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On fixed points for α–η–ψ-contractive multi-valued mappings in partial metric spaces*

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Abstract. Recently, Samet et al. introduced the notion of α–ψ-contractive type mappings and established some fixed point theorems in complete metric spaces. Successively, Asl et al. introduced the notion of α_*–ψ-contractive multi-valued mappings and gave a fixed point result for these multi-valued mappings. In this paper, we establish results of fixed point for α_*-admissible mixed multi-valued mappings with respect to a function η and common fixed point for a pair (S,T) of mixed multi-valued mappings, that is, α_*-admissible with respect to a function η in partial metric spaces. An example is given to illustrate our result.

Keywords: partial metric space, α–η–ψ-contractive condition, α_*-admissible pair with respect to a function η, fixed point, common fixed point.

1 Introduction

The study of iterative schemes for various classes of contractive and nonexpansive mappings is a central topic in metric fixed point theory. This research started with the work of Banach [6] who proved a classical theorem, known as the Banach contraction principle, for the existence of a unique fixed point for a contraction. The importance of this result is also in the fact that it gives the convergence of an iterative scheme to a unique fixed point. Since Banach's result, there has been a lot of activity in this area and many developments have been taken place (see also [26]). Some authors have also provided results dealing with the existence and approximation of fixed points of certain classes of contractive multi-valued mappings [7, 8, 12, 17, 21, 22].

Let (X,d) be a metric space and let $CB(X)$ denote the collection of all nonempty closed and bounded subsets of X. For $A,B \in CB(X)$, define

$$H(A,B) := \max\Big\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\Big\},$$

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where $d(x,A) := \inf\{d(x,a)\colon a \in A\}$ is the distance of a point x to the set A. It is known that H is a metric on $CB(X)$, called the Hausdorff metric induced by the metric d.

Definition 1. Let (X,d) be a metric space. An element x in X is said to be a fixed point of a multi-valued mapping $T : X \to CB(X)$ if $x \in Tx$.

We recall that $T : X \to CB(X)$ is said to be a multi-valued contraction mapping if there exists $k \in [0,1)$ such that

$$H(Tx,Ty) \leqslant kd(x,y) \quad \text{for all } x,y \in X.$$

The study of fixed points for multi-valued contractions using the Hausdorff metric was initiated by Nadler [17] who proved the following theorem.

Theorem 1. (*See* [17].) *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued contraction mapping. Then there exists $x \in X$ such that $x \in Tx$.*

Later on, an interesting and rich fixed point theory was developed. The theory of multi-valued mappings has application in control theory, convex optimization, differential equations and economics (see also [11,15]). On the other hand, Matthews [16] introduced the concept of a partial metric as a part of the study of denotational semantics of dataflow networks. He gave a modified version of the Banach contraction principle, more suitable in this context (see also [2,3,10,13,19,20,27]). In fact, (complete) partial metric spaces constitute a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via domain theory (see, [9,14,16,23,25,28]). More recently, Aydi et al. [5] introduced a notion of partial Hausdorff metric type, associated to a partial metric, and proved an analogous to the well known Nadler's fixed point theorem [17] in the setting of partial metric spaces. Very recently, Romaguera [24] introduced the concept of mixed multi-valued mappings, so that both a self mapping $T : X \to X$ and a multi-valued mapping $T : X \to CB^p(X)$ (the family of all non-empty, closed and bounded subsets of a partial metric space X), are mixed multi-valued mappings. In this paper, we establish results of fixed point for α_*-admissible mixed multi-valued mappings with respect to a function η. Also, we prove results of common fixed point for a pair (S,T) of multi-valued mappings, that is, α_*-admissible with respect to a function η in the setting of partial metric spaces.

In the sequel, the letters $\mathbb{R}$ and $\mathbb{N}$ will denote the set of all real numbers and the set of all positive integer numbers, respectively.

2 Preliminaries

First, we recall some definitions of partial metric spaces that can be found in [10,16,18,19,23]. A partial metric on a nonempty set X is a function $p : X \times X \to [0,+\infty)$ such that for all $x,y,z \in X$:

(p1) $x = y \iff p(x,x) = p(x,y) = p(y,y)$;
(p2) $p(x,x) \leqslant p(x,y)$;

(p3) $p(x, y) = p(y, x)$;
(p4) $p(x, y) \leqslant p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that if $p(x, y) = 0$, then from (p1) and (p2) it follows that $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of partial metric space is the pair $([0, +\infty), p)$, where $p(x, y) = \max\{x, y\}$.

Each partial metric p on X generates a T_0 topology τ_p on X, which has as a base the family of open p-balls $\{B_p(x, \epsilon)\colon x \in X, \epsilon > 0\}$, where

$$B_p(x, \epsilon) = \bigl\{y \in X\colon p(x, y) < p(x, x) + \epsilon\bigr\} \tag{1}$$

for all $x \in X$, $\epsilon > 0$.

Let (X, p) be a partial metric space. A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to+\infty} p(x, x_n)$.

A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n, x_m)$. A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to+\infty} p(x_n, x_m)$.

A sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{n,m\to+\infty} p(x_n, x_m) = 0$. We say that (X, p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.

Now, we recall the definition of partial Hausdorff metric and some properties that can be found in [1]. Let $CB^p(X)$ be the family of all nonempty, closed and bounded subsets of the partial metric space (X, p), induced by the partial metric p. Note that closedness is taken from (X, τ_p) and boundedness is given as follows: A is a bounded subset in (X, p) if there exist $x_0 \in X$ and $M \geqslant 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$, that is, $p(x_0, a) < p(x_0, x_0) + M$.

For $A, B \in CB^p(X)$ and $x \in X$, define

$$\begin{aligned} p(x, A) &= \inf\bigl\{p(x, a)\colon a \in A\bigr\},\\ \delta_p(A, B) &= \sup\bigl\{p(a, B)\colon a \in A\bigr\},\\ \delta_p(B, A) &= \sup\bigl\{p(b, A)\colon b \in B\bigr\}. \end{aligned}$$

Remark 1. (See [4].) Let (X, p) be a partial metric space and A any nonempty set in (X, p), then

$$a \in \bar{A} \quad \text{if and only if } p(a, A) = p(a, a), \tag{2}$$

where $\bar{A}$ denotes the closure of A with respect to the partial metric p. Note that A is closed in (X, p) if and only if $A = \bar{A}$.

In the following proposition, we bring some properties of the mapping $\delta_p : CB^p(X) \times CB^p(X) \to [0, +\infty)$.

Proposition 1. (*See* [1, *Prop.* 2.2].) *Let* (X, p) *be a partial metric space. For any* $A, B, C \in CB^p(X)$*, we have the following*:

(i) $\delta_p(A, A) = \sup\{p(a, a)\colon\ a \in A\}$;
(ii) $\delta_p(A, A) \leqslant \delta_p(A, B)$;
(iii) $\delta_p(A, B) = 0$ *implies that* $A \subseteq B$;
(iv) $\delta_p(A, B) \leqslant \delta_p(A, C) + \delta_p(C, B) - \inf_{c\in C} p(c, c)$.

