Nonlinear Analysis: Modelling and Control, Vol. 20, No. 3, 395–416 ISSN 1392-5113
http://dx.doi.org/10.15388/NA.2015.3.6

Anti-periodic solution for fuzzy Cohen–Grossberg neural networks with time-varying and distributed delays*

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Received: November 26, 2013 / **Revised:** July 18, 2014 / **Published online:** March 23, 2015

Abstract. In this paper, by using a continuation theorem of coincidence degree theory and a differential inequality, we establish some sufficient conditions ensuring the existence and global exponential stability of anti-periodic solutions for a class of fuzzy Cohen–Grossberg neural networks with time-varying and distributed delays. In addition, we present an illustrative example to show the feasibility of obtained results.

Keywords: Cohen–Grossberg neural networks, exponential stability, anti-periodic solutions, coincidence degree.

1 Introduction

In 1983, Cohen and Grossberg (see [9]) proposed the following Cohen–Grossberg model:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -a_i(x_i)\left[b_i(x_i) - \sum_{j=1}^{n} t_{ij}s_j(x_j) - I_i\right], \quad i = 1, 2, \dots, n,$$

where $n \geqslant 2$ is the number of neurons in the network, $x_i(t)$ denotes the neuron state variable; $a_i(\cdot)$ is an amplification function; $b_i(\cdot)$ denotes a behaved function; $(t_{ij})_{n\times n}$ is the connection weight matrix, which denotes how the neurons are connected in the network; the activation function is $s_j(x)$ and I_i is the external input. Since then, Cohen–Grossberg neural networks (CGNNs) have been intensively studied due to their promising potential applications in classification, parallel computation, associative memory and optimization problems (see [2, 3, 5]). There have been many results on Cohen–Grossberg

*This research was supported by the National Natural Sciences Foundation of People's Republic of China under grant No. 11361072.

BAM neural networks. For example, in paper [23], authors established sufficient conditions to guarantee the uniqueness and global exponential stability of periodic solutions for a Cohen–Grossberg-type BAM neural networks by using suitable Lyapunov functions, the properties of M-matrix and some suitable mathematical transformation; the author in [26] considered the existence and global attractively of the equilibrium point for a class of Cohen–Grossberg neural networks based on M-matrix theory; authors in [14] studied the stability and existence of periodic solutions to delayed Cohen–Grossberg BAM neural networks with impulses on time scales. For other results on Cohen–Grossberg neural networks, readers may see [4, 11, 13, 34, 38] and reference therein.

Moreover, in mathematical modeling of real world problems, we will encounter some inconveniences, for example, the complexity and the uncertainty or vagueness. For the sake of taking vagueness into consideration, the fuzzy theory is considered as a suitable method. T. Yang and L. Yang proposed fuzzy cellular neural networks in 1996 ([35]). They integrated fuzzy logic into traditional cellular neural networks and maintained local connectedness among cells. Fuzzy neural networks have fuzzy logic between their template input and/or output besides the sum of product operation. Studies have revealed that fuzzy neural networks are very useful for image processing problems, which is a cornerstone in image processing and pattern recognition. In recent years, lots of results on fuzzy neural networks have been derived by many scholars (see [30, 32, 37, 40] and reference therein). For instance, authors in [15, 17, 19, 20, 22] obtained some sufficient conditions for the existence and stability of unique equilibrium point or periodic solution for some fuzzy neural networks; authors in [39, 41] studied the stability of fuzzy BAM neural networks and fuzzy Cohen–Grossberg BAM neural networks, respectively.

Arising from problems in applied sciences, it is well-known that anti-periodic problems of nonlinear differential equations have been extensively studied by many authors during the past twenty years (see [7, 8, 36] and reference therein) and the existence and stability of anti-periodic solutions are an important topic in nonlinear differential equations. For example, anti-periodic trigonometric polynomials are important in the study of interpolation problems (see [10, 12]) and anti-periodic wavelets are discussed in [6]. Since the signal transmission process of neural networks can often be described as an anti-periodic process, anti-periodic solutions for different classes of neural networks were discussed by many authors (see [16, 18, 21, 27, 28, 29, 33] and reference therein).

In this paper, we will integrate fuzzy operations into Cohen–Grossberg neural networks and maintain local connectedness among cells. By using a continuation theorem of coincidence degree theory which does not need to compute the topological degree and a differential inequality, we study the existence and global exponential stability of anti-periodic solutions for the following fuzzy Cohen–Grossberg neural network with time-varying and distributed delays:

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i\big(x_i(t)\big)\Bigg[&b_i\big(x_i(t)\big) - \sum_{j=1}^{n} k_{ij}(t) f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big)\\ &- \bigwedge_{j=1}^{n} \alpha_{ij}(t) f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big) - \bigvee_{j=1}^{n} \beta_{ij}(t) f_j - \big(x_j\big(t-\tau_{ij}(t)\big)\big)\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=1}^{n} e_{ij}(t) \int_0^{+\infty} v_{ij}(s) g_j\bigl(x_j(t-s)\bigr)\,\mathrm{d}s \\
&- \bigwedge_{j=1}^{n} \gamma_{ij}(t) \int_0^{+\infty} v_{ij}(s) g_j\bigl(x_j(t-s)\bigr)\,\mathrm{d}s \\
&- \bigvee_{j=1}^{n} \xi_{ij}(t) \int_0^{+\infty} v_{ij}(s) g_j\bigl(x_j(t-s)\bigr)\,\mathrm{d}s \\
&- \sum_{j=1}^{n} \delta_{ij}(t)\mu_j(t) - \bigwedge_{j=1}^{n} T_{ij}(t)\mu_j(t) - \bigvee_{j=1}^{n} H_{ij}(t)\mu_j(t) - I_i(t)\Biggr],
\end{aligned} \tag{1}
$$

where $i = 1, 2, \dots, n$, n is the number of neurons, $x_i(t)$ denotes the activation of the ith neuron at time t; $a_i(\cdot)$ is amplification function; $b_i(\cdot)$ represents the appropriately behaved function; f_j, g_j denote the activation functions of the jth neuron; $0 \leqslant \tau_{ij}(t) \leqslant \tau_{ij}$ is the transmission delay; $k_{ij}(t)$, $e_{ij}(t)$ are elements of feedback templates at time t and $\delta_{ij}(t)$ is the element of feed-forward templates at time t; $\alpha_{ij}(t)$, $\gamma_{ij}(t)$ denote elements of fuzzy feedback MIN templates at time t and $\beta_{ij}(t)$, $\xi_{ij}(t)$ are elements of fuzzy feedback MAX templates at time t; $T_{ij}(t)$ is the fuzzy feed-forward MIN template at time t and $H_{ij}(t)$ is the fuzzy feed-forward MAX templates at time t, respectively; $\mu_j(t)$ denotes the input of the jth neuron at time t; $v_{ij} : (0, +\infty) \to (0, +\infty)$ corresponds to the delay kernel function and satisfies $\int_0^{+\infty} v_{ij}(s)\,\mathrm{d}s \leqslant \bar{v}_{ij}$, where $\bar{v}_{ij}$ is a positive constant; $I_i(t)$ denotes biases of the ith neuron at time t, respectively, $i, j = 1, 2, \dots, n$; $\wedge$ and $\vee$ denote the fuzzy AND and fuzzy OR operations, respectively. To the best of our knowledge, there have been few papers published on the existence of anti-periodic solutions for fuzzy Cohen–Grossberg neural networks by using the symmetry continuation theorem which is used in this paper.

