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Common fixed point theorems via common limit range property in symmetric spaces under generalized Φ-contractions

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Abstract. The aim of this paper is to use common limit range property for a quadruple of non-self mappings for deriving common fixed point results under a generalized Φ-contraction condition in symmetric spaces. Some examples are given to exhibit different types of situations where these conditions can be used and to distinguish our results from the known ones. As an application, an existence result for certain systems of integral equations is presented.

Keywords: common fixed point, common limit range property, symmetric space, Φ-contractive condition.

1 Introduction and preliminaries

There exist a lot of generalizations of metric spaces, which showed themselves useful in obtaining more powerful fixed point and common fixed point results. Symmetric spaces are among the most important ones, since very often not the full power of metric requirements are needed in proving these results.

The notion of symmetric space goes back to Wilson [17].

Definition 1. A symmetric on a nonempty set X is a function $d : X \times X \to [0, +\infty)$ satisfying the following conditions:

1. $d(x, y) = 0$ if and only if $x = y$ for $x, y \in X$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.

The pair (X, d) is then called a symmetric space.

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Example 1. The set $l_p(\mathbb{R})$ with $0 < p < 1$, where $l_p(\mathbb{R}) = \{\{x_n\} \subset \mathbb{R}\colon \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ together with $d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to \mathbb{R}$,

$$d(x,y) := \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}, \quad \text{where } x = \{x_n\},\ y = \{y_n\} \in l_p(\mathbb{R}),$$

is a symmetric space.

Example 2. Let $X = [0,1] \cup \{2\}$ and let

$$d(x,y) = \begin{cases} |x-y|, & 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1, \\ |x|, & 0 < x \leqslant 1,\ y = 2, \end{cases}$$

and $d(0,2) = 1$. It is easy to see that (X,d) is a symmetric space.

Let d be a symmetric on X. For $x \in X$ and $\epsilon > 0$, let $\mathcal{B}(x,\epsilon) = \{y \in X\colon d(x,y) < \epsilon\}$. A topology $\tau(d)$ on X is defined as follows: $U \in \tau(d)$ if and only if for each $x \in U$, there exists an $\epsilon > 0$ such that $\mathcal{B}(x,\epsilon) \subset U$. A subset S of X is a neighbourhood of $x \in X$ if there exists $U \in \tau(d)$ such that $x \in U \subset S$. A symmetric d is a semimetric if for each $x \in X$ and each $\epsilon > 0$, $\mathcal{B}(x,\epsilon)$ is a neighbourhood of x in the topology $\tau(d)$. A symmetric (resp. semimetric) space (X,d) is a topological space whose topology $\tau(d)$ is induced by symmetric (resp. semimetric) d.

The obvious difference between a symmetric and a metric is engineered by the triangle inequality. Since a symmetric space need not be Hausdorff, therefore, in order to prove fixed point theorems, some additional assumptions are usually required. We will need the following axioms, which can be found, e.g., in [2, 4, 17]:

(W3) Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(x_n,y) = 0$ imply $x = y$ (see [17]);

(W4) Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(x_n,y_n) = 0$ imply $\lim_{n\to\infty} d(y_n,x) = 0$ (see [17]);

(HE) Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(y_n,x) = 0$ imply $\lim_{n\to\infty} d(x_n,y_n) = 0$ (see [2]);

(1C) Given $\{x_n\}$, x and $y \in X$, $\lim_{n\to\infty} d(x_n,x) = 0$ implies $\lim_{n\to\infty} d(x_n,y) = d(x,y)$ (such symmetric is usually called 1-continuous) (see [4]);

(CC) Given $\{x_n\}$, $\{y_n\}$ and $x, y \in X$, $\lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(y_n,y) = 0$ imply $\lim_{n\to\infty} d(x_n,y_n) = d(x,y)$ (such symmetric is usually called continuous) (see [4]).

In what follows, (X,d) stands for a symmetric space.

It is easy to see that (W4) $\Rightarrow$ (W3) and (1C) $\Rightarrow$ (W3). In general, all other implications between (W3), (1C) and (HE) are not true. However, (CC) implies all the remaining four conditions. For details, see an interesting note by Cho et al. [3], which contains some illustrative examples (in particular, it was shown in [3] that the space (X,d) from Example 2 satisfies axiom (HE), but does not satisfy any other of the previous axioms). Note that if (X, d) is a cone metric space over a normal cone and $D = \|d\|$ then (X, D) is a symmetric space, which satisfies axiom (CC), but is not in general a metric space (see [8]). Using these axioms, several authors proved various common fixed point theorems in symmetric or semi-metric spaces.

During the late 20th century, metrical common fixed point theory saw a trend of investigation, which moved around commuting nature of two maps. Several conditions were introduced, including weak commutativity [15], compatibility [9], weak compatibility [10] and many others, and a lot of respective common fixed point results were obtained. A survey of these notions and relationship among them can be seen in [11]. In particular, we recall that two mappings $A, S : X \to X$ are called weakly compatible if they commute at their coincidence points, that is, $ASx = SAx$ whenever $Ax = Sx$.

In the study of common fixed points of compatible-type mappings, we often require assumption of completeness of the space or continuity of mappings involved besides some contractive condition, but the study of fixed points of non-compatible mappings can be extended to the class of non-expansive or Lipschitz type mapping pairs even without assuming the continuity of the mappings involved or completeness of the space. Aamri and El Moutawakil [1] generalized the concept of non-compatibility by defining the notion of (E.A) property and proved common fixed point theorems under strict contractive condition. Although (E.A) property is a generalization of the concept of non-compatible maps, yet it requires either completeness of the whole space or some of the range spaces or continuity of maps. Most recently, Liu et al. [13] defined a common (E.A) property for two pairs of mappings.

As a further generalization, new notion of (CLR$_g$) property, recently given by Sintunavarat and Kumam [16], does not impose such conditions. The importance of (CLR$_g$) property is that it ensures that one does not require the closedness of range of subspaces. Recently, Imdad et al. [7] extended the notion of common limit range property to two pairs of self mappings, which further relaxes the requirement on closedness of the subspaces. Since then, a number of fixed point theorems have been established by several researchers in different settings under common limit range property. We refer the reader to [5, 6, 12] and the references therein.