Let (X, p) be a partial metric space. For $A, B \in CB^p(X)$, define

$$H_p(A, B) = \max\bigl\{\delta_p(A, B),\ \delta_p(B, A)\bigr\}.$$

In the following proposition, we bring some properties of the mapping H_p.

Proposition 2. (*See* [1, *Prop*. 2.3].) *Let* (X, p) *be a partial metric space. For all* $A, B, C \in CB^p(X)$, *we have*:

(h1) $H_p(A, A) \leqslant H_p(A, B)$;
(h2) $H_p(A, B) = H_p(B, A)$;
(h3) $H_p(A, B) \leqslant H_p(A, C) + H_p(C, B) - \inf_{c\in C} p(c, c)$.

Corollary 1. (*See* [1, *Cor*. 2.4].) *Let* (X, p) *be a partial metric space. For* $A, B \in CB^p(X)$ *the following holds*:

$$H_p(A, B) = 0 \quad \textit{implies that } A = B.$$

Remark 2. The converse of Corollary 1 is not true in general as shown by the following example.

Example 1. (See [1, Ex. 2.6].) Let $X = [0, 1]$ be endowed with the partial metric $p : X \times X \to [0, +\infty)$ defined by

$$p(x, y) = \max\{x, y\} \quad \text{for all } x, y \in X.$$

From (i) of Proposition 1, we have

$$H_p(X, X) = \delta_p(X, X) = \sup\{x : 0 \leqslant x \leqslant 1\} = 1 \neq 0.$$

In view of Proposition 2 and Corollary 1, we call the mapping $H_p : CB^p(X) \times CB^p(X) \to [0, +\infty)$, a partial Hausdorff metric induced by p.

Remark 3. It is easy to show that any Hausdorff metric is a partial Hausdorff metric. The converse is not true (see Example 1).

3 Main results

In [24], Romaguera introduced the concept of mixed multi-valued mappings as follows.

Definition 2. Let (X, p) be a partial metric space. $T : X \to X \cup CB^p(X)$ is called a mixed multi-valued mapping on X if T is a multi-valued mapping on X such that for each $x \in X$, $Tx \in X$ or $Tx \in CB^p(X)$.

As said above, both a self mapping $T : X \to X$ and a multi-valued mapping $T : X \to CB^p(X)$, are mixed multi-valued mappings. This approach is motivated, in part, by the fact that $CB^p(X)$ may be empty.

Now, we consider the family

$$\Psi = \{(\psi_1, \dots, \psi_5)\colon \psi_i : [0, +\infty) \to [0, +\infty),\ i = 1, \dots, 5\}$$

such that:

(i) ψ_2, ψ_5 are nondecreasing and ψ_4 is increasing;
(ii) $\psi_1(t), \psi_2(t), \psi_3(t) \leqslant \psi_4(t)$ for all $t > 0$;
(iii) $\psi_4(s + t) \leqslant \psi_4(s) + \psi_4(t)$ for all $s, t > 0$;
(iv) $\psi_1(t), \psi_2(t), \psi_5(t)$ are continuous in $t = 0$ and $\psi_1(0) = \psi_2(0) = \psi_5(0) = 0$;
(v) $\sum_{n=1}^{+\infty} \psi_4^n(t) < +\infty$ for all $t > 0$.

The following lemma is obvious.

Lemma 1. *If $(\psi_1, \dots, \psi_5) \in \Psi$, then $\psi_4(t) < t$ for all $t > 0$.*

Let (X, p) be a partial metric space and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions with η bounded. In the sequel we denote

$$\alpha_*(A, B) = \inf_{x \in A,\, y \in B} \alpha(x, y) \quad \text{and} \quad \eta_*(A, B) = \sup_{x \in A,\, y \in B} \eta(x, y)$$

for every $A, B \subset X$.

Definition 3. Let (X, p) be a partial metric space, $T : X \to X \cup CB^p(X)$ a mixed multi-valued mapping and $\alpha : X \times X \to [0, +\infty)$ a function. We say that T is an α_*-admissible mixed multi-valued mapping if

$$\alpha(x, y) \geqslant 1 \quad \text{implies} \quad \alpha_*(Tx, Ty) \geqslant 1, \quad x, y \in X.$$

Definition 4. Let (X, p) be a partial metric space, $S, T : X \to X \cup CB^p(X)$ be two mixed multi-valued mappings and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions with η bounded. We say that the pair (S, T) is α_*-admissible with respect to η if:

$$\alpha(x, y) \geqslant \eta(x, y) \quad \text{implies} \quad \alpha_*(Sx, Ty) \geqslant \eta_*(Sx, Ty), \quad x, y \in X.$$

We say that T is an α_*-admissible mixed multi-valued mapping with respect to η if the pair (T, T) is α_*-admissible with respect to η.

If we take, $\eta(x, y) = 1$ for all $x, y \in X$, then the definition of α_*-admissible mixed multi-valued mapping with respect to η reduces to Definition 3.

The following theorem is one of our main results.

Theorem 2. *Let (X, p) be a 0-complete partial metric space and let $T : X \to X \cup CB^p(X)$ be a mixed multi-valued mapping. Assume that there exist $(\psi_1, \dots, \psi_5) \in \Psi$*

and two functions $\alpha, \eta : X \times X \to [0, +\infty)$ *with* η *bounded, such that*

$$\inf_{u \in Tx} \eta(x, u) \leqslant \alpha(x, y) \quad \textit{implies}$$

$$H(Tx, Ty) \leqslant \max\left\{\psi_1\big(p(x, y)\big),\ \psi_2\big(p(x, Tx)\big),\ \psi_3\big(p(y, Ty)\big),\ \frac{\psi_4(p(x, Ty)) + \psi_5(p(y, Tx) - p(y, y))}{2}\right\} \tag{3}$$

for all $x, y \in X$*. Also suppose that the following assertions hold*:

(i) T *is an* α_**-admissible mixed multi-valued mapping with respect to* η;

(ii) *there exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geqslant \eta(x_0, x_1)$;

(iii) *for a sequence* $\{x_n\} \subset X$ *such that* $\alpha(x_n, x_{n+1}) \geqslant \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x$ *as* $n \to +\infty$*, then either*

$$\inf_{u_n \in Ty_n} \eta(y_n, u_n) \leqslant \alpha(y_n, x) \quad \textit{or} \quad \inf_{v_n \in Tz_n} \eta(z_n, v_n) \leqslant \alpha(z_n, x)$$

holds for all $n \in \mathbb{N}$*, where* $\{y_n\}$ *and* $\{z_n\}$ *are two given sequences such that* $y_n \in Tx_n$ *and* $z_n \in Ty_n$ *for all* $n \in \mathbb{N}$*.*