The initial condition of (1) is of the form

$$x_i(s) = \varphi_i(s), \quad s \in (-\infty, 0],\ i = 1, 2, \dots, n,$$

where $\varphi_i(\cdot)$ denotes positive real-valued continuous functions on $(-\infty, 0]$.

Denote $R^+ = (0, +\infty)$. Throughout this paper, we assume the following conditions hold:

- (H1) $k_{ij}(t)$, $\alpha_{ij}(t)$, $\tau_{ij}(t)$, $T_{ij}(t)$, $\beta_{ij}(t)$, $H_{ij}(t)$, $\mu_j(t)$, $I_i(t)$, $e_{ij}(t)$, $\gamma_{ij}(t)$, $\xi_{ij}(t)$ are $\omega/2$-anti-periodic continuous functions for $t \in R$, $i, j = 1, 2, \dots, n$;
- (H2) $a_i \in C(R, R^+)$, $a_i(-u) = a_i(u)$ and there exist positive constants a_i^m, a_i^M such that $a_i^m \leqslant a_i(u) \leqslant a_i^M$ for all $u \in R$, $i, j = 1, 2, \dots, n$;
- (H3) $b_i \in C(R, R)$ is differentiable, $b_i(0) = 0$, $b_i(-u) = -b_i(u)$ and there exist positive constants ρ_i, δ_i such that $0 < \rho_i \leqslant b_i'(u) \leqslant \delta_i$ for all $u \in R$, $i, j = 1, 2, \dots, n$;

(H4) $f_j, g_j \in C(R,R)$, $f_j(-u) = -f_j(u)$, $g_j(-u) = -g_j(u)$ and there exist positive constants L_j^f and M_j such that, for $u, v \in R$,

$$\left|f_j(u) - f_j(v)\right| \leqslant L_j^f |u-v| \quad \text{and} \quad \left|g_j(u)\right| \leqslant M_j, \quad i,j = 1,2,\dots,n.$$

For convenience, we denote

$$\bar{h} = \max_{t\in[0,\omega]} \left|h(t)\right|, \qquad \|h\|_2 = \left(\int_0^\omega \left|h(t)\right|^2 \mathrm{d}t\right)^{1/2},$$

where h is an $\omega/2$-anti-periodic function, and we denote

$$\hat{I}_i(t) = \sum_{j=1}^n \delta_{ij}(t)\mu_j(t) + \bigwedge_{j=1}^n T_{ij}(t)\mu_j(t) + \bigvee_{j=1}^n H_{ij}(t)\mu_j(t) + I_i(t),$$

$i = 1,2,\dots,n$.

The organization of the rest of this paper is as follows: in Section 2, we introduce some notations and preliminary results which are needed in later sections. In Sections 3 and 4, we establish some sufficient conditions for the existence and global exponential stability of the anti-periodic solution of (1), respectively. In Section 5, as a special case of our results obtained in Sections 3 and 4, we give sufficient conditions for the existence and global exponential stability of the fuzzy Cohen–Grossberg BAM neural networks with time-varying and distributed delays. In Section 6, we give an example to illustrate the feasibility of our results obtained in previous sections.

2 Preliminaries

In this section, we state some notions and preliminary results.

Definition 1. Let $u \in C(R,R)$. A function $u(t)$ is said to be $\omega/2$-anti-periodic on R if $u(t+\omega/2) = -u(t)$ for all $t \in R, \omega > 0$ is a constant.

A matrix or a vector $A \geqslant 0$ (or $A > 0$) means that all the elements of A are greater than or equal to (or greater than) zero. For matrices or vectors A, B, $A \geqslant B$ (or $A > B$), means that all entries of A are greater than or equal to (or greater than) corresponding entries of B.

Definition 2. (See [1].) A real matrix $A = (a_{ij})_{n\times n}$ is said to be an M-matrix if $a_{ij} \leqslant 0$, $i,j = 1,2,\dots,n, i \neq j$, and all successive principal minors of A are positive.

Lemma 1. (*See* [1].) *Let $A = (a_{ij})_{n\times n}$ be an matrix with nonpositive off-diagonal elements, then the following statements are equivalent*:

(i) *A is an M-matrix*;
(ii) *There exists a vector $\eta > 0$ such that $A\eta > 0$*;
(iii) *There exists a vector $\xi > 0$ such that $\xi^{\mathrm{T}} A > 0$.*

Lemma 2. (*See* [24].) *Let matrix $A \geqslant 0$. Then the following statements are equivalent*:

(i) $\rho(A) < 1$;

(ii) $(E - A)^{-1} \geqslant 0$, *where E denotes the identity matrix*;

(iii) *There is a constant vector $c > 0$ such that $c > cA$*;

(iv) *There is a constant vector $c > 0$ such that $c > cA^{\mathrm{T}}$*.

Lemma 3. (*See* [22].) *Let f_j be defined on R, $j = 1, 2, \dots, m$. Then for any $a_{ij} \in R$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, we have the following estimations*:

$$\left| \bigwedge_{j=1}^{m} a_{ij} f_j(u_j) - \bigwedge_{j=1}^{m} a_{ij} f_j(v_j) \right| \leqslant \sum_{j=1}^{m} |a_{ij}| \bigl| f_j(u_j) - f_j(v_j) \bigr|$$

and

$$\left| \bigvee_{j=1}^{m} a_{ij} f_j(u_j) - \bigvee_{j=1}^{m} a_{ij} f_j(v_j) \right| \leqslant \sum_{j=1}^{m} |a_{ij}| \bigl| f_j(u_j) - f_j(v_j) \bigr|,$$

where $u_j, v_j \in R$, $j = 1, 2, \dots, m$.

