) is complete.

Recently, Jleli and Samet [12] gave the following theorems.

Theorem 4. (*See* [12].) *Let (X,G) be a G-metric space. Let $d : X \times X \to [0,\infty)$ be the function defined by $d(x,y) = G(x,y,y)$. Then*:

(i) *(X,d) is a quasi-metric space*;
(ii) *$\{x_n\} \subset X$ is G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X,d)*;
(iii) *$\{x_n\} \subset X$ is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X,d)*;
(iv) *(X,G) is G-complete if and only if (X,d) is complete.*

Every quasi-metric induces a metric, that is, if (X,d) is a quasi-metric space, then the function $\delta : X \times X \to [0,\infty)$ defined by

$$\delta(x,y) = \max\bigl\{d(x,y), d(y,x)\bigr\} \tag{29}$$

is a metric on X [12].

Theorem 5. (*See* [12].) *Let* (X,G) *be a G-metric space. Let* $\delta : X \times X \to [0,\infty)$ *be the function defined by* $\delta(x,y) = \max\{G(x,y,y), G(y,x,x)\}$. *Then*:

(i) (X,δ) *is a metric space*;
(ii) $\{x_n\} \subset X$ *is G-convergent to* $x \in X$ *if and only if* $\{x_n\}$ *is convergent to* x *in* (X,δ);
(iii) $\{x_n\} \subset X$ *is G-Cauchy if and only if* $\{x_n\}$ *is Cauchy in* (X,δ);
(iv) (X,G) *is G-complete if and only if* (X,δ) *is complete.*

We need the following definition of Alghamdi and Karapınar [4,5] which is the analog of Definition 9.

Definition 19. (See [4].) For a nonempty set X, let $T : X \to X$ and $\beta : X^3 \to [0,\infty)$ be mappings. We say that the self-mapping T on X is β-admissible if for all $x,y \in X$, we have

$$\beta(x,y,y) \geqslant 1 \quad \Longrightarrow \quad \beta(Tx,Ty,Ty) \geqslant 1. \tag{30}$$

It is also known the following.

Lemma 1. (*See* [4,5].) *Let* $f : X \to X$*, where* X *is non-empty set. It is clear that the self-mapping* f *is* β*-admissible if and only if* f *is* α*-admissible.*

Now, we can give the following results on G-metric spaces.

Theorem 6. *Let* (X,G) *be a complete G-metric space and* $f : X \to X$ *be such that*

$$\begin{aligned} &F\bigl(\beta(x,y,y)G(fx,fy,fy), G(x,y,y), G(x,fx,fx), G(y,fy,fy),\\ &\quad G(x,fy,fy), G(y,fx,fx)\bigr) \leqslant 0 \end{aligned} \tag{31}$$

for all $x,y \in X$*, where* $\beta : X^3 \to |0,\infty)$ *and* $F \in \Gamma$. *Suppose that*:

(i) f *is* β*-admissible*;
(ii) *there exists* $x_0 \in X$ *such that* $\beta(x_0,fx_0,fx_0) \geqslant 1$ *and* $\beta(fx_0,x_0,x_0) \geqslant 1$;
(iii) f *is continuous.*

Then there exists a $u \in X$ *such that* $fu = u$.

Proof. It suffices to take the quasi-metric $d(x,y) = G(x,y,y)$ and $\alpha(x,y) = \beta(x,y,y)$. Due to (31), we get (4). Then due to Lemma 1, the result follows from Theorem 1. □

Alghamdi and Karapınar [4,5] also defined the following hypothesis.

(W) If $\{x_n\}$ is a sequence in X such that $\beta(x_n,x_{n+1},x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\beta(x_{n(k)},x,x) \geqslant 1$ for all k.

Theorem 7. *Let* (X,G) *be a complete G-metric space and* $f : X \to X$ *be such that*

$$\begin{aligned} &F\bigl(\beta(x,y,y)G(fx,fy,fy), G(x,y,y), G(x,fx,fx), G(y,fy,fy),\\ &\quad G(x,fy,fy), G(y,fx,fx)\bigr) \leqslant 0 \end{aligned} \tag{32}$$

for all $x, y \in X$, *where* $\beta : X^3 \to [0, \infty)$ *and* $F \in \Gamma$. *Suppose that*:

(i) f *is* β*-admissible*;
(ii) *there exists* $x_0 \in X$ *such that* $\beta(x_0, fx_0, fx_0) \geqslant 1$ *and* $\beta(fx_0, x_0, x_0) \geqslant 1$;
(iii) (W) *is verified.*

Then there exists a $u \in X$ *such that* $fu = u$.

