Nonlinear Analysis: Modelling and Control, Vol. 20, No. 3, 367–376 ISSN 1392-5113
http://dx.doi.org/10.15388/NA.2015.3.4

Existence of positive solutions to an arbitrary order fractional differential equation via a mixed monotone operator method*

Mohamed Jleli, Bessem Samet

Department of Mathematics, College of Science, King Saud University
P.O. Box 2455, Riyadh 11451, Saudi Arabia
jleli@ksu.edu.sa; bsamet@ksu.edu.sa

Received: March 16, 2014 / **Revised:** June 24, 2014 / **Published online:** March 20, 2015

Abstract. In this paper, using a mixed monotone operator method, we study the existence and uniqueness of positive solutions to a nonlinear arbitrary order fractional differential equation. An example is provided to illustrate the main result.

Keywords: fractional differential equation, positive solution, cone, mixed monotone.

1 Introduction

Fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, biology, chemistry, physics, etc., see [7,9,11,14]. In recent years, there has been a significant development in fractional differential equations. One can see the monographs of Kilbas et al. [9], Lakshmikantham et al. [10], Miller and Ross [12], Podlubny [13], and others. On the other hand, the existence of positive solutions to BVPs of fractional differential equations has been extensively studied by many researchers in recent years; see, for example, [1,2,3,4,5,6,8,15,16] and the references therein.

In [5], Goodrich studied the following fractional differential equation:

$$\begin{aligned} &D_{0^+}^{\alpha} x(t) + f\bigl(t, x(t)\bigr) = 0, \quad t \in (0,1),\ n-1 < \alpha \leqslant n, \\ &x^{(i)}(0) = 0, \quad i = 0,1,2,\dots,n-2, \\ &\bigl[D_{0^+}^{\beta} x(t)\bigr]_{t=1} = 0, \quad 2 \leqslant \beta \leqslant n-2, \end{aligned} \tag{1}$$

where $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, $D_{0^+}^{\alpha}$ is the standard Riemann–Liouville fractional derivative of order $n-1 < \alpha \leqslant n$, $n > 3$ ($n \in \mathbb{N}$), and $x^{(i)}$ represents the ith

*The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for the funding of this research through the Research Group Project No. RGP-VPP-237.

(ordinary) derivative of x. Using Krasnosel'skii fixed point theorem in cones, he derived sufficient conditions for the existence of positive solutions to the problem. Note that the uniqueness of positive solutions is not studied in [5].

In this paper, we are concerned with the existence and uniqueness criterions for positive solutions to the following nonlinear arbitrary order fractional differential equation:

$$\begin{aligned} &D_{0+}^{\alpha}x(t) + f\bigl(t, x(t), x(t)\bigr) + g\bigl(t, x(t)\bigr) = 0, \quad t \in (0,1),\ n-1 < \alpha \leqslant n, \\ &x^{(i)}(0) = 0, \quad i = 0,1,2,\dots,n-2, \\ &\bigl[D_{0+}^{\beta}x(t)\bigr]_{t=1} = 0, \quad 2 \leqslant \beta \leqslant n-2, \end{aligned} \tag{2}$$

where $n > 3$ ($n \in \mathbb{N}$), $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ and $g : [0,1] \times [0,\infty) \to [0,\infty)$ are given continuous functions. Using a mixed monotone operator method, we determine sufficient conditions under which problem (2) has a unique positive solution.

2 Preliminaries

For the convenience of the reader, we present here some definitions, lemmas and basic results that will be used in the proof of our main result.

Definition 1. (See [13].) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $\varphi : (0,+\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha}\varphi(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{(n)} \int_0^t \frac{\varphi(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0,+\infty)$.

Lemma 1. (*See* [5].) *Assume that $\xi : [0,1] \to \mathbb{R}$ is a continuous function, then the fractional differential equation*

$$\begin{aligned} &D_{0+}^{\alpha}x(t) + \xi(t) = 0, \quad t \in (0,1),\ n-1 < \alpha \leqslant n, \\ &x^{(i)}(0) = 0, \quad i = 0,1,\dots,n-2, \\ &\bigl[D_{0+}^{\beta}x(t)\bigr]_{t=1} = 0, \quad 2 \leqslant \beta \leqslant n-2, \end{aligned} \tag{3}$$

has the unique solution

$$x(t) = \int_0^1 G_{\alpha,\beta}(t,s)\xi(s)\,\mathrm{d}s,$$

where

$$G_{\alpha,\beta}(t,s) = \begin{cases} (t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1})/\Gamma(\alpha), & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}/\Gamma(\alpha), & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{4}$$

is the Green's function of (3).