Definition 3. The anti-periodic solution $z^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^{\mathrm{T}}$ with initial value $\phi^*(s) = (\varphi_1^*(s), \varphi_2^*(s), \dots, \varphi_n^*(s))^{\mathrm{T}}$ of (1) is said to be globally exponentially stable, if for any solution $z(t) = (x_1(t), x_2(t), \dots, x_n(t))^{\mathrm{T}}$ with initial value $\phi(s) = (\varphi_1(s), \varphi_2(s), \dots, \varphi_n(s))^{\mathrm{T}}$ of (1), there exist constants $\delta > 0$ and $r \geqslant 1$ such that

$$\sum_{i=1}^{n} \bigl| x_i(t) - x_i^*(t) \bigr| \leqslant r \mathrm{e}^{-\delta t} \| \varphi - \varphi^* \|_1$$

holds for all $t \geqslant 0$, where $\| \varphi - \varphi^* \|_1 = \sum_{i=1}^{n} \sup_{s \in (-\infty, 0]} |\varphi_i(s) - x_i^*(s)|$.

Lemma 4. (*See* [31].) *Let $z(t) = (x_1(t), x_2(t), \dots, x_n(t))^{\mathrm{T}}$ be a solution of the differential inequality*

$$D^+ z(t) \leqslant C z(t) + D \bar{z}(t), \quad t \geqslant 0,$$

where $\bar{z}(t) = (\bar{x}_1(t), \bar{x}_2(t), \dots, \bar{x}_n(t))^{\mathrm{T}}$, $\bar{x}_i(t) = \sup_{-\infty \leqslant s \leqslant t} \{x_i(s)\}$, $i = 1, 2, \dots, n$. If the conditions

(i) $C = (c_{ij})_{n \times n}$, $c_{ij} \geqslant 0$ $(i \neq j)$, $i, j = 1, 2, \dots, n$; $\sum_{k=1}^{n} \bar{z}_k(0) > 0$;

(ii) $-(C + D)$ *is an M-matrix*

hold, then there exist constants $\delta > 0$, $r_k \geqslant 1$ $(k = 1, 2, \dots, n)$ such that

$$z_k(t) \leqslant r_k \sum_{k=1}^{n} \bar{z}_k(0) \mathrm{e}^{-\delta t}, \quad t \geqslant 0.$$

The following fixed point theorem of coincidence degree is crucial in the arguments of our main results.

Lemma 5. (*See* [25].) *Let $\mathbb{X}$, $\mathbb{Y}$ be two Banach spaces, $\Omega \subset \mathbb{X}$ be open bounded and symmetric with $0 \in \Omega$. Suppose that $L : D(L) \subset \mathbb{X} \to \mathbb{Y}$ is a linear Fredholm operator of index zero with $D(L) \cap \bar{\Omega} \neq \emptyset$ and $N : \bar{\Omega} \to \mathbb{Y}$ is L-compact. Further, we also assume that*:

(H) $Lx - Nx \neq \lambda(-Lx - N(-x))$ *for all* $x \in D(L) \cap \partial\Omega$, $\lambda \in (0, 1]$.

Then equation $Lx = Nx$ has at least one solution on $D(L) \cap \bar{\Omega}$.

3 Existence of anti-periodic solutions

Theorem 1. *Assume that* (H1)–(H4) *hold. Suppose further that*

(H5) $G := G_1 - G_2$ *is an M-matrix, where* $G_1 = \operatorname{diag}(a_i^m - \theta_i a_i^m a_i^M \omega)_{n\times n}$, $G_2 = (\nu_{ij})_{n\times n}$, $\nu_{ij} = a_i^M (1/\rho_i + a_i^m \omega) \sum_{j=1}^m (\bar{k}_{ji} + \bar{\alpha}_{ji} + \bar{\beta}_{ji}) L_j^f$, $i, j = 1, 2, \ldots, n$.

Then (1) *has at least one $\omega/2$-anti-periodic solution.*

Proof. Let $C^k[0,\omega] = \{u = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} : [0,\omega] \to R^n | u^k(t)$ is a continuous map$\}$, $k = 0, 1$. Take

$$X = Y = \left\{ u \in C[0,\omega]\colon\ u\left(t + \frac{\omega}{2}\right) = -u(t) \text{ for all } t \in \left[0, \frac{\omega}{2}\right] \right\},$$

then X and Y are Banach spaces with the norm $\|u\|_X = \|u\|_Y = \sum_{i=1}^n |x_i|_0$, in which $|x_i|_0 = \max_{t\in[0,\omega]} |x_i(t)|$, $i = 1, 2, \ldots, n$. Set

$$L : \operatorname{Dom} L \cap X \to Y, \quad u \to u',$$

where

$$\operatorname{Dom} L = \left\{ u \in C^1[0,\omega]\colon\ u\left(t + \frac{\omega}{2}\right) = -u(t) \text{ for all } t \in \left[0, \frac{\omega}{2}\right] \right\}$$

and

$$N : X \to Y, \quad Nu = \big(A_1(t), A_2(t), \ldots, A_n(t)\big)^{\mathrm{T}},$$

where

$$\begin{aligned} A_i(t) = -a_i\big(x_i(t)\big)\Bigg[& b_i(x_i(t)) - \sum_{j=1}^n k_{ij}(t) f_j\big(x_j\big(t - \tau_{ij}(t)\big)\big) \\ & - \bigwedge_{j=1}^n \alpha_{ij}(t) f_j\big(x_j\big(t - \tau_{ij}(t)\big)\big) - \bigvee_{j=1}^n \beta_{ij}(t) f_j\big(x_j\big(t - \tau_{ij}(t)\big)\big) \\ & - \sum_{j=1}^n e_{ij}(t) \int_0^{+\infty} v_{ij}(s) g_j\big(x_j(t-s)\big)\,\mathrm{d}s - \bigwedge_{j=1}^n \gamma_{ij}(t) \int_0^{+\infty} v_{ij}(s) g_j\big(x_j(t-s)\big)\,\mathrm{d}s \\ & - \bigvee_{j=1}^n \xi_{ij}(t) \int_0^{+\infty} v_{ij}(s) g_j\big(x_j(t-s)\big)\,\mathrm{d}s - \hat{I}_i(t) \Bigg], \quad i = 1, 2, \ldots, n. \end{aligned}$$

It is easy to see that

$$\operatorname{Ker} L = \{0\} \quad \text{and} \quad \operatorname{Im} L = \left\{ z \in Y\colon \int_0^{\omega} z(s)\,\mathrm{d}s = 0 \right\} \equiv Y.$$

Thus $\dim \operatorname{Ker} L = 0 = \operatorname{codim} \operatorname{Im} L$, and L is a linear Fredholm operator of index zero.