Proof. As in the proof of Theorem 6, we derive the result from Theorem 2. □

Corollary 6. *Let* (X, G) *be a complete G-metric space and* $f : X \to X$ *be such that*

$$\beta(x, y, y)G(fx, fy, fy) \leqslant k \max\bigl\{G(x, y, y), G(x, fx, fx), G(y, fy, fy), G(x, fy, fy), G(y, fx, fx)\bigr\} \tag{33}$$

for all $x, y \in X$, *where* $k \in [0, 1/2)$. *Suppose that*:

(i) f *is* β*-admissible*;
(ii) *there exists* $x_0 \in X$ *such that* $\beta(x_0, fx_0, fx_0) \geqslant 1$ *and* $\beta(fx_0, x_0, x_0) \geqslant 1$;
(iii) f *is continuous or* (W) *is verified.*

Then, there exists a $u \in X$ *such that* $fu = u$.

Proof. It is similarly as Corollary 2. It follows from Theorem 6 and Theorem 7. □

Corollary 7. *Let* (X, G) *be a complete G-metric space and* $f : X \to X$ *be a mapping. Suppose that there exists a function* $F \in \Gamma$ *such that*

$$F\bigl(G(fx, fy, fy), G(x, y, y), G(x, fx, fx), G(y, fy, fy), G(x, fy, fy), G(y, fx, fx)\bigr) \leqslant 0 \tag{34}$$

for all $x, y \in X$. *Then* f *has a unique fixed point.*

Proof. Consider the case where $\beta(x, y, y) = 1$ for all $x, y \in X$ in Theorem 7. The uniqueness follows from Theorem 3. □

As Corollary 4, we obtain from Corollary 7 the following:

Corollary 8. *Let* (X, G) *be a complete G-metric space and* $f : X \to X$ *a given mapping. Suppose that*

$$G(fx, fy, fy) \leqslant k \max\bigl\{G(x, y, y), G(x, fx, fx), G(y, fy, fy), G(x, fy, fy), G(y, fx, fx)\bigr\} \tag{35}$$

for all $x, y \in X$, *where* $k \in [0, 1/2)$. *Then* f *has a unique fixed point.*

4 Application

In this section, we provide an application to solve the nonlinear integral equation

$$x(t) = \int_a^t K\big(t, s, x(s)\big)\,\mathrm{d}s, \tag{36}$$

where $t \in J = [a, b]$ and $K : J \times J \times \mathbb{R} \to \mathbb{R}$ is continuous. Let $X = \mathcal{C}(J, \mathbb{R})$ with the usual supremum norm, that is,

$$\|x\| = \max_{t \in J}\big|x(t)\big|.$$

Note that the existence for the unique solution of (36) is based on Corollary 4.

Theorem 8. *Suppose the following conditions hold*:

(i) *there exists a continuous function* $p : J \times J \to \mathbb{R}_+$ *such that*

$$\big|K(t, s, u)\big| \leqslant \frac{p(t, s)}{b - a}|u|$$

for each $t, s \in J$ *and* $u \in \mathbb{R}$;

(ii) *if* $u, v \in X$ *with* $u \neq v$*, we have* $\int_a^t K(t, s, u(s))\,\mathrm{d}s \neq \int_a^t K(t, s, v(s))\,\mathrm{d}s$ *for each* $t \in J$;

(iii) $\sup_{t \in J} p(t, s) = k < 1/2$.

Then the integral equation (36) *has a unique solution* $x \in \mathcal{C}(J, \mathbb{R})$.

Proof. Consider the quasi-metric $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \|x\| \quad \text{if } x \neq y \quad \text{and} \quad d(x, y) = 0 \quad \text{if } x = y.$$

It is clear that (X, d) is a complete quasi-metric space. Consider the mapping $T : X \to X$ defined by

$$Tx(t) = \int_a^t K\big(t, s, x(s)\big)\,\mathrm{d}s$$

for all $x \in X$. We have to prove that T has a unique fixed point.

For all $x \in X$, we have

$$\big|Tx(t)\big| \leqslant \int_a^t \big|K\big(t, s, x(s)\big)\big|\,\mathrm{d}s \leqslant \int_a^b \frac{p(t, s)}{b - a}\big|x(s)\big|\,\mathrm{d}s \leqslant \|x\| \int_a^b \frac{k}{b - a}\,\mathrm{d}s = k\|x\|,$$

so $\|Tx\| \leqslant k\|x\|$. For all $x, y \in X$ with $x \neq y$, we get under assumption that $Tx \neq Ty$. Thus,

$$\begin{aligned} d(Tx, Ty) &= \|Tx\| \leqslant k\|x\| = kd(x, y) \\ &\leqslant k \max\big\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big\}. \end{aligned} \tag{37}$$

On the other hand, obviously (37) holds in the case $x = y$. So all hypotheses of Corollary 4 are satisfied, and so T has a unique fixed point, that is, the problem (36) has a unique solution. □