Then T *has a fixed point.*

Proof. By (ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geqslant \eta(x_0, x_1)$. This implies that $\alpha(x_0, x_1) \geqslant \eta(x_0, x_1) \geqslant \inf_{y \in Tx_0} \eta(x_0, y)$. If $x_0 = x_1$ or $x_1 \in Tx_1$, then x_1 is a fixed point of T. Assume that $x_1 \notin Tx_1$ and that Tx_1 is not a singleton. Therefore, from (3), we have

$$\begin{aligned}
0 < p(x_1, Tx_1) &\leqslant H(Tx_0, Tx_1) \\
&\leqslant \max\left\{\psi_1\big(p(x_0, x_1)\big),\ \psi_2(p(x_0, Tx_0)),\ \psi_3(p(x_1, Tx_1)),\ \frac{\psi_4(p(x_0, Tx_1)) + \psi_5(p(x_1, Tx_0) - p(x_1, x_1))}{2}\right\} \\
&\leqslant \max\left\{\psi_1\big(p(x_0, x_1)\big),\ \psi_2\big(p(x_0, x_1)\big),\ \psi_3\big(p(x_1, Tx_1)\big),\ \frac{\psi_4(p(x_0, x_1)) + \psi_4(p(x_1, Tx_1))}{2}\right\} \\
&\leqslant \max\big\{\psi_1\big(p(x_0, x_1)\big),\ \psi_2\big(p(x_0, x_1)\big),\ \psi_3\big(p(x_1, Tx_1)\big),\\
&\qquad \max\big\{\psi_4\big(p(x_0, x_1)\big),\ \psi_4\big(p(x_1, Tx_1)\big)\big\}\big\} \\
&= \max\big\{\psi_4\big(p(x_0, x_1)\big),\ \psi_4\big(p(x_1, Tx_1)\big)\big\}.
\end{aligned}$$

Now, if

$$\max\big\{\psi_4\big(p(x_0, x_1)\big),\ \psi_4\big(p(x_1, Tx_1)\big)\big\} = \psi_4\big(p(x_1, Tx_1)\big),$$

then

$$0 < p(x_1, Tx_1) \leqslant H(Tx_0, Tx_1) \leqslant \psi_4\big(p(x_1, Tx_1)\big) < p(x_1, Tx_1),$$

which is a contradiction. Hence,

$$0 < p(x_1, Tx_1) \leqslant H(Tx_0, Tx_1) \leqslant \psi_4\big(p(x_0, x_1)\big).$$

If $q > 1$, then

$$0 < p(x_1, Tx_1) < qH(Tx_0, Tx_1) \leqslant q\psi_4\big(p(x_0, x_1)\big).$$

So there exists $x_2 \in Tx_1$ such that

$$0 < p(x_1, x_2) < qH(Tx_0, Tx_1) \leqslant q\psi_4\big(p(x_0, x_1)\big). \tag{4}$$

If $Tx_1 = \{x_2\}$ is a singleton, again by (3), we get

$$0 < p(x_1, x_2) \leqslant H(Tx_0, Tx_1) \leqslant \psi_4\big(p(x_0, x_1)\big)$$

and so (4) holds.

Note that $x_1 \neq x_2$. Also, since T is α_*-admissible with respect to η, we have $\alpha_*(Tx_0, Tx_1) \geqslant \eta_*(Tx_0, Tx_1)$. This implies

$$\alpha(x_1, x_2) \geqslant \alpha_*(Tx_0, Tx_1) \geqslant \eta_*(Tx_0, Tx_1) \geqslant \eta(x_1, x_2) \geqslant \inf_{y \in Tx_1} \eta(x_1, y).$$

Therefore, from (3), we have

$$\begin{aligned} H(Tx_1, Tx_2) &\leqslant \max\bigg\{\psi_1\big(p(x_1, x_2)\big),\ \psi_2\big(p(x_1, Tx_1)\big),\ \psi_3\big(p(x_2, Tx_2)\big), \\ &\qquad \frac{\psi_4(p(x_1, Tx_2)) + \psi_5(p(x_2, Tx_1) - p(x_2, x_2))}{2}\bigg\} \\ &\leqslant \psi_4\big(p(x_1, x_2)\big). \end{aligned} \tag{5}$$

Put $t_0 = p(x_0, x_1) > 0$. Then from (4), we deduce that $p(x_1, x_2) < q\psi_4(t_0)$. Now, since ψ_4 is increasing, we deduce $\psi_4(p(x_1, x_2)) < \psi_4(q\psi_4(t_0))$. Put

$$q_1 = \frac{\psi_4(q\psi_4(t_0))}{\psi_4(p(x_1, x_2))} > 1.$$

If $x_2 \in Tx_2$, then x_2 is a fixed point of T. Hence, we suppose that $x_2 \notin Tx_2$. Then

$$0 < p(x_2, Tx_2) \leqslant H(Tx_1, Tx_2) < q_1 H(Tx_1, Tx_2).$$

So there exists $x_3 \in Tx_2$ (obviously $x_3 = Tx_2$ if Tx_2 is a singleton) such that

$$0 < p(x_2, x_3) < q_1 H(Tx_1, Tx_2)$$

and from (5), we get

$$0 < p(x_2, x_3) < q_1 H(Tx_1, Tx_2) \leqslant q_1 \psi_4\big(p(x_1, x_2)\big) = \psi_4\big(q\psi_4(t_0)\big).$$

Again, since ψ_4 is increasing, then $\psi_4(p(x_2, x_3)) < \psi_4(\psi_4(q\psi_4(t_0)))$. Put

$$q_2 = \frac{\psi_4(\psi_4(q\psi_4(t_0)))}{\psi_4(p(x_2, x_3))} > 1.$$

If $x_3 \in Tx_3$, then x_3 is a fixed point of T. Hence, we assume that $x_3 \notin Tx_3$. Then

$$0 < p(x_3, Tx_3) \leqslant H(Tx_2, Tx_3) < q_2 H(Tx_2, Tx_3).$$

So there exists $x_4 \in Tx_3$ (obviously $x_4 = Tx_3$ if Tx_3 is a singleton) such that

$$0 < p(x_3, x_4) < q_2 H(Tx_2, Tx_3). \tag{6}$$

Clearly, $x_2 \neq x_3$. Again, since T is α_*-admissible with respect to η,

$$\alpha(x_2, x_3) \geqslant \alpha_*(Tx_1, Tx_2) \geqslant \eta_*(Tx_1, Tx_2) \geqslant \eta(x_2, x_3) \geqslant \inf_{y \in Tx_2} \eta(x_2, y).$$

Then from (3), we have

$$\begin{aligned} H(Tx_2, Tx_3) \leqslant \max\bigg\{ & \psi_1\big(p(x_2, x_3)\big),\ \psi_2\big(p(x_2, Tx_2)\big),\ \psi_3\big(p(x_3, Tx_3)\big), \\ & \frac{\psi_4(p(x_2, Tx_3)) + \psi_5(p(x_3, Tx_2) - p(x_3, x_3))}{2} \bigg\} \\ \leqslant \psi_4\big(p(x_2, x_3)\big). & \end{aligned} \tag{7}$$