Now we give definitions of the mentioned properties for non-self mappings.

Definition 2. Let Y be an arbitrary set, (X, d) be a symmetric space and let A, B, S, T be mappings from Y into X. Then:

1. The pair (A, S) is said to satisfy the property (E.A) [1] if there exists a sequence $\{x_n\}$ in Y such that
$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = u$$
for some $u \in X$;
2. The pairs (A, S) and (B, T) are said to share the common property (E.A) [13] if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that
$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = u$$
for some $u \in X$;

3. The pair (A, S) is said to have the common limit range property with respect to the mapping S (denoted by (CLR$_S$)) [16] if there exists a sequence $\{x_n\}$ in Y such that
$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = u,$$
where $u \in S(Y)$;
4. The pairs (A, S) and (B, T) are said to have the common limit range property with respect to mappings S and T (denoted by (CLR$_{ST}$)) [7] if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that
$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = u,$$
where $u \in S(Y) \cap T(Y)$.

Remark 1.
(i) If $A = B$ and $S = T$, then condition 4 reduces to condition 3.
(ii) (CLR$_{ST}$) property implies the common property (E.A), but not conversely.

In 2013, Pathak and Tiwari [14] proved a common fixed point theorem for weakly compatible mappings, assuming closedness of range subspaces for two pairs of mappings satisfying a new generalized Φ-contraction condition in metric spaces.

In this paper, an attempt is made to derive new common fixed point results under generalized Φ-contraction condition in symmetric spaces satisfying common limit range property for a quadruple of non-self mappings relaxing the requirement of closedness of the subspaces. Some examples are given to exhibit different types of situations where these conditions can be used and to distinguish our results from the known ones. We extend our main result to four finite families of mappings in symmetric spaces using the notion of pairwise commuting mappings. As an application, we present an existence result for certain systems of integral equations.

2 Main results

The attempted improvements in this paper are the following:

(i) The results are proved in symmetric spaces.
(ii) The condition on containment of ranges amongst the involved mappings is relaxed.
(iii) Continuity requirements of all the involved mappings are completely relaxed.
(iv) The (E.A) property is replaced by (CLR$_{ST}$) property, which is the most general among all existing weak commutativity concepts.
(v) The condition on completeness of the whole space is relaxed.

In what follows, following [14], we denote by Φ the collection of all functions $\varphi : [0, \infty) \to [0, \infty)$, which are upper semicontinuous from the right, non-decreasing and satisfy $\limsup_{s\to t^+} \varphi(s) < t$, $\varphi(t) < t$ for all $t > 0$.

Let (X,d) be a symmetric space and let Y be a non-empty set. We will consider the following possible condition, satisfied by four mappings $A,B,S,T:Y\to X$:

$$\begin{aligned}&\big[d^p(Ax,By)+ad^p(Sx,Ty)\big]d^p(Ax,By)\\&\quad\leqslant a\max\big\{d^p(Ax,Sx)d^p(By,Ty),\ d^q(Ax,Ty)d^{q'}(By,Sx)\big\}\\&\qquad+\max\Big\{\varphi_1\big(d^{2p}(Sx,Ty)\big),\ \varphi_2\big(d^r(Ax,Sx)d^{r'}(By,Ty)\big),\\&\qquad\qquad\varphi_3\big(d^s(Ax,Ty)d^{s'}(By,Sx)\big),\\&\qquad\qquad\varphi_4\bigg(\frac{1}{2}\big[d^l(Ax,Ty)d^{l'}(Ax,Sx)+d^l(By,Sx)d^{l'}(By,Ty)\big]\bigg)\Big\}\end{aligned}\tag{1}$$

for all $x,y\in X$ and some $\varphi_i\in\Phi$ $(i=1,2,3,4)$, $a,p,q,q',r,r',s,s',l,l'\geqslant 0$ such that $2p=q+q'=r+r'=s+s'=l+l'\leqslant 1$. Condition (1) is commonly called a generalized Φ-contraction.

Now we state and prove our main result.

Theorem 1. *Let (X,d) be a symmetric space where d satisfies conditions (1C) and (HE). Let Y be an arbitrary non-empty set, and let $A,B,S,T:Y\to X$. Suppose that condition (1) holds. If the pairs (A,S) and (B,T) share the (CLR$_{ST}$) property, then (A,S) and (B,T) have a coincidence point each.*

If, moreover, $Y=X$ and both pairs (A,S) and (B,T) are weakly compatible, then A,B,S and T have a unique common fixed point.

Proof. Since the pairs (A,S) and (B,T) share the (CLR$_{ST}$) property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that

$$\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=\lim_{n\to\infty}Ty_n=\lim_{n\to\infty}By_n=\zeta,$$

where $\zeta\in S(Y)\cap T(Y)$. As $\zeta\in S(Y)$, there exists a point $\upsilon\in Y$ such that $S\upsilon=\zeta$. In order to prove that also $A\upsilon=\zeta$, suppose the contrary. Putting $x=\upsilon$ and $y=y_n$ in condition (1), we get

$$\begin{aligned}&\big[d^p(A\upsilon,By_n)+ad^p(S\upsilon,Ty_n)\big]d^p(A\upsilon,By_n)\\&\quad\leqslant a\max\big\{d^p(A\upsilon,S\upsilon)d^p(By_n,Ty_n),\ d^q(A\upsilon,Ty_n)d^{q'}(By_n,S\upsilon)\big\}\\&\qquad+\max\Big\{\varphi_1\big(d^{2p}(S\upsilon,Ty_n)\big),\ \varphi_2\big(d^r(A\upsilon,S\upsilon)d^{r'}(By_n,Ty_n)\big),\\&\qquad\qquad\varphi_3\big(d^s(A\upsilon,Ty_n)d^{s'}(By_n,S\upsilon)\big),\\&\qquad\qquad\varphi_4\bigg(\frac{1}{2}\big[d^l(A\upsilon,Ty_n)d^{l'}(A\upsilon,S\upsilon)+d^l(By_n,S\upsilon)d^{l'}(By_n,Ty_n)\big]\bigg)\Big\}.\end{aligned}\tag{2}$$