Lemma 2. *The Green function* (4) *has the following property*:

$$t^{\alpha-1}(1-s)^{\alpha-\beta-1}\big[1-(1-s)^{\beta}\big] \leqslant \Gamma(\alpha)G_{\alpha,\beta}(t,s) \leqslant t^{\alpha-1}(1-s)^{\alpha-\beta-1}, \quad t,s\in[0,1].$$

Proof. The right inequality is trivial. We have only to prove the left inequality. If $0 \leqslant s \leqslant t \leqslant 1$, then we have

$$t-s \leqslant t-ts = (1-s)t,$$

which implies that

$$(t-s)^{\alpha-1} \leqslant (1-s)^{\alpha-1}t^{\alpha-1}.$$

Then

$$\begin{aligned}\Gamma(\alpha)G_{\alpha,\beta}(t,s) &= t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}\\ &\geqslant t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}t^{\alpha-1}\\ &= t^{\alpha-1}\big[(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}\big]\\ &= t^{\alpha-1}(1-s)^{\alpha-\beta-1}\big[1-(1-s)^{\beta}\big].\end{aligned}$$

If $0 \leqslant t \leqslant s \leqslant 1$, then we have

$$\begin{aligned}\Gamma(\alpha)G_{\alpha,\beta}(t,s) &= t^{\alpha-1}(1-s)^{\alpha-\beta-1}\\ &\geqslant t^{\alpha-1}(1-s)^{\alpha-\beta-1}\big[1-(1-s)^{\beta}\big].\end{aligned}$$

So the left inequality is proved. □

In the sequel, we present some basic concepts in ordered Banach spaces for completeness and a fixed point theorem, which we will be used later.

Suppose that $(E, \|\cdot\|)$ is a real Banach space, which is partially ordered by a cone $P \subset E$, i.e.,

$$x,y\in E, \quad x \preceq y \quad \Longleftrightarrow \quad y-x\in P.$$

If $x \preceq y$ and $x \neq y$, then we denote $x \prec y$ or $y \succ x$. By θ_E we denote the zero element of E. Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies:

(P1) $x\in P,\ \lambda \geqslant 0 \ \Rightarrow\ \lambda x \in P$;

(P2) $-x, x \in P \ \Rightarrow\ x = \theta_E$.

Putting $\operatorname{int}(P) = \{x \in P \mid x \text{ is an interior point of } P\}$, a cone P is said to be solid if its interior $\operatorname{int}(P)$ is nonempty. Moreover, P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta_E \preceq x \preceq y$ implies $\|x\| \leqslant N\|y\|$. In this case, the smallest constant satisfying this inequality is called the normality constant of P. For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that

$$\lambda y \preceq x \preceq \mu y.$$

Clearly, $\sim$ is an equivalence relation. Given $h \succ \theta_E$, we denote by P_h the set

$$P_h = \{x \in E \mid x \sim h\}.$$

It is easy to see that $P_h \subset P$.

Definition 2. An operator $A : E \to E$ is said to be increasing (resp. decreasing) if for all $x, y \in E$, $x \preceq y$ implies $Ax \preceq Ay$ (resp. $Ax \succeq Ay$).