Thus from (6) and (7), we deduce that

$$0 < p(x_3, x_4) < q_2 H(Tx_2, Tx_3) \leqslant q_2 \psi_4\big(p(x_2, x_3)\big) = \psi_4\big(\psi_4\big(q\psi_4(t_0)\big)\big).$$

By continuing this process, we obtain a sequence $\{x_n\} \subset X$ such that $x_n \in Tx_{n-1}$, $x_n \neq x_{n-1}$, $\alpha(x_{n-1}, x_n) \geqslant \eta(x_{n-1}, x_n)$ and $p(x_n, x_{n+1}) \leqslant \psi_4^{n-1}(q\psi_4(t_0))$ for all $n \in \mathbb{N}$. Now for all $m > n$, we can write

$$p(x_n, x_m) \leqslant \sum_{k=n}^{m-1} p(x_k, x_{k+1}) \leqslant \sum_{k=n}^{m-1} \psi_4^{k-1}\big(q\psi_4(t_0)\big).$$

Therefore, $\{x_n\}$ is a 0-Cauchy sequence. Since, (X, p) is a 0-complete partial metric space, then there exists $z \in X$ such that $p(x_n, z) \to p(z, z) = 0$ as $n \to +\infty$. Then from (iii), either

$$\inf_{u_n \in Ty_n} \eta(y_n, u_n) \leqslant \alpha(y_n, z) \quad \text{or} \quad \inf_{v_n \in Tz_n} \eta(z_n, v_n) \leqslant \alpha(z_n, z)$$

holds for all $n \in \mathbb{N}$, where $\{y_n\}$ and $\{z_n\}$ are two given sequences such that $y_n \in Tx_n$ and $z_n \in Ty_n$ for all $n \in \mathbb{N}$. Here $x_{n-1} \in Tx_{n-2}$ and $x_n \in Tx_{n-1}$.

Therefore, either

$$\inf_{u_n \in Tx_{n-1}} \eta(x_{n-1}, u_n) \leqslant \alpha(x_{n-1}, z) \quad \text{or} \quad \inf_{v_n \in Tx_n} \eta(x_n, v_n) \leqslant \alpha(x_n, z)$$

holds for all $n \in \mathbb{N}$. If $p(z, Tz) > 0$, from (3), we have

$$\begin{aligned} p(z,Tz) &\leqslant H(Tx_{n-1},Tz) + p(x_n,z) - p(x_n,x_n) \\ &\leqslant \max\biggl\{\psi_1\bigl(p(x_{n-1},z)\bigr),\, \psi_2\bigl(p(x_{n-1},Tx_{n-1})\bigr),\, \psi_3\bigl(p(z,Tz)\bigr), \\ &\qquad \frac{\psi_4(p(x_{n-1},Tz)) + \psi_5(p(z,Tx_{n-1}))}{2}\biggr\} + p(x_n,z) \\ &\leqslant \max\biggl\{\psi_1\bigl(p(x_{n-1},z)\bigr),\, \psi_2\bigl(p(x_{n-1},x_n)\bigr),\, \psi_3\bigl(p(z,Tz)\bigr), \\ &\qquad \frac{\psi_4(p(x_{n-1},z) + p(z,Tz)) + \psi_5(p(z,x_n))}{2}\biggr\} + p(x_n,z) \end{aligned}$$

or

$$\begin{aligned} p(z,Tz) &\leqslant H(Tx_n,Tz) + p(x_{n+1},z) - p(x_{n+1},x_{n+1}) \\ &\leqslant \max\biggl\{\psi_1\bigl(p(x_n,z)\bigr),\, \psi_2\bigl(p(x_n,Tx_n)\bigr),\, \psi_3\bigl(p(z,Tz)\bigr), \\ &\qquad \frac{\psi_4(p(x_n,Tz)) + \psi_5(p(z,Tx_n))}{2}\biggr\} + p(x_{n+1},z) \\ &\leqslant \max\biggl\{\psi_1\bigl(p(x_n,z)\bigr),\, \psi_2\bigl(p(x_n,x_{n+1})\bigr),\, \psi_3\bigl(p(z,Tz)\bigr), \\ &\qquad \frac{\psi_4(p(x_n,z) + p(z,Tz)) + \psi_5(p(z,x_{n+1}))}{2}\biggr\} + p(x_{n+1},z) \end{aligned}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to +\infty$ in the above inequalities, we get

$$p(z,Tz) \leqslant \psi_4\bigl(p(z,Tz)\bigr) < p(z,Tz)$$

a contradiction. Thus $p(z, Tz) = 0$. If Tz is a singleton, then $z = Tz$. If Tz is not a singleton, from $p(z, Tz) = 0 = p(z, z)$, by Remark 1, we deduce $z \in Tz$. Thus z is a fixed point of T. □

If in Theorem 2, we assume $\eta(x, y) = 1$ for all $x, y \in X$, then we obtain the following corollary.

Corollary 2. *Let (X, p) be a 0-complete partial metric space and let $T : X \to X \cup CB^p(X)$ be a mixed multi-valued mapping. Assume that there exist $(\psi_1, \dots, \psi_5) \in \Psi$*

and a function $\alpha : X \times X \to [0, +\infty)$*, such that*

$$H(Tx, Ty) \leqslant \max\biggl\{\psi_1\bigl(p(x, y)\bigr),\ \psi_2\bigl(p(x, Tx)\bigr),\ \psi_3\bigl(p(y, Ty)\bigr),\\ \frac{\psi_4(p(x, Ty)) + \psi_5(p(y, Tx) - p(y, y))}{2}\biggr\} \tag{8}$$

for all $x, y \in X$ *with* $\alpha(x, y) \geqslant 1$*. Also suppose the following assertions hold*:

(i) T *is an* α_**-admissible mixed multi-valued mapping*;

(ii) *there exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $\alpha(x_0, x_1) \geqslant 1$;

(iii) *for a sequence* $\{x_n\} \subset X$ *such that* $\alpha(x_n, x_{n+1}) \geqslant 1$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x$ *as* $n \to +\infty$*, then either*

$$\alpha(y_n, x) \geqslant 1 \quad \textit{or} \quad \alpha(z_n, x) \geqslant 1$$

holds for all $n \in \mathbb{N}$ *where* $\{y_n\}$ *and* $\{z_n\}$ *are two given sequences such that* $y_n \in Tx_n$ *and* $z_n \in Ty_n$ *for all* $n \in \mathbb{N}$.