Passing to the upper limit as $n \to \infty$ in condition (2) and using properties (1C) and (HE), we have

$$\begin{aligned}&\big[d^p(A\upsilon,\zeta)+ad^p(\zeta,\zeta)\big]d^p(A\upsilon,\zeta)\\&\leqslant a\max\big\{d^p(A\upsilon,\zeta)d^p(\zeta,\zeta),\ d^q(A\upsilon,\zeta)d^{q'}(\zeta,\zeta)\big\}\\&\quad+\max\bigg\{\varphi_1\big(d^{2p}(\zeta,\zeta)\big),\ \varphi_2\big(d^r(A\upsilon,\zeta)d^{r'}(\zeta,\zeta)\big),\ \varphi_3\big(d^s(A\upsilon,\zeta)d^{s'}(\zeta,\zeta)\big),\\&\qquad\varphi_4\bigg(\frac{1}{2}\big[d^l(A\upsilon,\zeta)d^{l'}(A\upsilon,\zeta)+d^l(\zeta,\zeta)d^{l'}(\zeta,\zeta)\big]\bigg)\bigg\},\end{aligned}$$

i.e.,

$$\begin{aligned}d^{2p}(A\upsilon,\zeta)&\leqslant\max\bigg\{\varphi_1(0),\ \varphi_2(0),\ \varphi_3(0),\ \varphi_4\bigg(\frac{1}{2}d^{l+l'}(A\upsilon,\zeta)\bigg)\bigg\}\\&\leqslant\varphi_4\bigg(\frac{1}{2}d^{2p}(A\upsilon,\zeta)\bigg)<\frac{1}{2}d^{2p}(A\upsilon,\zeta),\end{aligned}$$

a contradiction. Therefore, $A\upsilon=S\upsilon=\zeta$, which shows that υ is a coincidence point of the pair (A,S).

As $\zeta\in T(Y)$, there exists a point $\nu\in Y$ such that $T\nu=\zeta$. In order to prove that also $B\nu=\zeta$, suppose the contrary. Putting $x=\upsilon$ and $y=\nu$ in condition (1), we have

$$\begin{aligned}&\big[d^p(A\upsilon,B\nu)+ad^p(S\upsilon,T\nu)\big]d^p(A\upsilon,B\nu)\\&\leqslant a\max\big\{d^p(A\upsilon,S\upsilon)d^p(B\nu,T\nu),\ d^q(A\upsilon,T\nu)d^{q'}(B\nu,S\upsilon)\big\}\\&\quad+\max\bigg\{\varphi_1\big(d^{2p}(S\upsilon,T\nu)\big),\ \varphi_2\big(d^r(A\upsilon,S\upsilon)d^{r'}(B\nu,T\nu)\big),\\&\qquad\varphi_3\big(d^s(A\upsilon,T\nu)d^{s'}(B\nu,S\upsilon)\big),\\&\qquad\varphi_4\bigg(\frac{1}{2}\big[d^l(A\upsilon,T\nu)d^{l'}(A\upsilon,S\upsilon)+d^l(B\nu,S\upsilon)d^{l'}(B\nu,T\nu)\big]\bigg)\bigg\},\end{aligned}$$

which implies that

$$\begin{aligned}&\big[d^p(\zeta,B\nu)+ad^p(\zeta,\zeta)\big]d^p(\zeta,B\nu)\\&\leqslant a\max\big\{d^p(\zeta,\zeta)d^p(B\nu,\zeta),\ d^q(\zeta,\zeta)d^{q'}(B\nu,\zeta)\big\}\\&\quad+\max\bigg\{\varphi_1\big(d^{2p}(\zeta,\zeta)\big),\ \varphi_2\big(d^r(\zeta,\zeta)d^{r'}(B\nu,\zeta)\big),\ \varphi_3\big(d^s(\zeta,\zeta)d^{s'}(B\nu,\zeta)\big),\\&\qquad\varphi_4\bigg(\frac{1}{2}\big[d^l(\zeta,\zeta)d^{l'}(\zeta,\zeta)+d^l(B\nu,\zeta)d^{l'}(B\nu,\zeta)\big]\bigg)\bigg\},\end{aligned}$$

i.e.,

$$\begin{aligned}d^{2p}(\zeta,B\nu)&\leqslant\max\big\{\varphi_1(0),\ \varphi_2(0),\ \varphi_3(0),\ \varphi_4\big(d^{l+l'}(B\nu,\zeta)\big)\big\}\\&=\varphi_4\big(d^{l+l'}(B\nu,\zeta)\big)<d^{2p}(B\nu,\zeta),\end{aligned}$$

a contradiction. Thus, $B\nu = T\nu = \zeta$, showing that ν is a coincidence point of the pair (B, T).

Assume that $Y = X$. If both pairs (A, S) and (B, T) are weakly compatible, $A\upsilon = S\upsilon$ and $B\nu = T\nu$, imply that $A\zeta = AS\upsilon = SA\upsilon = S\zeta$ and $B\zeta = BT\nu = TB\nu = T\zeta$. In order to prove that $\zeta = A\zeta$, suppose the contrary. Putting $x = \zeta$ and $y = \nu$ in condition (1), we have

$$\begin{aligned} &\big[d^p(A\zeta, B\nu) + ad^p(S\zeta, T\nu)\big] d^p(A\zeta, B\nu) \\ &\quad\leqslant a \max\big\{d^p(A\zeta, S\zeta)d^p(B\nu, T\nu),\, d^q(A\zeta, T\nu)d^{q'}(B\nu, S\zeta)\big\} \\ &\qquad + \max\Big\{\varphi_1\big(d^{2p}(S\zeta, T\nu)\big),\, \varphi_2\big(d^r(A\zeta, S\zeta)d^{r'}(B\nu, T\nu)\big), \\ &\qquad\qquad \varphi_3\big(d^s(A\zeta, T\nu)d^{s'}(B\nu, S\zeta)\big), \\ &\qquad\qquad \varphi_4\bigg(\frac{1}{2}\big[d^l(A\zeta, T\nu)d^{l'}(A\zeta, S\zeta) + d^l(B\nu, S\zeta)d^{l'}(B\nu, T\nu)\big]\bigg)\Big\}, \end{aligned} \tag{3}$$