Definition 3. An operator $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in first component and decreasing in second component, i.e.,

$$(x,y),(u,v) \in P \times P, \quad x \preceq u, y \succeq v \quad \Longrightarrow \quad A(x,y) \preceq A(u,v).$$

Definition 4. An operator $A : P \to P$ is said to be sub-homogeneous if it satisfies

$$A(tx) \succeq tAx \quad \forall t \in (0,1),\ x \in P.$$

Lemma 3. (*See* [17].) *Let $\gamma \in (0,1)$. Let $A : P \times P \to P$ be a mixed monotone operator that satisfies*

$$A\bigl(tx, t^{-1}y\bigr) \succeq t^{\gamma} A(x,y), \quad t \in (0,1),\ x,y \in P.$$

Let $B : P \to P$ be an increasing sub-homogeneous operator. Assume that:

(i) *There is $h_0 \in P_h$ such that $A(h_0, h_0) \in P_h$ and $Bh_0 \in P_h$;*
(ii) *There exists a constant $\delta_0 > 0$ such that $A(x,y) \succeq \delta_0 Bx$ for all $x, y \in P$.*

Then:

(I) $A : P_h \times P_h \to P_h$, $B : P_h \to P_h$;
(II) *There exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*

$$rv_0 \preceq u_0 \prec v_0, \qquad u_0 \preceq A(u_0, v_0) + Bu_0 \preceq A(v_0, u_0) + Bv_0 \preceq v_0;$$

(III) *There exists a unique $x^* \in P_h$ such that $x^* = A(x^*, x^*) + Bx^*$;*
(IV) *For any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1},$$

$n = 1, 2, \ldots$, *we have*

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|y_n - x^*\| = 0.$$

3 Main result

Let $E = C([0,1])$ be the Banach space of real continuous functions defined in $[0,1]$ endowed with the norm

$$\|y\| = \max\bigl\{\bigl|y(t)\bigr|\colon t \in [0,1]\bigr\}$$

and $P \subset E$ be the positive cone defined by

$$P = \bigl\{y \in C\bigl([0,1]\bigr)\colon y(t) \geqslant 0,\ t \in [0,1]\bigr\}.$$

Our main result in this paper is the following.

Theorem 1. *Suppose that*:

(i) $f:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ *and* $g:[0,1]\times[0,+\infty)\to[0,+\infty)$ *are continuous with*

$$m\bigl(\bigl\{t\in[0,1]\colon\ g(t,0)\neq 0\bigr\}\bigr)>0,$$

where for some measurable set Ξ, $m(\Xi)$ *denotes the Lebesgue measure of* Ξ;

(ii) $f(t,x,y)$ *is increasing in* $x\in[0,+\infty)$ *for fixed* $t\in[0,1]$ *and* $y\in[0,+\infty)$*, decreasing in* $y\in[0,+\infty)$ *for fixed* $t\in[0,1]$ *and* $x\in[0,+\infty)$*, and* $g(t,x)$ *is increasing in* $x\in[0,+\infty)$ *for fixed* $t\in[0,1]$;

(iii) $g(t,\lambda x)\geqslant\lambda g(t,x)$ *for all* $\lambda\in(0,1)$, $t\in[0,1]$, $x\in[0,+\infty)$;

(iv) *There exists a constant* $\gamma\in(0,1)$ *such that*

$$f\bigl(t,\lambda x,\lambda^{-1}y\bigr)\geqslant\lambda^{\gamma}f(t,x,y),\quad \lambda\in(0,1),\ t\in[0,1],\ x,y\in[0,+\infty);$$

(v) *There exists a constant* $\delta_0>0$ *such that*

$$f(t,x,y)\geqslant\delta_0 g(t,x),\quad t\in[0,1],\ x,y\in[0,+\infty).$$

Then:

(I) *There exist* $u_0,v_0\in P_h$ *and* $r\in(0,1)$ *such that* $rv_0\preceq u_0\prec v_0$ *and*

$$u_0(t)\leqslant\int_0^1 G_{\alpha,\beta}(t,s)f\bigl(s,u_0(s),v_0(s)\bigr)\,\mathrm{d}s+\int_0^1 G_{\alpha,\beta}(t,s)g\bigl(s,u_0(s)\bigr)\,\mathrm{d}s,$$

$$v_0(t)\geqslant\int_0^1 G_{\alpha,\beta}(t,s)f\bigl(s,v_0(s),u_0(s)\bigr)\,\mathrm{d}s+\int_0^1 G_{\alpha,\beta}(t,s)g\bigl(s,v_0(s)\bigr)\,\mathrm{d}s,$$

where $h(t)=t^{\alpha-1}$, $t\in[0,1]$;