Then T *has a fixed point.*

Example 2. Let $X = \{1, 2, 3, 4\}$ and $p : X \times X \to [0, +\infty)$ be defined by $p(1, 1) = p(2, 2) = p(4, 4) = 1/6$, $p(3, 3) = 0$, $p(1, 2) = p(1, 4) = p(2, 4) = p(3, 4) = 1/2$, $p(1, 3) = 1/4$, $p(2, 3) = 1/3$ and $p(x, y) = p(y, x)$ for all $x, y \in X$. Let $T : X \to CB^p(X)$ be defined by $T1 = \{3\}$, $T2 = \{1\}$, $T3 = \{3\}$ and $T4 = \{1, 4\}$. Clearly, (X, p) is a 0-complete partial metric space and Tx is a bounded closed subset of X for all $x \in X$. Let $\alpha : X \times X \to [0, +\infty)$ be defined by $\alpha(1, 1) = \alpha(1, 3) = \alpha(2, 3) = \alpha(3, 3) = \alpha(3, 1) = \alpha(3, 2) = 1$ and $\alpha(x, y) = 0$ otherwise. Now, let $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5 : [0, +\infty) \to [0, +\infty)$ be defined by $\psi_1(t) = t/2$, $\psi_2(t) = 2t/3$, $\psi_3(t) = t/2$, $\psi_4(t) = 3t/4$ and $\psi_5(t) = 5t/6$, then $(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) \in \varPsi$.

Now, we have:

$$\begin{aligned}
H(T1, T1) &= H\bigl(\{3\}, \{3\}\bigr) = 0 \leqslant \psi_1\bigl(p(1, 1)\bigr),\\
H(T1, T3) &= H\bigl(\{3\}, \{3\}\bigr) = 0 \leqslant \psi_1\bigl(p(1, 3)\bigr),\\
H(T2, T3) &= H\bigl(\{1\}, \{3\}\bigr) = 0.25 \leqslant \psi_3\bigl(p\bigl(2, \{1\}\bigr)\bigr),\\
H(T3, T3) &= H\bigl(\{3\}, \{3\}\bigr) = 0 \leqslant \psi_1\bigl(p(3, 3)\bigr).
\end{aligned}$$

This implies

$$H(Tx, Ty) \leqslant \max\biggl\{\psi_1\bigl(p(x, y)\bigr),\ \psi_2\bigl(p(x, Tx)\bigr),\ \psi_3\bigl(p(y, Ty)\bigr),\\ \frac{\psi_4(p(x, Ty)) + \psi_5[p(y, Tx) - p(y, y)]}{2}\biggr\}$$

for all $x, y \in X$ with $\alpha(x, y) \geqslant 1$. T is an α_*-admissible mixed multi-valued mapping and $x_0 = 1$ satisfies condition (ii). Now, we note that for a sequence $\{x_n\} \subset X$ such that

$\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, we have $x = 3$ and this ensures that (iii) holds. Thus, by Corollary 2 the mixed multi-valued mapping T has a fixed point.

We note that

$$H(T2, T4) = \frac{1}{2} > \max\biggl\{\psi_1\bigl(p(2,4)\bigr),\ \psi_2\bigl(p(2,T2)\bigr),\ \psi_3\bigl(p(4,T4)\bigr), \frac{\psi_4(p(2,T4)) + \psi_5(p(4,T2) - p(4,4))}{2}\biggr\}.$$

4 Common fixed point results

Let (X,p) be a partial metric space, let $\alpha, \eta : X \times X \to [0,+\infty)$ be two functions with η bounded and let $S, T : X \to 2^X$ be two multi-valued mappings on X. We denote

$$\Gamma(Sx, Ty) = \min\Bigl\{\inf_{u \in Sx} \eta(x,u),\ \inf_{v \in Ty} \eta(y,v)\Bigr\} = \Gamma(Ty, Sx).$$

Let $\Phi = \{(\psi_1, \dots, \psi_5)\colon\ \psi_i : [0,+\infty) \to [0,+\infty),\ i = 1, \dots, 5\}$ such that:

(i) ψ_2, ψ_3 are nondecreasing and ψ_4, ψ_5 are increasing;
(ii) $\psi_1(t), \psi_2(t), \psi_3(t) \leqslant \min\{\psi_4(t), \psi_5(t)\}$ for all $t > 0$;
(iii) $\psi_i(s+t) \leqslant \psi_i(s) + \psi_i(t)$ $(i = 4,5)$ for all $s, t > 0$;
(iv) $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ are continuous in $t = 0$ and $\psi_1(0) = \psi_2(0) = \psi_3(0) = 0$;
(v) $\sum_{n=1}^{+\infty} \psi_5^n(t) < +\infty$ for all $t > 0$;
(vi) $\psi_4(t) < t$ for all $t > 0$;
(vii) $\psi_4(\psi_5(t)) = \psi_5(\psi_4(t))$ for all $t > 0$.

The following theorem is our main result on the existence of common fixed point for multi-valued mappings.

Theorem 3. *Let (X,p) be a 0-complete partial metric space and let $S,T : X \to X \cup CB^p(X)$ be two mixed multi-valued mappings on X. Assume that there exist $(\psi_1, \dots, \psi_5) \in \Phi$ and two functions $\alpha, \eta : X \times X \to [0,+\infty)$ with η bounded such that*

$$H(Sx, Ty) \leqslant \max\biggl\{\psi_1\bigl(p(x,y)\bigr),\ \psi_2\bigl(p(x,Sx)\bigr),\ \psi_3\bigl(p(y,Ty)\bigr), \frac{\psi_4(p(x,Ty) - p(x,x)) + \psi_5(p(y,Tx) - p(y,y))}{2}\biggr\} \tag{9}$$

for all $x, y \in X$ with $\alpha(x,y) \geqslant \Gamma(Sx,Ty)$. Also suppose the following assertions hold:

(i) *the pair (S,T) is α_*-admissible with respect to η;*
(ii) *there exist $x_0 \in X$ and $x_1 \in Sx_0$ such that $\alpha(x_0,x_1) \geqslant \eta(x_0,x_1)$;*
(iii) *$\alpha(x,x) \geqslant \Gamma(Sx,Tx)$ for all $x \in X$, which is a fixed point of S or T;*

(iv) *for a sequence* $\{x_n\} \subset X$ *such that* $\alpha(x_n, x_{n+1}) \geqslant \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x$ *as* $n \to +\infty$*, then either*

$$\inf_{u_n \in Sy_n} \eta(y_n, u_n) \leqslant \alpha(y_n, x) \quad \textit{or} \quad \inf_{v_n \in Tz_n} \eta(z_n, v_n) \leqslant \alpha(z_n, x)$$

holds for all $n \in \mathbb{N}$ *where* $\{y_n\}$ *and* $\{z_n\}$ *are two given sequences such that* $y_n \in Tx_n$ *and* $z_n \in Sy_n$ *for all* $n \in \mathbb{N}$.