which implies that

$$\begin{aligned} &\big[d^p(A\zeta, \zeta) + ad^p(A\zeta, \zeta)\big] d^p(A\zeta, \zeta) \\ &\quad\leqslant a \max\big\{0,\, d^q(A\zeta, \zeta)d^{q'}(\zeta, A\zeta)\big\} \\ &\qquad + \max\big\{\varphi_1\big(d^{2p}(A\zeta, \zeta)\big),\, \varphi_2(0),\, \varphi_3\big(d^s(A\zeta, \zeta)d^{s'}(\zeta, A\zeta)\big),\, \varphi_4(0)\big\}, \end{aligned}$$

i.e.,

$$\begin{aligned} &(1 + a)d^{2p}(A\zeta, \zeta) \\ &\quad\leqslant ad^{2p}(\zeta, A\zeta) + \max\big\{\varphi_1\big(d^{2p}(A\zeta, \zeta)\big),\, \varphi_2(0),\, \varphi_3\big(d^{s+s'}(\zeta, A\zeta)\big)\big\} \\ &\quad= ad^{2p}(\zeta, A\zeta) + \varphi_1\big(d^{2p}(A\zeta, \zeta)\big) \\ &\quad< (1 + a)d^{2p}(\zeta, A\zeta), \end{aligned}$$

a contradiction. Thus, $\zeta = A\zeta = S\zeta$. Therefore, ζ is a common fixed point of the pair (A, S). In order to prove that $B\zeta = \zeta$, suppose the contrary. Putting $x = \upsilon$ and $y = \zeta$ in condition (1), we have

$$\begin{aligned} &\big[d^p(A\upsilon, B\zeta) + ad^p(S\upsilon, T\zeta)\big] d^p(A\upsilon, B\zeta) \\ &\quad\leqslant a \max\big\{d^p(A\upsilon, S\upsilon)d^p(B\zeta, T\zeta),\, d^q(A\upsilon, T\zeta)d^{q'}(B\zeta, S\upsilon)\big\} \\ &\qquad + \max\Big\{\varphi_1\big(d^{2p}(S\upsilon, T\zeta)\big),\, \varphi_2\big(d^r(A\upsilon, S\upsilon)d^{r'}(B\zeta, T\zeta)\big), \\ &\qquad\qquad \varphi_3\big(d^s(A\upsilon, T\zeta)d^{s'}(B\zeta, S\upsilon)\big), \\ &\qquad\qquad \varphi_4\bigg(\frac{1}{2}\big[d^l(A\upsilon, T\zeta)d^{l'}(A\upsilon, S\upsilon) + d^l(B\zeta, S\upsilon)d^{l'}(B\zeta, T\zeta)\big]\bigg)\Big\}. \end{aligned} \tag{4}$$

From (4), we obtain

$$\begin{aligned} & \big[d^p(\zeta, B\zeta) + ad^p(\zeta, B\zeta)\big] d^p(\zeta, B\zeta) \\ c \quad & \leqslant a \max\big\{0, d^q(\zeta, B\zeta) d^{q'}(B\zeta, \zeta)\big\} \\ & \quad + \max\bigg\{\varphi_1\big(d^{2p}(\zeta, B\zeta)\big),\ \varphi_2(0),\ \varphi_3\big(d^s(\zeta, B\zeta) d^{s'}(B\zeta, \zeta)\big), \\ & \qquad \varphi_4\bigg(\frac{1}{2}\big[d^l(\zeta, B\zeta) d^{l'}(\zeta, B\upsilon) + d^l(B\zeta, \zeta) \cdot 0\big]\bigg)\bigg\}, \end{aligned}$$

i.e.,

$$\begin{aligned} & (1+a) d^{2p}(\zeta, B\zeta) \\ & \quad \leqslant ad^{q+q'}(B\zeta, \zeta)\} \\ & \quad + \max\bigg\{\varphi_1\big(d^{2p}(\zeta, B\zeta)\big),\ \varphi_2(0),\ \varphi_3(d^{s+s'}(B\zeta, \zeta),\ \varphi_4\bigg(\frac{1}{2} d^{l+l'}(\zeta, B\upsilon)\bigg)\bigg\}, \end{aligned}$$

hence,

$$(1+a) d^{2p}(\zeta, B\zeta) < (a+1) d^{2p}(B\zeta, \zeta),$$

a contradiction. Therefore, $B\zeta = T\zeta = \zeta$ and we can conclude that ζ is a common fixed point of A, B, S and T.

Let ξ be any other common fixed point of A, B, S and T. That is, $B\xi = T\xi = S\xi = A\xi = \xi$. Putting $x = \zeta$ and $y = \xi$ in (1), we have

$$\begin{aligned} & \big[d^p(A\zeta, B\xi) + ad^p(S\zeta, T\xi)\big] d^p(A\zeta, B\xi) \\ & \quad \leqslant a \max\big\{d^p(A\zeta, S\zeta) d^p(B\xi, T\xi),\ d^q(A\zeta, T\xi) d^{q'}(B\xi, S\zeta)\big\} \\ & \qquad + \max\bigg\{\varphi_1\big(d^{2p}(S\zeta, T\xi)\big),\ \varphi_2\big(d^r(A\zeta, S\zeta) d^{r'}(B\xi, T\xi)\big), \\ & \qquad \varphi_3\big(d^s(A\zeta, T\xi) d^{s'}(B\xi, S\zeta)\big), \\ & \qquad \varphi_4\bigg(\frac{1}{2}\big[d^l(A\zeta, T\xi) d^{l'}(A\zeta, S\zeta) + d^l(B\xi, S\zeta) d^{l'}(B\xi, T\xi)\big]\bigg)\bigg\}, \end{aligned}$$

which implies that

$$\begin{aligned} & (1+a) d^{2p}(S\zeta, T\xi) \\ & \quad \leqslant ad^{q+q'}(\xi, \zeta) + \max\big\{\varphi_1\big(d^{2p}(\zeta, \xi)\big),\ \varphi_2(0),\ \varphi_3\big(d^{s+s'}(\xi, \zeta)\big),\ \varphi_4(0)\big\} \\ & \quad = ad^{2p}(\xi, \zeta) + \max\big\{\varphi_1\big(d^{2p}(\zeta, \xi)\big),\ \varphi_2(0),\ \varphi_3\big(d^{2p}(\xi, \zeta)\big),\ \varphi_4(0)\big\} \\ & \quad = ad^{2p}(\xi, \zeta) + \max\big\{\varphi_1\big(d^{2p}(\zeta, \xi)\big),\ \varphi_2(0),\ \varphi_3\big(d^{2p}(\xi, \zeta)\big),\ \varphi_4(0)\big\} \\ & \quad < (1+a) d^{2p}(\xi, \zeta), \end{aligned}$$

unless $\zeta = \xi$. Hence, A, B, S and T have unique common fixed point. □

Remark 2. Theorem 1 extends Theorem 3.1 of Pathak et al. [14].