(II) (2) *has a unique positive solution* $x^*\in P_h$;

(III) *For any* $x_0,y_0\in P_h$*, constructing successively the sequences*

$$x_n(t)=\int_0^1 G_{\alpha,\beta}(t,s)f\bigl(s,x_{n-1}(s),y_{n-1}(s)\bigr)\,\mathrm{d}s+\int_0^1 G_{\alpha,\beta}(t,s)g\bigl(s,x_{n-1}(s)\bigr)\,\mathrm{d}s,$$

$$y_n(t)=\int_0^1 G_{\alpha,\beta}(t,s)f\bigl(s,y_{n-1}(s),x_{n-1}(s)\bigr)\,\mathrm{d}s+\int_0^1 G_{\alpha,\beta}(t,s)g\bigl(s,y_{n-1}(s)\bigr)\,\mathrm{d}s,$$

we have

$$\lim_{n\to\infty}\|x_n-x^*\|=\lim_{n\to\infty}\|y_n-x^*\|=0.$$

Proof. From Lemma 1, (2) has an integral formulation given by

$$x(t) = \int_0^1 G_{\alpha,\beta}(t,s) f\big(s, x(s), x(s)\big)\,\mathrm{d}s + \int_0^1 G_{\alpha,\beta}(t,s) g\big(s, x(s)\big)\,\mathrm{d}s.$$

Consider the operators $A : P \times P \to E$ and $B : P \to E$ defined by

$$A(u,v)(t) = \int_0^1 G_{\alpha,\beta}(t,s) f\big(s, u(s), v(s)\big)\,\mathrm{d}s, \quad t \in [0,1],$$
$$Bu(t) = \int_0^1 G_{\alpha,\beta}(t,s) g\big(s, u(s)\big)\,\mathrm{d}s, \quad t \in [0,1].$$

Clearly, x is a solution to (2) if and only if $A(x,x)+Bx = x$. Further, it follows from (ii) that A is mixed monotone and B is increasing. On the other hand, for any $\lambda \in (0,1)$, $u, v \in P$, from (iv) we have

$$\begin{aligned} A\big(\lambda u, \lambda^{-1} v\big)(t) &= \int_0^1 G_{\alpha,\beta}(t,s) f\big(s, \lambda u(s), \lambda^{-1} v(s)\big)\,\mathrm{d}s \\ &\geqslant \lambda^{\gamma} \int_0^1 G_{\alpha,\beta}(t,s) f\big(s, u(s), v(s)\big)\,\mathrm{d}s = \lambda^{\gamma} A(u,v)(t). \end{aligned}$$

Thus we have

$$A\big(\lambda u, \lambda^{-1} v\big) \succeq \lambda^{\gamma} A(u,v), \quad \lambda \in (0,1),\ u, v \in P.$$

From (iii), for all $\lambda \in (0,1)$, $u \in P$, we have

$$\begin{aligned} B(\lambda u)(t) &= \int_0^1 G_{\alpha,\beta}(t,s) g\big(s, \lambda u(s)\big)\,\mathrm{d}s \geqslant \lambda \int_0^1 G_{\alpha,\beta}(t,s) g\big(s, u(s)\big)\,\mathrm{d}s \\ &= \lambda Bu(t). \end{aligned}$$

Thus we have,

$$B(\lambda u) \succeq \lambda Bu, \quad \lambda \in (0,1),\ u \in P,$$

which implies that B is a sub-homogeneous operator. Let $h \in P$ be defined by

$$h(t) = t^{\alpha-1}, \quad t \in [0,1].$$

Using Lemma 2 and (ii), we have

$$
\begin{aligned}
A(h,h)(t) &= \int_0^1 G_{\alpha,\beta}(t,s) f\bigl(s, h(s), h(s)\bigr)\,\mathrm{d}s \\
&\leqslant \frac{h(t)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} f(s,1,0)\,\mathrm{d}s, \quad t \in [0,1].
\end{aligned}
$$