Then S *and* T *have a common fixed point.*

Proof. From (iii) and (9) it follows that the mixed multi-valued mappings S and T have the same fixed points. Let $x_0 \in X$ and $x_1 \in Sx_0$ be such that $\alpha(x_0, x_1) \geqslant \eta(x_0, x_1)$, then

$$\alpha(x_0, x_1) \geqslant \eta(x_0, x_1) \geqslant \inf_{u \in Sx_0} \eta(x_0, u) \geqslant \Gamma(Sx_0, Tx_1).$$

If $x_0 = x_1$, then x_0 is a common fixed point of S and T. The same holds if $x_1 \in Tx_1$. Hence, we assume that $x_0 \neq x_1$ and $x_1 \notin Tx_1$. Assume that Tx_1 is not a singleton, from (9), we have

$$\begin{aligned}
0 &< p(x_1, Tx_1) \leqslant H(Sx_0, Tx_1) \\
&\leqslant \max\biggl\{\psi_1\bigl(p(x_0, x_1)\bigr),\ \psi_2\bigl(p(x_0, Sx_0)\bigr),\ \psi_3\bigl(p(x_1, Tx_1)\bigr), \\
&\qquad \frac{\psi_4(p(x_0, Tx_1) - p(x_0, x_0)) + \psi_5(p(x_1, Sx_0) - p(x_1, x_1))}{2}\biggr\} \\
&\leqslant \max\biggl\{\psi_1\bigl(p(x_0, x_1)\bigr),\ \psi_2\bigl(p(x_0, x_1)\bigr),\ \psi_3\bigl(p(x_1, Tx_1)\bigr), \\
&\qquad \frac{\psi_4(p(x_0, x_1) + p(x_1, Tx_1) - p(x_1, x_1) - p(x_0, x_0))}{2}\biggr\} \\
&\leqslant \max\bigl\{\psi_1\bigl(p(x_0, x_1)\bigr),\ \psi_2\bigl(p(x_0, x_1)\bigr),\ \psi_3\bigl(p(x_1, Tx_1)\bigr), \\
&\qquad \max\bigl\{\psi_4\bigl(p(x_0, x_1)\bigr),\ \psi_4\bigl(p(x_1, Tx_1)\bigr)\bigr\}\bigr\} \\
&= \max\bigl\{\psi_4\bigl(p(x_0, x_1)\bigr),\ \psi_4\bigl(p(x_1, Tx_1)\bigr)\bigr\}.
\end{aligned}$$

Now, if $\max\{\psi_4(p(x_0, x_1)), \psi_4(p(x_1, Tx_1))\} = \psi_4(p(x_1, Tx_1))$, then

$$0 < p(x_1, Tx_1) \leqslant H(Sx_0, Tx_1) \leqslant \psi_4\bigl(p(x_1, Tx_1)\bigr) < p(x_1, Tx_1),$$

which is a contradiction. Hence,

$$\max\bigl\{\psi_4\bigl(p(x_0, x_1)\bigr),\ \psi_4\bigl(p(x_1, Tx_1)\bigr)\bigr\} = \psi_4\bigl(p(x_0, x_1)\bigr).$$

If $q > 1$, then

$$0 < p(x_1, Tx_1) \leqslant H(Sx_0, Tx_1) < qH(Sx_0, Tx_1)$$

and hence there exists $x_2 \in Tx_1$ such that

$$0 < p(x_1, x_2) < qH(Sx_0, Tx_1) \leqslant q\psi_4\big(p(x_0, x_1)\big). \tag{10}$$

If $Tx_1 = \{x_2\}$ is a singleton, again by (9), we get

$$0 < p(x_1, x_2) \leqslant H(Sx_0, Tx_1) \leqslant \psi_4\big(p(x_0, x_1)\big)$$

and so (10) holds. Note that $x_1 \neq x_2$. Also, since the pair (S, T) is α_*-admissible with respect to η, then $\alpha_*(Sx_0, Ty_1) \geqslant \eta_*(Sx_0, Ty_1)$. This implies

$$\begin{aligned}\alpha(x_1, x_2) &\geqslant \alpha_*(Sx_0, Tx_1) \geqslant \eta_*(Sx_0, Tx_1) \geqslant \eta(x_1, x_2)\\ &\geqslant \inf_{y \in Tx_1} \eta(x_1, y) \geqslant \Gamma(Sx_2, Tx_1).\end{aligned}$$

If $x_2 \in Sx_2$, then x_2 is a common fixed point of S and T. Assume that $x_2 \notin Sx_2$ and that Sx_2 is not a singleton, from (9), we have

$$\begin{aligned}0 &< p(x_2, Sx_2) \leqslant H(Sx_2, Tx_1)\\ &\leqslant \max\bigg\{\psi_1\big(p(x_2, x_1)\big),\ \psi_2\big(p(x_2, Sx_2)\big),\ \psi_3\big(p(x_1, Tx_1)\big),\\ &\qquad \frac{\psi_4(p(x_2, Tx_1) - p(x_2, x_2)) + \psi_5(p(x_1, Sx_2) - p(x_1, x_1))}{2}\bigg\}\\ &\leqslant \max\bigg\{\psi_1\big(p(x_1, x_2)\big),\ \psi_2\big(p(x_2, Sx_2)\big),\ \psi_3\big(p(x_1, x_2)\big),\\ &\qquad \frac{\psi_5(p(x_1, x_2) + p(x_2, Sx_2) - p(x_2, x_2) - p(x_1, x_1))}{2}\bigg\}\\ &\leqslant \max\big\{\psi_5\big(p(x_1, x_2)\big),\ \psi_5\big(p(x_2, Sx_2)\big)\big\}.\end{aligned}$$

Now, if $\max\{\psi_5(p(x_1, x_2)), \psi_5(p(x_2, Sx_2))\} = \psi_5(p(x_2, Sx_2))$, then

$$0 < p(x_2, Sx_2) \leqslant H(Sx_2, Tx_1) \leqslant \psi_5\big(p(x_2, Sx_2)\big) < p(x_2, Sx_2),$$

which is a contradiction. Hence,

$$0 < p(x_2, Sx_2) \leqslant H(Sx_2, Tx_1) \leqslant \psi_5\big(p(x_1, x_2)\big). \tag{11}$$

The same is worth also if Sx_2 is a singleton. Put $t_0 = p(x_0, x_1)$. Then from (10), we have $p(x_1, x_2) < q\psi_4(t_0)$ where $t_0 > 0$. Now, since ψ_5 is increasing, then $\psi_5(p(x_1, x_2)) < \psi_5(q\psi_4(t_0))$. Put

$$q_1 = \frac{\psi_5(q\psi_4(t_0))}{\psi_5(p(x_1, x_2))} > 1.$$

Since $x_2 \in Tx_1$ or $x_2 = Tx_1$, we have

$$0 < p(x_2, Sx_2) \leqslant H(Sx_2, Tx_1) < q_1 H(Sx_2, Tx_1)$$

and hence there exists $x_3 \in Sx_2$ or $x_3 = Sx_2$ such that

$$0 < p(x_2, x_3) \leqslant q_1 H(Sx_2, Tx_1).$$

Now, from (11), we deduce

$$0 < p(x_2, x_3) < q_1 H(Sx_2, Tx_1) \leqslant q_1 \psi_5\big(p(x_1, x_2)\big) = \psi_5\big(q\psi_4(t_0)\big).$$