Define the continuous projector $P : X \to \operatorname{Ker} L$ and the averaging projector $Q\colon Y \to Y$ by

$$Pu = \int_0^{\omega} u(s)\,\mathrm{d}s = 0, \qquad Qz = \frac{1}{\omega}\int_0^{\omega} z(s)\,\mathrm{d}s.$$

Hence $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$. Denoting by $L_P^{-1} : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$ the inverse of $L|_{D(L)\cap \operatorname{Ker} P}$, we have

$$L_P^{-1} z = \int_0^{t} z(s)\,\mathrm{d}s - \frac{1}{2}\int_0^{\omega/2} z(s)\,\mathrm{d}s. \tag{2}$$

Similar to the proof of Theorem 3.1 in [14], it is not difficult to show that $QN(\bar{\Omega})$, $L_P^{-1}(I - Q)N(\bar{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Therefore, N is L-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset X$.

In order to apply Lemma 5, we need to find an appropriate open bounded subset Ω in X. Corresponding to the operator equation $Lu - Nu = \lambda(-Lu - N(-u))$, $\lambda \in (0, 1]$, we have

$$x_i'(t) = \frac{1}{1+\lambda} G_i(t, x) - \frac{\lambda}{1+\lambda} G_i(t, -x), \quad i = 1, 2, \dots, n, \tag{3}$$

where

$$\begin{aligned} G_i(t,x) = -a_i\big(x_i(t)\big)\Bigg[& b_i\big(x_i(t)\big) - \sum_{j=1}^{n} k_{ij}(t) f_j\big(x_j\big(t - \tau_{ij}(t)\big)\big) \\ & - \bigwedge_{j=1}^{n} \alpha_{ij}(t) f_j\big(x_j\big(t - \tau_{ij}(t)\big)\big) - \bigvee_{j=1}^{n} \beta_{ij}(t) f_j\big(x_j\big(t - \tau_{ij}(t)\big)\big) \\ & - \sum_{j=1}^{n} e_{ij}(t) \int_0^{+\infty} v_{ij}(s) g_j\big(x_j(t-s)\big)\,\mathrm{d}s \\ & - \bigwedge_{j=1}^{n} \gamma_{ij}(t) \int_0^{+\infty} v_{ij}(s) g_j\big(x_j(t-s)\big)\,\mathrm{d}s \\ & - \bigvee_{j=1}^{n} \xi_{ij}(t) \int_0^{+\infty} v_{ij}(s) g_j\big(x_j(t-s)\big)\,\mathrm{d}s - \hat{I}_i(t)\Bigg], \quad i = 1, 2, \dots, n, \end{aligned}$$

and

$$\begin{aligned}
G_i(t,-x) = -a_i\big(-x_i(t)\big)\bigg[&b_i\big(-x_i(t)\big) - \sum_{j=1}^{n} k_{ij}(t) f_j\big(-x_j\big(t-\tau_{ij}(t)\big)\big) \\
&- \bigwedge_{j=1}^{n} \alpha_{ij}(t) f_j\big(-x_j\big(t-\tau_{ij}(t)\big)\big) - \bigvee_{j=1}^{n} \beta_{ij}(t) f_j\big(-x_j\big(t-\tau_{ij}(t)\big)\big) \\
&- \sum_{j=1}^{n} e_{ij}(t) \int_0^{+\infty} v_{ij}(s) g_j\big(-x_j(t-s)\big)\,\mathrm{d}s \\
&- \bigwedge_{j=1}^{n} \gamma_{ij}(t) \int_0^{+\infty} v_{ij}(s) g_j\big(-x_j(t-s)\big)\,\mathrm{d}s \\
&- \bigvee_{j=1}^{n} \xi_{ij}(t) \int_0^{+\infty} v_{ij}(s) g_j\big(-x_j(t-s)\big)\,\mathrm{d}s - \hat{I}_i(t)\bigg], \quad i = 1,2,\dots,n.
\end{aligned}$$

In view of the first equation of (3), for $i = 1,2,\dots,n$, we get from (H2)–(H4) and Lemma 3 that

$$\begin{aligned}
\int_0^{\omega} \big|x_i'(t)\big|\,\mathrm{d}t &= \int_0^{\omega} \bigg|\frac{1}{1+\lambda} G_i(t,x) - \frac{\lambda}{1+\lambda} G_i(t,-x)\bigg|\,\mathrm{d}t \\
&\leqslant \bigg[\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}\bigg] \int_0^{\omega} \max\big\{\big|G_i(t,x)\big|,\ \big|G_i(t,-x)\big|\big\}\,\mathrm{d}t \\
&\leqslant a_i^M\bigg[\int_0^{\omega} \big|b_i\big(x_i(t)\big)\big|\,\mathrm{d}t + \sum_{j=1}^{n} \bar{k}_{ij} \int_0^{\omega} \big|f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big)\big|\,\mathrm{d}t \\
&\quad + \int_0^{\omega} \bigwedge_{j=1}^{n} \bar{\alpha}_{ij} \big|f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big)\big|\,\mathrm{d}t + \int_0^{\omega} \bigvee_{j=1}^{n} \bar{\beta}_{ij} \big|f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big)\big|\,\mathrm{d}t \\
&\quad + \sum_{j=1}^{n} \bar{e}_{ij} \int_0^{\omega} \int_0^{+\infty} v_{ij}(s) \big|g_j\big(x_j(t-s)\big)\big|\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \int_0^{\omega} \bigwedge_{j=1}^{n} \bar{\gamma}_{ij} \int_0^{+\infty} v_{ij}(s) \big|g_j\big(x_j(t-s)\big)\big|\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \int_0^{\omega} \bigvee_{j=1}^{n} \bar{\xi}_{ij} \int_0^{+\infty} v_{ij}(s) \big|g_j\big(x_j(t-s)\big)\big|\,\mathrm{d}s\,\mathrm{d}t + \bar{\hat{I}}_i\omega
\end{aligned}$$