In Theorem 1, if we put $a = 0$ and $\varphi_i(t) = ht$ ($i = 1, 2, 3, 4$), where $0 < h < 1$, we get the following consequence.

Corollary 1. *Let (X, d) be a symmetric space, where d satisfies conditions (1C) and (HE), and let Y be an arbitrary non-empty set. Let $A, B, S, T : Y \to X$ be such that*

$$\begin{aligned} &d^{2p}(Ax, By) \\ &\quad\leqslant h \max\Big\{ d^{2p}(Sx, Ty),\ d^{r}(Ax, Sx)d^{r'}(By, Ty),\ d^{s}(Ax, Ty)d^{s'}(By, Sx), \\ &\qquad\qquad \frac{1}{2}\big[d^{l}(Ax, Ty)d^{l'}(Ax, Sx) + d^{l}(By, Sx)d^{l'}(By, Ty)\big]\Big\} \end{aligned} \tag{5}$$

for all $x, y \in X$ and some $h \in (0, 1)$, $p, r, r', s, s', l, l' \geqslant 0$ with $2p = r + r' = s + s' = l + l' \leqslant 1$. If the pairs (A, S) and (B, T) satisfy the (CLR$_{ST}$) property, then (A, S) and (B, T) have a coincidence point each. If, moreover $Y = X$ and both pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

The following proposition will help us to get further results.

Proposition 1. *Let (X, d) be a symmetric space, where d satisfies condition (CC), and let Y be an arbitrary non-empty set. Let $A, B, S, T : Y \to X$ be mappings such that the following hypotheses hold*:

1. *The pair (A, S) satisfies the (CLR$_S$) property*;
2. $A(Y) \subset T(Y)$;
3. *$T(Y)$ is a closed subset of X*;
4. *$\{By_n\}$ converges for every sequence $\{y_n\}$ in Y such that $\{Ty_n\}$ converges*;
5. *A, B, S, T satisfy condition* (1).

Then the pairs (A, S) and (B, T) share the (CLR$_{ST}$) property.

The same conclusion holds if, in conditions (1)–(4)*, A and B, as well as S and T, change places.*

Proof. If the pair (A, S) satisfy the (CLR$_S$) property, then there exists a sequence $\{x_n\}$ in Y such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \zeta,$$

where $\zeta \in S(Y)$. By (2), $A(Y) \subset T(Y)$ (where $T(Y)$ is a closed subset of X) and for each $\{x_n\} \subset Y$, there corresponds a sequence $\{y_n\} \subset Y$ such that $Ax_n = Ty_n$. Therefore,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = \zeta,$$

where $\zeta \in S(Y) \cap T(Y)$. Thus, we have

$$\lim_{n\to+\infty} d(Ax_n, \zeta) = \lim_{n\to+\infty} d(Sx_n, \zeta) = \lim_{n\to+\infty} d(Ty_n, \zeta) = 0.$$

Therefore, by (HE) we have

$$\lim_{n\to\infty} d(Ax_n, Sx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(Sx_n, Ty_n) = 0.$$

By (4), the sequence $\{By_n\}$ converges; we need to show that $By_n \to \zeta$ as $n \to \infty$. By (CC), we get $\lim_{n\to\infty} d(Ax_n, By_n) = d(\zeta, \lim_{n\to\infty} By_n)$, $\lim_{n\to\infty} d(Sx_n, By_n) = d(\zeta, \lim_{n\to\infty} By_n)$ and $\lim_{n\to\infty} d(By_n, Ty_n) = d(\lim_{n\to\infty} By_n, \zeta)$. Putting $x = x_n$ and $y = y_n$ in condition (1), we get

$$\begin{aligned}
&\big[d^p(Ax_n, By_n) + ad^p(Sx_n, Ty_n)\big]d^p(Ax_n, By_n)\\
&\quad \leqslant a\max\big\{d^p(Ax_n, Sx_n)d^p(By_n, Ty_n),\ d^q(Ax_n, Ty_n)d^{q'}(By, Sx_n)\big\}\\
&\qquad + \max\Big\{\varphi_1\big(d^{2p}(Sx_n,\ Ty_n)\big),\ \varphi_2\big(d^r(Ax_n, Sx_n)d^{r'}(By_n, Ty_n)\big),\\
&\qquad\qquad \varphi_3\big(d^s(Ax_n, Ty_n)d^{s'}(By_n, Sx_n)\big),\\
&\qquad\qquad \varphi_4\Big(\frac{1}{2}\big[d^l(Ax_n, Ty_n)d^{l'}(Ax_n, Sx_n) + d^l(By_n, Sx_n)d^{l'}(By_n, Ty_n)\big]\Big)\Big\}, \qquad (6)
\end{aligned}$$

Passing to the upper limit as $n \to \infty$ in the inequality (6), we have

$$\begin{aligned}
&\Big[d^p\Big(\zeta, \lim_{n\to\infty} By_n\Big) + a\cdot 0\Big]d^p\Big(\zeta, \lim_{n\to\infty} By_n\Big)\\
&\quad \leqslant a\max\Big\{0\cdot d^p\Big(\lim_{n\to\infty} By_n, \zeta\Big),\ 0\cdot d^{q'}\Big(\lim_{n\to\infty} By_n, \zeta\Big)\Big\}\\
&\qquad + \max\Big\{\varphi_1(0),\ \varphi_2\Big(0\cdot d^{r'}\Big(\lim_{n\to\infty} By_n, \zeta\Big)\Big),\ \varphi_3\Big(0\cdot d^{s'}\Big(\lim_{n\to\infty} By_n, \zeta\Big)\Big),\\
&\qquad\qquad \varphi_4\Big(\frac{1}{2}\cdot 0 + d^l\Big(\lim_{n\to\infty} By_n, \zeta\Big)\Big)d^{l'}\Big(\lim_{n\to\infty} By_n, \zeta\Big)\Big\}. \qquad (7)
\end{aligned}$$