Clearly, $x_2 \neq x_3$. Again, since the pair (S, T) is α_*-admissible with respect to η, then

$$\begin{aligned}\alpha(x_2, x_3) &\geqslant \alpha_*(Tx_1, Sx_2) \geqslant \eta_*(Tx_1, Sx_2) \geqslant \eta(x_2, x_3)\\ &\geqslant \inf_{y \in Sx_2} \eta(x_2, y) \geqslant \Gamma(Sx_2, Tx_3).\end{aligned}$$

If $x_3 \in Tx_3$ or $x_3 = Tx_3$, then x_3 is a common fixed point of S and T. Assume that $x_3 \notin Tx_3$. Now, from (9) we deduce

$$\begin{aligned}0 &< p(x_3, Tx_3) \leqslant H(Sx_2, Tx_3)\\ &\leqslant \max\bigg\{\psi_1\big(p(x_2, x_3)\big), \psi_2\big(p(x_2, Sx_2)\big), \psi_3\big(p(x_3, Tx_3)\big),\\ &\qquad \frac{\psi_4(p(x_2, Tx_3) - p(x_2, x_2)) + \psi_5(p(x_3, Sx_2) - p(x_3, x_3))}{2}\bigg\}\\ &\leqslant \max\bigg\{\psi_1\big(p(x_2, x_3)\big), \psi_2\big(p(x_2, x_3)\big), \psi_3\big(p(x_3, Tx_3)\big),\\ &\qquad \frac{\psi_4(p(x_2, x_3) + p(x_3, Tx_3) - p(x_3, x_3) - p(x_2, x_2))}{2}\bigg\}\\ &\leqslant \max\big\{\psi_4\big(p(x_2, x_3)\big), \psi_4\big(p(x_3, Tx_3)\big)\big\}.\end{aligned}$$

If $\max\{\psi_4(p(x_2, x_3)), \psi_4(p(x_3, Tx_3))\} = \psi_4(p(x_3, Tx_3))$, then

$$0 < p(x_3, Tx_3) \leqslant H(Sx_2, Tx_3) \leqslant \psi_4\big(p(x_3, Tx_3)\big) < p(x_3, Tx_3),$$

which is a contradiction. Hence,

$$\max\big\{\psi_4\big(p(x_2, x_3)\big), \psi_4\big(p(x_3, Tx_3)\big)\big\} = \psi_4\big(p(x_2, x_3)\big)$$

and so

$$0 < p(x_3, Tx_3) \leqslant H(Sx_2, Tx_3) \leqslant \psi_4\big(p(x_2, x_3)\big). \tag{12}$$

Again, since ψ_4 is increasing, we deduce that

$$\psi_4\big(p(x_2, x_3)\big) < \psi_4\big(\psi_5\big(q\psi_4(t_0)\big)\big).$$

Put

$$q_2 = \frac{\psi_4(\psi_5(q\psi_4(t_0)))}{\psi_4(p(x_2, x_3))} > 1.$$

Then

$$0 < p(x_3, Tx_3) \leqslant H(Sx_2, Tx_3) < q_2 H(Sx_2, Tx_3)$$

and hence there exists $x_4 \in Tx_3$ or $x_4 = Tx_3$ such that

$$0 < p(x_3, x_4) < q_2 H(Sx_2, Tx_3) \leqslant q_2 \psi_4\bigl(p(x_2, x_3)\bigr). \tag{13}$$

Now, from (12) and (13), we deduce that

$$0 < p(x_3, x_4) < q_2 H(Sx_2, Tx_3) \leqslant q_2 \psi_4\bigl(p(x_2, x_3)\bigr) = \psi_4\bigl(\psi_5\bigl(q\psi_4(t_0)\bigr)\bigr).$$

By continuing this process, we obtain a sequence $\{x_n\}$ in X such that $x_{2n} \in Tx_{2n-1}$, $x_{2n+1} \in Sx_{2n}$ and

$$\begin{aligned} p(x_{2n-1}, x_{2n}) &\leqslant (\psi_4\psi_5)^{n-1}\bigl(q\psi_4(t_0)\bigr) \\ p(x_{2n}, x_{2n+1}) &\leqslant \psi_5\bigl[(\psi_4\psi_5)^{n-1}\bigl(q\psi_4(t_0)\bigr)\bigr]. \end{aligned}$$

Now, for all $m > n$, we can write

$$\begin{aligned} p(x_{2n}, x_{2m}) &\leqslant \sum_{k=n}^{m-1} p(x_{2k}, x_{2k+1}) + \sum_{k=n}^{m-1} p(x_{2k+1}, x_{2k+2}) \\ &\leqslant \sum_{k=n}^{m-1} \psi_5^k\bigl(\psi_4^{k-1}\bigl(q\psi_4(t_0)\bigr)\bigr) + \sum_{k=n}^{m-1} \psi_5^k\bigl(\psi_4^k\bigl(q\psi_4(t_0)\bigr)\bigr) \\ &\leqslant 2\sum_{k=n}^{m-1} \psi_5^k\bigl(q\psi_4(t_0)\bigr). \end{aligned}$$

Since $\sum_{k=1}^{+\infty} \psi_5^k(q\psi_4(t_0)) < +\infty$, we get $\lim_{n\to+\infty} p(x_{2n}, x_{2m}) = 0$. Similary, we obtain

$$\lim_{n\to+\infty} p(x_{2n+1}, x_{2m+1}) = 0, \qquad \lim_{n\to+\infty} p(x_{2n+1}, x_{2m}) = 0,$$
$$\lim_{n\to+\infty} p(x_{2n}, x_{2m+1}) = 0.$$

This implies that $\lim_{n,m\to+\infty} p(x_n, x_m) = 0$ and so $\bigl\{x_n\bigr\}$ is a 0-Cauchy sequence. Since (X, p) is a 0-complete partial metric space, then there exists $z \in X$ with $p(z, z) = 0$ such that $x_n \to z$ as $n \to +\infty$. Then from (ii) either

$$\inf_{u\in Sy_n} \eta(y_n, u) \leqslant \alpha(y_n, z) \quad \text{or} \quad \inf_{v\in Tz_n} \eta(z_n, v) \leqslant \alpha(z_n, z)$$

holds for all $n \in \mathbb{N}$, where $\{y_n\}$ and $\{z_n\}$ are two given sequences such that $y_n \in Tx_n$ and $z_n \in Sy_n$ for all $n \in \mathbb{N}$. Here $x_{2n} \in Tx_{2n-1}$ and $x_{2n+1} \in Sx_{2n}$. Therefore, either

$$\inf_{u\in Sx_{2n}} \eta(x_{2n}, u) \leqslant \alpha(x_{2n}, z) \quad \text{or} \quad \inf_{v\in Tx_{2n+1}} \eta(x_{2n+1}, v) \leqslant \alpha(x_{2n+1}, z)$$