$$
\begin{aligned}
&\leqslant a_i^M\Bigg[\int_0^{\omega}\big|b_i\big(x_i(t)\big)\big|\,\mathrm{d}t+\sum_{j=1}^{n}\bar{k}_{ij}\int_0^{\omega}\big|f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big)-f_j(0)\big|\,\mathrm{d}t\\
&\quad+\sum_{j=1}^{n}\bar{\alpha}_{ij}\int_0^{\omega}\big|f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big)-f_j(0)\big|\,\mathrm{d}t\\
&\quad+\sum_{j=1}^{n}\int_0^{\omega}\bar{\beta}_{ij}\big|f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big)-f_j(0)\big|\,\mathrm{d}t+\bar{\hat{I}}_i\omega\\
&\quad+\sum_{j=1}^{n}\big((\bar{b}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\big|f_j(0)\big|+(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j\big)\Bigg]\\
&\leqslant a_i^M\Bigg[\theta_i\sqrt{\omega}\|x_i\|_2+\sum_{j=1}^{n}(\bar{k}_{ij}+\bar{\alpha}_{ji}+\bar{\beta}_{ij})L_j^f\sqrt{\omega}\,\|x_j\|_2+\bar{\hat{I}}_i\omega\\
&\quad+\sum_{j=1}^{n}(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\big|f_j(0)\big|+\sum_{j=1}^{n}(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j\Bigg].
\end{aligned}
\tag{4}
$$

Integrating (3) from 0 to ω, for $i=1,2,\dots,n$, we have that

$$
\begin{aligned}
&\left|\int_0^{\omega}\left[\frac{a_i(x_i(t))b_i(x_i(t))}{1+\lambda}-\frac{\lambda a_i(-x_i(t))b_i(-x_i(t))}{1+\lambda}\right]\mathrm{d}t\right|\\
&\quad=\left|\int_0^{\omega}\left[\frac{a_i(x_i(t))b_i(x_i(t))}{1+\lambda}+\frac{\lambda a_i(x_i(t))b_i(x_i(t))}{1+\lambda}\right]\mathrm{d}t\right|=\left|\int_0^{\omega}a_i\big(x_i(t)\big)b_i\big(x_i(t)\big)\,\mathrm{d}t\right|\\
&\quad\leqslant a_i^M\int_0^{\omega}\Bigg[\sum_{j=1}^{n}\bar{k}_{ij}\big|f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big)-f_j(0)\big|\\
&\qquad+\sum_{j=1}^{n}\bar{\alpha}_{ij}\big|f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big)-f_j(0)\big|+\bar{\hat{I}}_i\\
&\qquad+\sum_{j=1}^{n}\bar{\beta}_{ij}\big|f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big)-f_j(0)\big|\Bigg]\,\mathrm{d}t\\
&\qquad+a_i^M\sum_{j=1}^{n}(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\big|f_j(0)\big|+a_i^M\sum_{j=1}^{n}(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j\\
&\quad\leqslant a_i^M\Bigg[\sum_{j=1}^{n}(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})L_j^f\sqrt{\omega}\,\|x_j\|_2+\bar{\hat{I}}_i\omega\\
&\qquad+\sum_{j=1}^{n}(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\big|f_j(0)\big|+\sum_{j=1}^{n}(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j\Bigg].
\end{aligned}
$$

By the mean value theorem of integration and (H_3), for $i = 1, 2, \dots, n$, we obtain that

$$\left|\int_0^\omega a_i\big(x_i(t)\big)x_i(t)\,\mathrm{d}t\right| \leqslant \frac{a_i^M}{\rho_i}\left[\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})L_j^f\sqrt{\omega}\,\|x_j\|_2+\bar{\hat{I}}_i\omega + \sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\big|f_j(0)\big| + \sum_{j=1}^n(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j\right]. \tag{5}$$

For any $\zeta_i, \eta_i \in [0,\omega]$, $i = 1, 2, \dots, n$, we have

$$\int_0^\omega a_i\big(x_i(t)\big)x_i(t)\,\mathrm{d}t \leqslant \int_0^\omega a_i\big(x_i(t)\big)x_i(\zeta_i)\,\mathrm{d}t + \int_0^\omega a_i\big(x_i(t)\big)\left(\int_0^\omega \big|x_i'(t)\big|\,\mathrm{d}t\right)\mathrm{d}t,$$

and

$$\int_0^\omega a_i\big(x_i(t)\big)x_i(t)\,\mathrm{d}t \geqslant \int_0^\omega a_i\big(x_i(t)\big)x_i(\eta_i)\,\mathrm{d}t - \int_0^\omega a_i\big(x_i(t)\big)\left(\int_0^\omega \big|x_i'(t)\big|\,\mathrm{d}t\right)\mathrm{d}t,$$

where $i = 1, 2, \dots, n$. Dividing by $\int_0^\omega a_i(x_i(t))\,\mathrm{d}t$ on the two sides of above two inequalities, respectively, we obtain that, for $i = 1, 2, \dots, n$,

$$x_i(\zeta_i) \geqslant \frac{1}{\int_0^\omega a_i(x_i(t))\,\mathrm{d}t}\int_0^\omega a_i\big(x_i(t)\big)x_i(t)\,\mathrm{d}t - \int_0^\omega \big|x_i'(t)\big|\,\mathrm{d}t, \tag{6}$$

and

$$x_i(\eta_i) \leqslant \frac{1}{\int_0^\omega a_i(x_i(t))\,\mathrm{d}t}\int_0^\omega a_i\big(x_i(t)\big)x_i(t)\,\mathrm{d}t + \int_0^\omega \big|x_i'(t)\big|\,\mathrm{d}t. \tag{7}$$

Let $\bar{t}_i$, $\underline{t}_i \in [0,\omega]$ such that $x_i(\bar{t}_i) = \max_{t\in[0,\omega]} x_i(t)$, $x_i(\underline{t}_i) = \min_{t\in[0,\omega]} x_i(t)$, by the arbitrariness of ζ_i, η_i, we obtain from (4)–(7) that for $i = 1, 2, \dots, n$,