Hence, the inequality (7) implies

$$\begin{aligned}
d^{2p}\Big(\zeta, \lim_{n\to\infty} By_n\Big) &\leqslant \max\Big\{\varphi_1(0),\ \varphi_2(0),\ \varphi_3(0),\ \varphi_4\Big(d^{l+l'}\Big(\lim_{n\to\infty} By_n, \zeta\Big)\Big)\Big\}\\
&\leqslant \varphi_4\Big(d^{l+l'}\Big(\lim_{n\to\infty} By_n, \zeta\Big)\Big)\\
&< d^{l+l'}\Big(\lim_{n\to\infty} By_n, \zeta\Big) = d^{2p}\Big(\lim_{n\to\infty} By_n, \zeta\Big),
\end{aligned}$$

unless $d(\zeta, \lim_{n\to\infty} By_n) = 0$. Hence, $By_n \to \zeta$ as $n \to \infty$, which shows that the pairs (A, S) and (B, T) share the (CLR$_{ST}$) property. □

The converse of Proposition 1 is not true. For a counterexample, see [7, Ex. 3.5].

If we replace properties (1C) and (HE) by (CC) property in Theorem 1 and use Proposition 1, we have the following result.

Theorem 2. *Let (X, d) be a symmetric space, where d satisfies condition (CC), and let Y be a non-empty set. Let $A, B, S, T : Y \to X$ be mappings, such that conditions* (1)–(5) *of Proposition* 1 *hold. Then (A, S) and (B, T) have a coincidence point each.*

If, moreover, $Y = X$ and both pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. In follows from Proposition 1 that the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property. Therefore, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = \zeta,$$

where $\zeta \in S(Y) \cap T(Y)$. The rest of the proof is similar to the proof of Theorem 1. □

Obviously, if the pairs (A, S) and (B, T) satisfy the common property (E.A), and, at the same time, $S(Y)$ and $T(Y)$ are closed subsets of X, then the pairs (A, S) and (B, T) share the (CLR_{ST}) property. Hence, we have the following variant of Theorem 1.

Theorem 3. *Let (X, d) be a symmetric space, where d satisfies conditions (1C) and (HE), and let Y be an arbitrary non-empty set. Let $A, B, S, T : Y \to X$ be such mappings that inequality* (1) *and the following assumptions hold*:

1. *The pairs (A, S) and (B, T) satisfy the common property (E.A)*;
2. *$S(Y)$ and $T(Y)$ are closed subsets of X.*

Then (A, S) and (B, T) have a coincidence point each. If, moreover, $Y = X$ and both pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Next, we state two more variants of our results, which can be proved similarly as Theorems 2 and 3.

Corollary 2. *The conclusions of Theorem* 3 *remain true if condition* 2 *is replaced by the following*:

$2'$. *$\overline{A(Y)} \subset T(Y)$ and $\overline{B(Y)} \subset S(Y)$.*

Corollary 3. *The conclusions of Theorem* 3 *remain true if condition* 2 *is replaced by the following*:

$2''$. *$A(Y)$ and $B(Y)$ are closed subsets of X, and $A(Y) \subset T(Y)$, $B(Y) \subset S(Y)$.*

By choosing A, B, S and T suitably in Theorem 1, we can deduce some corollaries for a pair as well as for a triple of self mappings. Since the formulations of these results are similar to those in [6, 7], we omit the details here. We just state the following result for four families of mappings.

Corollary 4. *Let (X, d) be a symmetric space, where d satisfies conditions $(1C)$ and (HE), and let Y be an arbitrary non-empty set. Let $\{A_i\}_{i=1}^m$, $\{B_j\}_{j=1}^n$, $\{S_t\}_{t=1}^v$ and $\{T_u\}_{u=1}^w$ be four finite families of mappings from Y to X, where $A = A_1A_2\cdots A_m$, $B = B_1B_2\cdots B_n$, $S = S_1S_2\cdots S_v$ and $T = T_1T_2\cdots T_w$ satisfy condition (1), and the pairs (A, S) and (B, T) satisfy the (CLR$_{ST}$) property. Then (A, S) and (B, T) have a point of coincidence each. Moreover, if $Y = X$, then $\{A_i\}_{i=1}^m, \{B_j\}_{r=1}^n, \{S_t\}_{t=1}^v$ and $\{T_u\}_{u=1}^w$ have a unique common fixed point provided the pairs of families $(\{A_i\}, \{S_t\})$ and $(\{B_j\}, \{T_u\})$ commute pairwise, where $i \in \{1, 2, \dots, m\}$, $t \in \{1, 2, \dots, v\}$, $j \in \{1, 2, \dots, n\}$ and $u \in \{1, 2, \dots, w\}$.*

3 Illustrative examples

Now we furnish examples demonstrating the validity of the hypotheses and degree of generality of our results over some recently established results.

The following example exhibits the validity of conditions of Theorem 1. This example is inspired by Imdad et al. [7].

Example 3. Let $Y = [1, 10) \subset [1, +\infty) = X$, and let X, Y be equipped with the symmetric $d(x, y) = (x - y)^2$ for all $x, y \in X$, which obviously satisfies (1C) and (HE). Consider the mappings $A, B, S, T : Y \to X$ given by

$$Sx = \begin{cases} 1 & \text{if } x = 1, \\ 10 & \text{if } x \in (1, 3], \\ (4x - 5)/7 & \text{if } x \in (3, 10), \end{cases} \qquad Tx = \begin{cases} 1 & \text{if } x = 1, \\ 10 & \text{if } x \in (1, 3], \\ (x + 4)/7 & \text{if } x \in (3, 10), \end{cases}$$

$$Ax = \begin{cases} 1 & \text{if } x \in \{1\} \cup (3, 10), \\ 8 & \text{if } x \in (1, 3], \end{cases} \qquad Bx = \begin{cases} 1 & \text{if } x \in \{1\} \cup (3, 10), \\ 5 & \text{if } x \in (1, 3]. \end{cases}$$

Then we have $A(Y) = \{1, 8\} \nsubseteq [1, 2) \cup \{10\} = T(Y)$ and $B(Y) = \{1, 5\} \nsubseteq [1, 5) \cup \{10\} = S(Y)$. The pairs (A, S) and (B, T) satisfy the (CLR$_{ST}$) property. Indeed, consider two sequences, $\{x_n\} = \{1\}$ and $\{y_n\} = \{3 + 1/n\}_{n \in \mathbb{N}}$. Then we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1,$$

where $1 \in S(Y) \cap T(Y)$; we note that $S(Y)$ and $T(Y)$ are not closed subsets of X.