Again, using Lemma 2 and (ii), we have

$$
\begin{aligned}
A(h,h)(t) &= \int_0^1 G_{\alpha,\beta}(t,s) f\bigl(s, h(s), h(s)\bigr)\,\mathrm{d}s \\
&\geqslant \frac{h(t)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} \bigl[1-(1-s)^{\beta}\bigr] f(s,0,1)\,\mathrm{d}s, \quad t \in [0,1].
\end{aligned}
$$

Denote

$$
\mu_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} \bigl[1-(1-s)^{\beta}\bigr] f(s,0,1)\,\mathrm{d}s
$$

and

$$
\mu_2 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} f(s,1,0)\,\mathrm{d}s.
$$

Then we have

$$
\mu_1 h \preceq A(h,h) \preceq \mu_2 h.
$$

On the other hand, from (ii) and (v), we have

$$
f(s,1,0) \geqslant f(s,0,1) \geqslant \delta_0 g(s,0) \geqslant 0.
$$

Since $m(\{t \in [0,1]\colon\ g(t,0) \neq 0\}) > 0$, we have

$$
\mu_2 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} f(s,1,0)\,\mathrm{d}s \geqslant \frac{\delta_0}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} g(s,0)\,\mathrm{d}s > 0
$$

and

$$
\begin{aligned}
\mu_1 &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} \bigl[1-(1-s)^{\beta}\bigr] f(s,0,1)\,\mathrm{d}s \\
&\geqslant \frac{\delta_0}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} \bigl[1-(1-s)^{\beta}\bigr] g(s,0)\,\mathrm{d}s > 0.
\end{aligned}
$$

Thus we proved that $A(h,h) \in P_h$. Similarly,

$$\frac{h(t)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1}\bigl[1-(1-s)^{\beta}\bigr] g(s,0)\,\mathrm{d}s \leqslant Bh(t) \leqslant \frac{h(t)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} g(s,1)\,\mathrm{d}s, \quad t \in [0,1].$$

Denote

$$\lambda_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1}\bigl[1-(1-s)^{\beta}\bigr] g(s,0)\,\mathrm{d}s$$

and

$$\lambda_2 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} g(s,1)\,\mathrm{d}s.$$

Then we have

$$\lambda_1 h \preceq Bh \preceq \lambda_2 h.$$

From (ii) and the condition: $m(\{t \in [0,1]\colon g(t,0) \neq 0\}) > 0$, we have $\lambda_1 > 0$ and $\lambda_2 > 0$. Thus we proved that $Bh \in P_h$. Let $u, v \in P$. From (v), we have

$$A(u,v)(t) = \int_0^1 G_{\alpha,\beta}(t,s) f\bigl(s,u(s),v(s)\bigr)\,\mathrm{d}s \geqslant \delta_0 \int_0^1 G_{\alpha,\beta}(t,s) g\bigl(s,u(s)\bigr)\,\mathrm{d}s = \delta_0 Bu(t), \quad t \in [0,1].$$

Thus we have

$$A(u,v) \succeq \delta_0 Bu, \quad u \in P.$$

Finally, applying Lemma 3, we obtain the desired results. □

We end the paper with the following example.

Example. Consider the boundary value problem

$$\begin{aligned} &D_{0^+}^{9/2} x(t) + 2\bigl(t^2 + \sqrt{x(t)}\,\bigr) + \frac{1}{\sqrt{x(t)+1}} = 0, \quad t \in (0,1), \\ &x(0) = x'(0) = x''(0) = x'''(0) = 0, \\ &\bigl[D_{0^+}^{5/2} x(t)\bigr]_{t=1} = 0. \end{aligned} \tag{5}$$

Consider the functions $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ and $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ defined by

$$f(t,u,v) = t^2 + \sqrt{u} + \frac{1}{\sqrt{v+1}}, \quad g(t,u) = \sqrt{u} + t^2, \quad t \in [0,1],\ u,v \geqslant 0.$$

Then (5) is equivalent to

$$\begin{gathered} D_{0^+}^{9/2} x(t) + f\big(t, x(t), x(t)\big) + g\big(t, x(t)\big) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = x''(0) = x'''(0) = 0, \\ \big[D_{0^+}^{5/2} x(t)\big]_{t=1} = 0. \end{gathered} \tag{6}$$