$$\begin{aligned} x_i(\underline{t}_i) &\geqslant \frac{1}{\int_0^\omega a_i(x_i(t))\,\mathrm{d}t}\int_0^\omega a_i\big(x_i(t)\big)x_i(t)\,\mathrm{d}t - \int_0^\omega \big|x_i'(t)\big|\,\mathrm{d}t \\ &\geqslant -\frac{1}{\int_0^\omega a_i(x_i(t))\,\mathrm{d}t}\left|\int_0^\omega a_i(x_i(t))x_i(t)\,\mathrm{d}t\right| - \int_0^\omega \big|x_i'(t)\big|\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&\geqslant -\frac{a_i^M}{a_i^m\rho_i\omega}\Bigg[\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})L_j^f\sqrt{\omega}\,\|x_j\|_2+\bar{\hat{I}}_i\omega \\
&\quad+\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\big|f_j(0)\big|+\sum_{j=1}^n(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j\Bigg] \\
&\quad-a_i^M\Bigg[\theta_i\sqrt{\omega}\,\|x_i\|_2+\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})L_j^f\sqrt{\omega}\,\|x_j\|_2+\bar{\hat{I}}_i\omega \\
&\quad+\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\big|f_j(0)\big|+\sum_{j=1}^n(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
x_i(\bar{t}_i)&\leqslant\frac{1}{\int_0^\omega a_i(x_i(t))\,\mathrm{d}t}\int_0^\omega a_i\big(x_i(t)\big)x_i(t)\,\mathrm{d}t+\int_0^\omega\big|x_i'(t)\big|\,\mathrm{d}t \\
&\leqslant\frac{1}{\int_0^\omega a_i\big(x_i(t)\big)\,\mathrm{d}t}\left|\int_0^\omega a_i\big(x_i(t)\big)x_i(t)\,\mathrm{d}t\right|+\int_0^\omega\big|x_i'(t)\big|\,\mathrm{d}t \\
&\leqslant\frac{a_i^M}{a_i^m\rho_i\omega}\Bigg[\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})L_j^f\sqrt{\omega}\,\|x_j\|_2+\bar{\hat{I}}_i\omega \\
&\quad+\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\big|f_j(0)\big|+\sum_{j=1}^n(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j\Bigg] \\
&\quad+a_i^M\Bigg[\theta_i\sqrt{\omega}\,\|x_i\|_2+\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})L_j^f\sqrt{\omega}\,\|x_j\|_2+\bar{\hat{I}}_i\omega \\
&\quad+\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\big|f_j(0)\big|+\sum_{j=1}^n(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j\Bigg].
\end{aligned}
$$

Thus, we have that for $i=1,2,\dots,n$,

$$
\begin{aligned}
|x_i|_0=\max_{t\in[0,\omega]}\big|x_i(t)\big|&\leqslant\frac{a_i^M}{a_i^m\rho_i\omega}\Bigg[\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ji}+\bar{\beta}_{ij})L_j^f\sqrt{\omega}\,\|x_j\|_2+\bar{\hat{I}}_i\omega \\
&\quad+\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\big|f_j(0)\big|+\sum_{j=1}^n(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j\Bigg] \\
&\quad+a_i^M\Bigg[\theta_i\sqrt{\omega}\,\|x_i\|_2+\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})L_j^f\sqrt{\omega}\,\|x_j\|_2+\bar{\hat{I}}_i\omega \\
&\quad+\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\big|f_j(0)\big|+\sum_{j=1}^n(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j\Bigg].
\end{aligned}
\tag{8}
$$

In addition, we have that

$$\|x_i\|_2 = \left(\int_0^\omega \bigl|x_i(s)\bigr|^2\,\mathrm{d}s\right)^{1/2} \leqslant \sqrt{\omega}\max_{t\in[0,\omega]}\bigl|x_i(t)\bigr| = \sqrt{\omega}\,|x_i|_0, \quad i=1,2,\dots,n.$$

By (8), we have for $i=1,2,\dots,n$,

$$\begin{aligned}
a_i^m\omega|x_i|_0 \leqslant{}& \frac{a_i^M}{\rho_i}\Biggl[\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})L_j^f\sqrt{\omega}\,\|x_j\|_2 \\
&+\bar{\hat{I}}_i\omega+\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\bigl|f_j(0)\bigr|+\sum_{j=1}^n(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j\Biggr] \\
&+a_i^m\omega a_i^M\Biggl[\theta_i\sqrt{\omega}\,\|x_i\|_2+\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})L_j^f\sqrt{\omega}\,\|x_j\|_2+\bar{\hat{I}}_i\omega \\
&+\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\bigl|f_j(0)\bigr|+\sum_{j=1}^n(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j\Biggr] \\
\leqslant{}& \frac{a_i^M}{\rho_i}\Biggl[\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})L_j^f\omega|x_j|_0+\bar{\hat{I}}_i\omega \\
&+\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\bigl|f_j(0)\bigr|+\sum_{j=1}^n(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j\Biggr] \\
&+a_i^m\omega a_i^M\Biggl[\theta_i\omega|x_i|_0+\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})L_j^f\omega|x_j|_0+\bar{\hat{I}}_i\omega \\
&+\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\bigl|f_j(0)\bigr|+\sum_{j=1}^n(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j\Biggr],
\end{aligned}$$

which imply that

$$\begin{aligned}
&\bigl(a_i^m-\theta_i a_i^m a_i^M\omega\bigr)|x_i|_0-a_i^M\left(\frac{1}{\rho_i}+a_i^m\omega\right)\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})L_j^f|x_j|_0 \\
&\leqslant a_i^M\left(\frac{1}{\rho_i}+a_i^m\omega\right)\Biggl(\sum_{j=1}^n(\bar{k}_{ij}+\bar{\alpha}_{ij}+\bar{\beta}_{ij})\omega\bigl|f_j(0)\bigr| \\
&\quad+\sum_{j=1}^n(\bar{e}_{ij}+\bar{\gamma}_{ij}+\bar{\xi}_{ij})\bar{v}_{ij}\omega M_j+\bar{\hat{I}}_i\Biggr)\triangleq N_i, \quad i=1,2,\dots,n. \qquad (9)
\end{aligned}$$

Denote

$$|u|_0=\bigl(|x_1|_0,|x_2|_0,\dots,|x_n|_0\bigr)^{\mathrm{T}} \quad \text{and} \quad N=(N_1,N_2,\dots,N_n)^{\mathrm{T}}.$$

Then (9) can be rewritten in the matrix form

$$E|u|_0 \leqslant N.$$

Since E is a nonsingular M matrix, we have that

$$|u|_0 \leqslant E^{-1}N \triangleq (Q_1, Q_2, \ldots, Q_n)^{\mathrm{T}}.$$

Let $Q = \sum_{k=1}^{n} Q_i + 1$ (clearly, Q is independent of λ). Take $\Omega = \{u \in X\colon \|u\|_X < Q\}$. It is clear that Ω satisfies all the requirements in Lemma 5 and condition (H) is satisfied. Hence, we conclude from Lemma 5 that system (1) has at least one $\omega/2$-anti-periodic solution. This completes the proof. □

4 Exponential stability of anti-periodic solution

In this section, we study the global exponential stability of the anti-periodic solution of (1) obtained in Section 2.