Now, define functions $\varphi_i : [0, +\infty) \to [0, +\infty)$ by

$$\varphi_i(t) = ht, \quad \text{with } \frac{49}{64} < h < 1, \text{ for } i \in \{1, 2, 3, 4\} \text{ and all } t \geqslant 0$$

and take $p = r = r' = s = s' = l = l' = 1/2$. Clearly, $\varphi_i \in \Phi$. By a routine calculation, one can check that the inequality (5) is satisfied for all $x, y \in Y$. Thus, all the conditions of Corollary 1 are satisfied (except $Y = X$), and 1 is a unique common fixed point of the pairs (A, S) and (B, T). Note that all the involved mappings are discontinuous at their unique common fixed point.

In the following illustration the importance of weakly compatible assumption for validity of the result is shown.

Example 4. Let $Y = [0, 5) \subset [0, +\infty) = X$ be equipped with the symmetric $d(x, y) = (x-y)^2$ for $x, y \in X$, which satisfies (1C) and (HE). Consider the mappings $A, B, S, T : Y \to X$ given by

$$Ax = Bx = x + 3 \quad \text{and} \quad Sx = Tx = 2(1 + x).$$

Then the pairs (A, S), (B, T) satisfy the (CLR$_{ST}$) property. Indeed, consider two sequences, $\{x_n\} = \{1 + 1/n\}_{n\in\mathbb{N}}$, $\{y_n\} = \{1 - 1/n\}_{n\in\mathbb{N}}$. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 4,$$

where $4 \in S(Y) \cap T(Y)$; however, $S(Y)$ and $T(Y)$ are not closed subsets of X.

By a routine calculation, for a suitable value of h, one can check that inequality (5) is satisfied. Thus, all the conditions of the first part of Corollary 1 are satisfied. It can be noted that, indeed, 1 is a coincidence point of (A, S), as well as of (B, T). However, these pairs are not weakly compatible and there is no common fixed point of the pairs (A, S) and (B, T).

Theorem 2 cannot be applied in the case of mappings from Example 3 since conditions 2 and 3 of Proposition 1 are not fulfilled. The following example shows the situation when Theorem 2 can be used.

Example 5. In the setting of Example 3, replace the mappings S and T by the following:

$$Sx = Tx = \begin{cases} 1 & \text{if } x = 1, \\ 10 & \text{if } x \in (1, 3], \\ (9x - 20)/7 & \text{if } x \in (3, 10). \end{cases}$$

Then $A(Y) = \{1, 8\} \subset [1, 10] = T(Y)$ and $B(Y) = \{1, 5\} \subset [1, 10] = S(Y)$; $S(Y)$ and $T(Y)$ are now closed subsets of X. Thus, all the conditions of Theorem 2 are satisfied (except $Y = X$), and 1 is a unique common fixed point of the pairs (A, S) and (B, T).

Now we furnish an example demonstrating that condition 1 of Theorem 1 is only sufficient and not necessary.

Example 6. Consider $X = Y = [2, 20]$, equipped with the symmetric $d(x, y) = (x-y)^2$ for all $x, y \in Y$. Consider the mappings $A, B, S, T : Y \to X$ given by

$$Sx = Tx = \begin{cases} 2 & \text{if } x = 2, \\ 7 & \text{if } 2 < x \leqslant 5, \\ (x + 1)/3 & \text{if } 5 < x \leqslant 20, \end{cases} \qquad Ax = Bx = \begin{cases} 2 & \text{if } x = 2, \\ 7 & \text{if } 2 < x \leqslant 5, \\ 2 & \text{if } 5 < x \leqslant 20. \end{cases}$$

Then the pairs (A, S) and (B, T) satisfy all the conditions of Theorem 1, except the inequality (1) (take, e.g., $x \in (2, 5]$ and $y = 2$). However, these four mappings have a coincidence at $x = 2$, which also remains their common fixed point. This confirms that condition 1 of Theorem 1 is sufficient and not necessary.

Our last example highlights the non-closedness of ranges of S and T in X in Corollaries 2 and 3.

Example 7. In the setting of Example 3, replace the mappings S and T by the following:

$$Sx = Tx = \begin{cases} 1 & \text{if } x = 1, \\ 10 & \text{if } x \in (1,3], \\ (8x-17)/7 & \text{if } x \in (3,10). \end{cases}$$

Then $A(Y) = \{1,8\} \subset [1,9) \cup \{10\} = T(Y)$ and $B(Y) = \{1,5\} \subset [1,9) \cup \{10\} = S(Y)$. Now, $S(Y)$ and $T(Y)$ are not closed subspaces of X, but condition $2'$ (resp. $2''$) of Corollary 2 (resp. 3) is satisfied, except $Y = X$. Again, 1 is a unique common fixed point of A, B, S and T.

4 Application to systems of integral equations

Consider the following system of integral equations:

$$\begin{aligned} u(t) &= \int_0^T K_1\bigl(t,s,u(s)\bigr)\,\mathrm{d}s + g(t), \\ u(t) &= \int_0^T K_2\bigl(t,s,u(s)\bigr)\,\mathrm{d}s + g(t), \\ u(t) &= \int_0^T K_3\bigl(t,s,u(s)\bigr)\,\mathrm{d}s + g(t), \end{aligned} \tag{8}$$

$t \in I = [0,T]$, where $T > 0$. The purpose of this section is to give an existence theorem for a solution of the system (8) using Corollary 1.