holds for all $n \in \mathbb{N}$. So from (9) and $p(z, z) = 0$ we have

$$\begin{aligned}0 < p(z,Tz) &\leqslant H(Sx_{2n},Tz) + p(x_{2n+1},z) - p(x_{2n+1},x_{2n+1}) \\ &\leqslant \max\biggl\{\psi_1\bigl(p(x_{2n},z)\bigr),\ \psi_2\bigl(p(x_{2n},Sx_{2n})\bigr),\ \psi_3\bigl(p(z,Tz)\bigr), \\ &\qquad \frac{\psi_4(p(x_{2n},Tz) - p(x_{2n},x_{2n})) + \psi_5\bigl(p(z,Sx_{2n})\bigr)}{2}\biggr\} \\ &\quad + p(x_{2n+1},z)\end{aligned}$$

or

$$\begin{aligned}0 < p(z,Sz) &\leqslant H(Tx_{2n+1},Sz) + p(x_{2n+2},z) - p(x_{2n+2},x_{2n+2}) \\ &\leqslant \max\biggl\{\psi_1\bigl(p(x_{2n+1},z)\bigr),\ \psi_2\bigl(p(z,Sz)\bigr),\ \psi_3\bigl(p(x_{2n+1},Tx_{2n+1})\bigr), \\ &\qquad \frac{\psi_4(p(z,Tx_{2n+1})) + \psi_5(p(x_{2n+1},Sz) - p(x_{2n+1},x_{2n+1}))}{2}\biggr\} \\ &\quad + p(x_{2n+2},z)\end{aligned}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to +\infty$ in above inequalities we get

$$p(z,Tz) \leqslant \psi_4\bigl(p(z,Tz)\bigr) \quad \text{or} \quad p(z,Sz) \leqslant \psi_5\bigl(p(z,Sz)\bigr)$$

and hence $p(z,Tz) = 0$ or $p(z,Sz) = 0$. This implies that z is a fixed point of T or S, and hence z is a common fixed point of the mixed multi-valued mappings S and T. □

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References

1. M. Abbas, B. Ali, C. Vetro, A Suzuki type fixed point theorem for a generalized multivalued mapping on partial Hausdorff metric spaces, *Topology Appl.*, **160**:553–563, 2013.

2. M. Abbas, T. Nazir, Fixed point of generalized weakly contractive mappings in ordered partial metric spaces, *Fixed Point Theory Appl.*, **2012**:1, 19 pp., 2012.

3. I. Altun, H. Simsek, Some fixed point theorems on dualistic partial metric spaces, *J. Adv. Math. Stud.*, **1**:1–8, 2008.

4. I. Altun, F. Sola, H. Simsek, Generalized contractions on partial metric spaces, *Topology Appl.*, **157**:2778–2785, 2010.

5. H. Aydi, M. Abbas, C. Vetro, Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces, *Topology Appl.*, **159**:3234–3242, 2012.

6. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.*, **3**:133–181, 1922.

7. I. Beg, A. Azam, Fixed points of multivalued locally contractive mappings, *Boll. Unione Mat. Ital., Ser. VII, A*, **4**:227–233, 1990.
8. I. Beg, A. Azam, Fixed points of asymptotically regular multivalued mappings, *J. Aust. Math. Soc. A*, **53**:313–326, 1992.
9. M.A. Bukatin, S.Yu. Shorina, Partial metrics and co-continuous valuations, in M. Nivat, et al. (Eds.), *Foundations of Software Science and Computation Structure*, Lect. Notes Comput. Sci., Vol. 1378, Springer, 1998, pp. 125–139.
10. Lj. Ćirić, B. Samet, H. Aydi, C. Vetro, Common fixed points of generalized contractions on partial metric spaces and an application, *Appl. Math. Comput.*, **218**:2398–2406, 2011.
11. B. Damjanovic, B. Samet, C. Vetro, Common fixed point theorems for multi-valued maps, *Acta Math. Sci., Ser. B, Engl. Ed.*, **32**:818–824, 2012.
12. F.S. De Blasi, J. Myjak, S. Reich, A.J. Zaslavski, Generic existence and approximation of fixed points for nonexpansive set-valued maps, *Set-Valued Var. Anal.*, **17**:97–112, 2009.
13. C. Di Bari, P. Vetro, Fixed points for weak φ-contractions on partial metric spaces, *Int. J. of Engineering, Contemporary Mathematics and Sciences*, **1**:5–13, 2011.
14. R. Heckmann, Approximation of metric spaces by partial metric spaces, *Appl. Categ. Struct.*, **7**:71–83, 1999.
15. M. Jleli, V. Čojbašić Rajić, B. Samet, C. Vetro, Fixed point theorems on ordered metric spaces and applications to nonlinear elastic beam equations, *J. Fixed Point Theory Appl.*, **12**:175–192, 2012.
16. S.G. Matthews, Partial metric topology, *Proceedings of the 8th Summer Conference on General Topology and Applications, Ann. N. Y. Acad. Sci.*, **728**:183–197, 1994.
17. S.B. Nadler, Multivalued contraction mappings, *Pac. J. Math.*, **30**:475–488, 1969.
18. S.J. O'Neill, Partial metrics, valuations and domain theory, *Proceeding of the 11th Summer Conference on General Topology and Applications, Ann. N. Y. Acad. Sci.*, **806**:304–315, 1996.
19. D. Paesano, P. Vetro, Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces, *Topology Appl.*, **159**:475–488, 2012.
20. D. Paesano, P. Vetro, Common fixed points in a partially ordered partial metric space, *Int. J. Anal.*, **2013**, Article ID 428561, 8 pp., 2013.
21. S. Reich, A.J. Zaslavski, Existence and approximation of fixed points for set-valued mappings, *Fixed Point Theory Appl.*, **2010**, Article ID 351531, 10 pp., 2010.
22. S. Reich, A.J. Zaslavski, Convergence of inexact iterative schemes for nonexpansive set-valued mappings, *Fixed Point Theory Appl.*, **2010**, Article ID 518243, 10 pp., 2012.
23. S. Romaguera, A Kirk type characterization of completeness for partial metric spaces, *Fixed Point Theory Appl.*, **2010**, Article ID 493298, 6 pp., 2010.
24. S. Romaguera, On Nadler's fixed point theorem for partial metric spaces, *Math. Sci. Appl. E-Notes*, **1**:1–8, 2013.
25. S. Romaguera, O. Valero, A quantitative computational model for complete partial metric spaces via formal balls, *Math. Struct. Comput. Sci.*, **19**:541–563, 2009.

26. B. Samet, C. Vetro, Berinde mappings in orbitally complete metric spaces, *Chaos Solitons Fractals*, **44**:1075–1079, 2011.

27. B. Samet, C. Vetro, F. Vetro, From metric spaces to partial metric spaces, *Fixed Point Theory Appl.*, **2013**:5, 2013.

28. M.P. Schellekens, The correspondence between partial metrics and semivaluations, *Theor. Comput. Sci.*, **315**:1075–1079, 2004.