Theorem 2. *Let* (H1)–(H5) *hold. Suppose further that*:

(H6) *There exist positive constants* M_j^f *such that* $|f_j(u)| \leqslant M_j^f$ *for all* $u \in R$, $j = 1, 2, \ldots, n$;

(H7) *There exist positive constants* L_i^a *such that*

$$\bigl|a_i(u) - a_i(v)\bigr| \leqslant L_i^a|u - v| \quad \forall u, v \in R,\ i = 1, 2, \ldots, n;$$

(H8) *There exist positive constants* l_i^a *such that*

$$\bigl(a_i(u)b_i(u) - a_i(v)b_i(v)\bigr)(u - v) \geqslant 0,$$
$$\bigl|a_i(u)b_i(u) - a_i(v)b_i(v)\bigr| \geqslant l_i^a|u - v|,$$

for all $u, v \in R$, $i = 1, 2, \ldots, n$;

(H9) $\Gamma := A - CF^{\mathrm{T}}L$ *is an* M*-matrix, where* $A = \operatorname{diag}(\zeta_i)_{n\times n}$, $\zeta_i = l_i^a + L_i^a(\sum_{j=1}^{n}(\bar{k}_{ij} + \bar{\alpha}_{ij} + \bar{\beta}_{ij})M_j^f + \bar{\hat{I}}_i)$, $C = \operatorname{diag}(a_1^M, a_2^M, \ldots, a_n^M)_{n\times n}$, $F = (F_{ij})_{n\times n}$, $F_{ij} = \bar{e}_{ij} + \bar{\gamma}_{ij} + \bar{\xi}_{ij}$, $L = \operatorname{diag}(L_1^f, L_2^f, \ldots, L_n^f)_{n\times n}$, $i, j = 1, 2, \ldots, n$.

Then (1) *has one* $\omega/2$*-anti-periodic solution, which is globally exponentially stable.*

Proof. From Theorem 1, it is clear that (1) has at least one $\omega/2$-anti-periodic solution. We denote this anti-periodic solution by $z^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^{\mathrm{T}}$ and the initial condition by $\phi^*(s) = (\varphi_1^*(s), \varphi_2^*(s), \ldots, \varphi_n^*(s))^{\mathrm{T}}$. Next, we will use a differential inequality to study the global exponential stability of this anti-periodic solution.

Suppose that $z(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}}$ is an arbitrary solution of (1) with initial value $\phi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_n(s))^{\mathrm{T}}$. Let $\Theta(t) = (u_1(t), u_2(t), \ldots, u_n(t))^{\mathrm{T}} =$

$z(t) - z^*(t)$, where $u_i(t) = x_i(t) - x_i^*(t), i = 1, 2, \dots, n$. Then (1) can be rewritten as

$$\begin{aligned}
\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} &= -\big[a_i\big(x_i(t)\big)b_i\big(x_i(t)\big) - a_i\big(x_i^*(t)\big)b_i\big(x_i^*(t)\big)\big] \\
&\quad + \big[a_i\big(x_i(t)\big) - a_i\big(x_i^*(t)\big)\big]\Bigg[\sum_{j=1}^n k_{ij}(t)f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big) \\
&\quad + \bigwedge_{j=1}^n \alpha_{ij}(t)f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big) + \bigvee_{j=1}^n \beta_{ij}(t)f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big) \\
&\quad + \sum_{j=1}^n e_{ij}(t)\int_0^{+\infty} v_{ij}(s)g_j\big(x_j(t-s)\big)\,\mathrm{d}s + \bigwedge_{j=1}^n \gamma_{ij}(t)\int_0^{+\infty} v_{ij}(s)g_j\big(x_j(t-s)\big)\,\mathrm{d}s \\
&\quad + \bigvee_{j=1}^n \xi_{ij}(t)\int_0^{+\infty} v_{ij}(s)g_j\big(x_j(t-s)\big)\,\mathrm{d}s\Bigg] \\
&\quad + a_i\big(x_i^*(t)\big)\Bigg[\sum_{j=1}^n k_{ij}(t)\tilde{f}_j\big(w_j\big(t-\tau_{ij}(t)\big)\big) + \bigwedge_{j=1}^n \alpha_{ij}(t)\tilde{f}_j\big(w_j\big(t-\tau_{ij}(t)\big)\big) \\
&\quad + \bigvee_{j=1}^n \beta_{ij}(t)\tilde{f}_j\big(w_j\big(t-\tau_{ij}(t)\big)\big) + \sum_{j=1}^n e_{ij}(t)\int_0^{+\infty} v_{ij}(s)\widetilde{g}_j\big(x_j(t-s)\big)\,\mathrm{d}s \\
&\quad + \bigwedge_{j=1}^n \gamma_{ij}(t)\int_0^{+\infty} v_{ij}(s)\widetilde{g}_j\big(x_j(t-s)\big)\,\mathrm{d}s + \bigvee_{j=1}^n \xi_{ij}(t)\int_0^{+\infty} v_{ij}(s)\widetilde{g}_j\big(x_j(t-s)\big)\,\mathrm{d}s\Bigg] \\
&\quad + \big[a_i\big(x_i(t)\big) - a_i\big(x_i^*(t)\big)\big]\hat{I}_i(t), \quad i = 1, 2, \dots, n, \qquad (10)
\end{aligned}$$

where

$$\widetilde{f}_j\big(w_j\big(t - \tau_{ij}(t)\big)\big) = f_j\big(x_j\big(t - \tau_{ij}(t)\big)\big) - f_j\big(x_j^*\big(t - \tau_{ij}(t)\big)\big)$$

and

$$\widetilde{g}_j\big(x_i(s)\big) = g_j\big(x_j(s)\big) - g_j\big(x_j^*(s)\big).$$

The initial condition of (10) is the following:

$$u_i(s) = \varphi_i(s) - \varphi_i^*(s), \quad s \in (-\infty, 0],\ i = 1, 2, \dots, n.$$

Calculating the upper right derivative of $D^+(|u_i(t)|)$, by (H2)–(H8), we have

$$\begin{aligned}
D^+\big(\big|u_i(t)\big|\big) \leqslant -l_i^a\big|u_i(t)\big| + L_i^a\Bigg[\sum_{j=1}^n (\bar{k}_{ij} + \bar{\alpha}_{ij} + \bar{\beta}_{ij})M_j^f + \bar{\hat{I}}_i\Bigg]\big|u_i(t)\big| \\
+ a_i^M\sum_{j=1}^n (\bar{k}_{ij} + \bar{\alpha}_{ij} + \bar{\beta}_{ij})\bar{v}_{ij}L_j^g\big|\bar{u}_j(t)\big|
\end{aligned}$$

$$= -\left[l_i^a + L_i^a\left(\sum_{j=1}^n(\bar{k}_{ij} + \bar{\alpha}_{ij} + \bar{\beta}_{ij})M_j^f + \bar{\hat{I}}_i\right)\right]\left|u_i(t)\right| + a_i^M\sum_{j=1}^n(\bar{k}_{ij} + \bar{\alpha}_{ij} + \bar{\beta}_{ij})\bar{\upsilon}_{ij}L_j^g\left|\bar{u}_j(t)\right|, \quad i = 1, 2, \ldots, n.$$