Consider the set

$$C(I) := \{u : I \to \mathbb{R} \mid u \text{ is continuous on } I\}$$

and define $d : C(I) \times C(I) \to \mathbb{R}$ by

$$d(u,v) = \Bigl[\max_{t \in I} \bigl|u(t) - v(t)\bigr|\Bigr]^2 \quad \forall u,v \in C(I).$$

Then $(C(I), d)$ is a symmetric space. Define further mappings $T_i : C(I) \to C(I)$ by

$$T_i x(t) = \int_0^T K_i\bigl(t,s,x(s)\bigr)\,\mathrm{d}s + g(t), \quad t \in I,\ i \in \{1,2,3\}.$$

Consider the following conditions:

(i) $K_1, K_2, K_3 : I \times I \times \mathbb{R} \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are continuous;

(ii) There exists a continuous function $G : I \times I \to \mathbb{R}_+$ such that

$$\begin{aligned}
&\bigl|K_1\bigl(t,s,u(t)\bigr) - K_2\bigl(t,s,v(t)\bigr)\bigr|^p \\
&\quad\leqslant G(t,s)\max\biggl\{\bigl|T_3u(t) - T_3v(t)\bigr|^{2p}, \bigl|T_1u(t) - T_3u(t)\bigr|^r\bigl|T_2v(t) - T_3v(t)\bigr|^{r'}, \\
&\qquad\bigl|T_1u(t) - T_3v(t)\bigr|^s\bigl|T_2v(t) - T_3u(t)\bigr|^{s'}, \\
&\qquad\frac{1}{\sqrt{2}}\bigl[\bigl|T_1u(t) - T_3v(t)\bigr|^l\bigl|T_1u(t) - T_3u(t)\bigr|^{l'} \\
&\qquad+ \bigl|T_2v(t) - T_3u(t)\bigr|^l\bigl|T_2v(t) - T_3v(t)\bigr|^{l'}\bigr]\biggr\}
\end{aligned}$$

for all $u, v \in C(I)$ and $t, s \in I$, where $p, r, r', s, s', l, l' \geqslant 0$ with $2p = r + r' = s + s' = l + l' \leqslant 1$;

(iii) $\max_{t\in I}\int_0^T G(t,s)\,\mathrm{d}s = \alpha < T^{-p}$;

(iv) There exist sequences $\{u_n\}$ and $\{v_n\}$ in $C(I)$ and $u^* \in C(I)$ such that

$$\lim_{n\to\infty} T_1u_n = \lim_{n\to\infty} T_3u_n = \lim_{n\to\infty} T_2v_n = \lim_{n\to\infty} T_3v_n = u^*$$

in $(C(I), d)$;

(v) $T_1T_3u = T_3T_1u$ whenever $T_1u = T_3u$ for some $u \in C(I)$, and $T_2T_3v = T_3T_2v$ whenever $T_2v = T_3v$ for some $v \in C(I)$.

We will prove the following result.

Theorem 4. *Suppose that hypotheses* (i)–(v) *hold. Then system* (8) *has a unique solution* $x^* \in C(I)$.

Proof. Notice first that the conditions (1C) and (HE) hold in $(C(I), d)$ trivially. By hypothesis (iv) the pairs (T_1, T_3) and (T_2, T_3) share the common limit range property with respect to T_3. Now, for all $u, v \in C(I)$, by (ii) and (iii), we have

$$\begin{aligned}
&\bigl|T_1u(t) - T_2v(t)\bigr| \\
&\quad\leqslant \int_0^T \bigl|K_1\bigl(t,s,u(s)\bigr) - K_2\bigl(t,s,v(s)\bigr)\bigr|\,\mathrm{d}s \\
&\quad\leqslant T\biggl(\int_0^T G(t,s)\,\mathrm{d}s\biggr)^{1/p} \\
&\qquad\times\max\biggl\{\bigl|T_3u(t) - T_3v(t)\bigr|^{2p}, \bigl|T_1u(t) - T_3u(t)\bigr|^r\bigl|T_2v(t) - T_3v(t)\bigr|^{r'}, \\
&\qquad\bigl|T_1u(t) - T_3v(t)\bigr|^s\bigl|T_2v(t) - T_3u(t)\bigr|^{s'}, \\
&\qquad\frac{1}{\sqrt{2}}\bigl[\bigl|T_1u(t) - T_3v(t)\bigr|^l\bigl|T_1u(t) - T_3u(t)\bigr|^{l'} \\
&\qquad+ \bigl|T_2v(t) - T_3u(t)\bigr|^l\bigl|T_2v(t) - T_3v(t)\bigr|^{l'}\bigr]\biggr\}^{1/p}
\end{aligned}$$

$$\begin{aligned}\leqslant T\alpha^{1/p}\max\Big\{& d^{2p}(T_3u,T_3v),\ d^r(T_1u,T_3u)d^{r'}(T_2v,T_3v),\\ & d^s(T_1u,T_3v)d^{s'}(T_2v,T_3u),\\ & \frac{1}{\sqrt{2}}\big[d^l(T_1u,T_3v)d^{l'}(T_1u,T_3u)+d^l(T_2v,T_3u)d^{l'}(T_2v,T_3v)\big]\Big\}^{1/(2p)}.\end{aligned}$$

On routine calculations, we get

$$\begin{aligned}& d^{2p}(T_1u,T_2v)\\ &\quad\leqslant \alpha^2T^{2p}\max\Big\{d^{2p}(T_3u,T_3v),\ d^r(T_1u,T_3u)d^{r'}(T_2v,T_3v),\\ &\qquad d^s(T_1u,T_3v)d^{s'}(T_2v,T_3u),\\ &\qquad \frac{1}{2}\big[d^l(T_1u,T_3v)d^{l'}(T_1u,T_3u)+d^l(T_2v,T_3u)d^{l'}(T_2v,T_3v)\big]\Big\}.\end{aligned}$$

Then, putting $A = T_1$, $B = T_2$ and $S = T = T_3$, Corollary 1 is applicable in the case $Y = X = C(I)$, where $\alpha^2T^{2p} \in (0, 1)$. Moreover, in view of hypothesis (v), the pairs (T_1, T_3) and (T_2, T_3) are weakly compatible, and so T_1, T_2 and T_3 have a unique common fixed point. Then there exists a unique $x^* \in C(I)$, a common fixed point of T and S, that is, x^* is a unique solution to (8). □

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