Let us check that all the required conditions of Theorem 1 are satisfied. Clearly, the functions $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ and $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous with

$$m\big(\big\{t \in [0,1]\colon\ g(t,0) \neq 0\big\}\big) = 1.$$

We observe easily that $f(t,x,y)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$ and $y \in [0,+\infty)$, decreasing in $y \in [0,+\infty)$ for fixed $t \in [0,1]$ and $x \in [0,+\infty)$, and $g(t,x)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$. For all $\lambda \in (0,1)$, $t \in [0,1]$ and $u \geqslant 0$, we have

$$g(t,\lambda u) = \sqrt{\lambda u} + t^2 = \sqrt{\lambda}\sqrt{u} + t^2 \geqslant \lambda\big(\sqrt{u} + t^2\big) = \lambda g(t,u).$$

For all $\lambda \in (0,1)$, $t \in [0,1]$, $u,v \geqslant 0$, we have

$$\begin{aligned} f(t,\lambda u,\lambda^{-1} v) &= t^2 + \sqrt{\lambda u} + \frac{1}{\sqrt{\lambda^{-1} v + 1}} = t^2 + \sqrt{\lambda u} + \frac{\sqrt{\lambda}}{\sqrt{v+\lambda}} \\ &\geqslant \sqrt{\lambda}\left(t^2 + \sqrt{u} + \frac{1}{\sqrt{v+1}}\right) = \lambda^{1/2} f(t,u,v). \end{aligned}$$

For all $t \in [0,1]$, $u,v \geqslant 0$, we have

$$f(t,u,v) = t^2 + \sqrt{u} + \frac{1}{\sqrt{v+1}} \geqslant t^2 + \sqrt{u} = 1 \cdot g(t,u).$$

Thus we proved that all the hypotheses of Theorem 1 are satisfied. Then we deduce that (5) has one and only one positive solution $x^* \in P_h$, where $h(t) = t^{7/2}$, $t \in [0,1]$.

References

1. C. Bai, Positive solutions for nonlinear fractional differential equations with coefficient that changes sign, *Nonlinear Anal., Theory Methods Appl.*, **64**:677–685, 2006.
2. C. Bai, Triple positive solutions for a boundary value problem of nonlinear fractional differential equation, *Electron. J. Qual. Theory Differ. Equ.*, **24**:1–10, 2008.

3. Z. Bai, H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **311**: 495–505, 2005.
4. C. Cheng, Z. Feng, Y. Su, Positive solutions of fractional differential equations with derivative terms, *Electron. J. Differ. Equ.*, **215**:1–27, 2012.
5. C.S. Goodrich, Existence of a positive solution to a class of fractional differential equations, *Appl. Math. Lett.*, **23**:1050–1055, 2010.
6. C.S. Goodrich, Existence of a positive solution to systems of differential equations of fractional order, *Comput. Math. Appl.*, **62**:1251–1268, 2011.
7. R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
8. M. Jleli, E. Karapinar, B. Samet, Positive solutions for multi-point boundary value problems for singular fractional differential equations, *J. Appl. Math.*, **2014**, Article ID 596123, 7 pp., 2014.
9. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Math. Stud., Vol. 204, Elsevier, Amsterdam, 2006.
10. V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, Cambridge, 2009.
11. G.A. Losa, D. Merlini, T.F. Nonnenmacher, E.R. Weibel (Eds.), *Fractals in Biology and Medicine, Vol. 2*, Birkhäuser, Basel, 1998.
12. K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.
13. I. Podlubny, *Fractional Differential Equations*, Math. Sci. Eng., Vol. 198, Academic Press, San Diego, CA, 1999.
14. S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.
15. X. Xu, D. Jiang, C. Yuan, Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation, *Nonlinear Anal., Theory Methods Appl.*, **71**(10):4676–4688, 2009.
16. W. Yang, Positive solutions for a coupled system of nonlinear fractional differential equations with integral boundary conditions, *Comput. Math. Appl.*, **63**:288–297, 2012.
17. C. Zhai, M. Hao, Fixed point theorems for mixed monotone operators with perturbation and applications to fractional differential equation boundary value problems, *Nonlinear Anal., Theory Methods Appl.*, **75**(4):2542–2551, 2012.