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References

1. R.P. Agarwal, E. Karapınar, A.-F. Roldán-López-de-Hierro, Fixed point theorems in quasi-metric spaces and applications to multidimensional fixed point theorems on G-metric spaces, *J. Nonlinear Convex Anal.*, **16**(9):1787–1816, 2015.
2. M.U. Ali, T. Kamran, E. Karapınar, On (α, ψ, η)-contractive multivalued mappings, *Fixed Point Theory Appl.*, **2014**, Article ID 7, 8 pp., 2014.
3. A. Aliouche, V. Popa, General common fixed point theorems for occasionally weakly compatible hybrid mappings and applications, *Novi Sad. J. Math.*, **39**(1):89–109, 2009.
4. M.A. Alghamdi, E. Karapınar, G-β-ψ contractive-type mappings and related fixed point theorems, *J. Inequal. Appl.*, **2013**, Article ID 70, 16 pp., 2013.
5. M.A. Alghamdi, E. Karapınar, G-β-ψ-contractive type mappings in G-metric spaces, *Fixed Point Theory Appl.*, **2013**, Article ID 123, 17 pp., 2013.
6. V. Berinde, Approximating fixed points of implicit almost contractions, *Hacet. J. Math. Stat.*, **41**(1):93–102, 2012.
7. V. Berinde, F. Vetro, Common fixed points of mappings satisfying implicit contractive conditions, *Fixed Point Theory Appl.*, **2012**, Article ID 105, 8 pp., 2012.
8. L.B. Ćirić, A generalization of Banach's contraction principle, *Proc. Am. Math. Soc.*, **45**(2):267–273, 1974.
9. M. Imdad, S. Kumar, M.S. Khan, Remarks on some fixed point theorems satisfying implicit relations, *Rad. Math.*, **11**(1):135–143, 2002.
10. M. Jleli, E. Karapınar, B. Samet, Best proximity points for generalized alpha-psi-proximal contractive type mappings, *J. Appl. Math.*, **2013**, Article ID 534127, 10 pp., 2013.
11. M. Jleli, E. Karapınar, B. Samet, Fixed point results for α-ψ_λ-contractions on gauge spaces and applications, *Abstr. Appl. Anal.*, **2013**, Article ID 730825, 7 pp., 2013.
12. M. Jleli, B. Samet, Remarks on G metric spaces and fixed point theorems, *Fixed Point Theory Appl.*, **2012**, Article ID 210, 7 pp., 2012.
13. E. Karapinar, B. Samet, Generalized α-ψ-contractive type mappings and related fixed point theorems with applications, *Abstr. Appl. Anal.*, **2012**, Article ID 793486, 17 pp. 2012.
14. E. Karapinar, Fixed point theory for cyclic weak ϕ-contraction, *Appl. Math. Lett.*, **24**:822–825, 2011.
15. V. La Rosa, P. Vetro, Common fixed points for α-ψ-φ-contractions in generalized metric spaces, *Nonlinear Anal. Model. Control*, **19**(1):43–54, 2014.

16. B. Mohammadi, Sh. Rezapour, N. Shahzad, Some results on fixed points of α-ψ-Ciric generalized multifunctions, *Fixed Point Theory Appl.*, **2013**, Article ID 24, 10 pp., 2013.

17. Z. Mustafa, B. Sims, A new approach to generalized metric spaces, *J. Nonlinear Convex Anal.*, **7**:289–297, 2006.

18. M. Păcurar, I.A. Rus, Fixed point theory for cyclic φ-contractions, *Nonlinear Anal., Theory Methods Appl., Ser. A*, **72**(3–4):1181–1187, 2010.

19. V. Popa, Fixed point theorems for implicit contractive mappings, *Stud. Cercet. Ştiinţ., Ser. Mat. Univ. Bacău*, **7**:129–133, 1997.

20. V. Popa, Some fixed point theorems for compatible mappings satisfying an implicit relation, *Demonstr. Math.*, **32**:157–163, 1999.

21. V. Popa, A general fixed point theorem for four weakly compatible mappings satisfying an implicit relation, *Filomat*, **19**:45–51, 2005.

22. V. Popa, A.M. Patriciu, A general fixed point theorem for mappings satisfying an ϕ-implicit relation in complete G-metric spaces, *Gazi Univ. J. Science*, **25**(2):403–408, 2012.

23. V. Popa, A.M. Patriciu, A General fixed point theorem for pairs of weakly compatible mappings in G-metric spaces, *J. Nonlinear Sci. App.*, **5**(2):151–160, 2012.

24. S. Reich, A.J. Zaslawski, Well-posedness of fixed point problems, *Far East J. Math. Sci.*, **Spec. Vol.**, Part III:393–401, 2001.

25. B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Anal., Theory Methods Appl., Ser. A*, **75**(4):2154–2165, 2012.