Let $\Lambda(t) = [\Theta(t)]^+$, where $[\Theta(t)]^+ = (|u_1(t)|, |u_2(t)|, \ldots, |u_n(t)|)^{\mathrm{T}}$. Then, we have the following inequality:

$$D^+\Lambda(t) \leqslant -A\Lambda(t) + CF^{\mathrm{T}}L\bar{\Lambda}(t),$$

where $\bar{\Lambda}(t) = (\bar{\Lambda}_1(t), \bar{\Lambda}_2(t), \ldots, \bar{\Lambda}_n(t))^{\mathrm{T}}$, $\bar{\Lambda}_i(t) = \sup_{0\leqslant s\leqslant +\infty}\{|u_i(t-s)|\}$, $i = 1, 2, \ldots, n$. According to Lemma 4, there exist constants $\delta > 0$ and $r_k \geqslant 1$ $(k = 1, 2, \ldots, n)$ such that

$$\Lambda_k(t) = \left|z_k(t) - z_k^*(t)\right| \leqslant r_k\sum_{k=1}^n\left|\bar{z}_k(0) - z_k^*(0)\right|\mathrm{e}^{-\delta t}, \quad t \geqslant 0,$$

that is

$$\left|z_k(t) - z_k^*(t)\right| \leqslant \bar{r}\|\phi - \phi^*\|_1\mathrm{e}^{-\delta t}, \quad \bar{r} = \max_{1\leqslant k\leqslant n}\{r_k\},\ t \geqslant 0.$$

Hence, we have

$$\sum_{i=1}^n\left|x_i(t) - x_i^*(t)\right| \leqslant r\mathrm{e}^{-\delta t}\|\varphi - \varphi^*\|_1, \quad r = (n)\bar{r},$$

holds for all $t \geqslant 0$. That is, the anti-periodic solution $z^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^{\mathrm{T}}$ of (1) is globally exponentially stable. This completes the proof. □

5 Anti-periodic solution for fuzzy Cohen–Grossberg BAM neural networks

Consider the following fuzzy Cohen–Grossberg BAM neural networks with time-varying and distributed delays

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i\big(x_i(t)\big)\Bigg[&b_i\big(x_i(t)\big) - \sum_{j=1}^m k_{ji}(t)f_j\big(y_j\big(t-\tau_{ji}(t)\big)\big)\\ &- \bigwedge_{j=1}^m \alpha_{ji}(t)f_j\big(y_j\big(t-\tau_{ji}(t)\big)\big) - \bigvee_{j=1}^m \beta_{ji}(t)f_j\big(y_j\big(t-\tau_{ji}(t)\big)\big)\\ &- \sum_{j=1}^m e_{ji}(t)\int_0^{+\infty}\upsilon_{ji}(s)g_j\big(y_j(t-s)\big)\,\mathrm{d}s\\ &- \bigwedge_{j=1}^m \gamma_{ji}(t)\int_0^{+\infty}\upsilon_{ji}(s)g_j\big(y_j(t-s)\big)\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&- \bigvee_{j=1}^{m} \xi_{ji}(t) \int_{0}^{+\infty} \upsilon_{ji}(s) g_j\big(y_j(t-s)\big)\,\mathrm{d}s - \sum_{j=1}^{m} \delta_{ji}(t)\mu_j(t) \\
&- \bigwedge_{j=1}^{m} T_{ji}(t)\mu_j(t) - \bigvee_{j=1}^{m} H_{ji}(t)\mu_j(t) - I_i(t)\Bigg], \quad i = 1,2,\dots,n, \qquad (11_1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}y_j(t)}{\mathrm{d}t} = {}& -c_j\big(y_j(t)\big)\Bigg[d_j\big(y_j(t)\big) - \sum_{i=1}^{n} p_{ij}(t) h_i\big(x_i\big(t-\theta_{ij}(t)\big)\big) \\
&- \bigwedge_{i=1}^{n} \zeta_{ij}(t) h_i\big(y_i\big(t-\theta_{ij}(t)\big)\big) - \bigvee_{i=1}^{n} q_{ij}(t) h_i\big(y_i\big(t-\theta_{ij}(t)\big)\big) \\
&- \sum_{i=1}^{n} r_{ij}(t) \int_{0}^{+\infty} \nu_{ij}(s)\tilde{h}_i\big(y_i(t-s)\big)\,\mathrm{d}s \\
&- \bigwedge_{i=1}^{n} \tilde{\gamma}_{ij}(t) \int_{0}^{+\infty} \nu_{ij}(s)\tilde{h}_i\big(y_i(t-s)\big)\,\mathrm{d}s \\
&- \bigvee_{i=1}^{n} \tilde{\xi}_{ij}(t) \int_{0}^{+\infty} \nu_{ij}(s)\tilde{h}_i\big(y_i(t-s)\big)\,\mathrm{d}s - \sum_{i=1}^{n} \tilde{\delta}_{ij}(t)\mu_i(t) \\
&- \bigwedge_{i=1}^{n} \tilde{T}_{ij}(t)\tilde{\mu}_i(t) - \bigvee_{i=1}^{n} \tilde{H}_{ij}(t)\tilde{\mu}_i(t) - J_j(t)\Bigg], \quad j = 1,2,\dots,m, \qquad (11_2)
\end{aligned}
$$

where n, m are the number of neurons in layers, $x_i(t)$ and $y_j(t)$ denote the activations of the ith neuron and the jth neuron at time t; $a_i(\cdot)$ and $c_j(\cdot)$ are amplification functions; $b_i(\cdot)$ and $d_j(\cdot)$ represent appropriately behaved functions; f_j, g_j, h_i, $\tilde{h}_i$ denote the activation functions of the jth neuron from F_Y and the ith neuron from F_X, respectively; $0 \leqslant \tau_{ji}(t) \leqslant \tau_{ji}$ and $0 \leqslant \theta_{ij}(t) \leqslant \theta_{ij}$ are transmission delays; $k_{ji}(t)$, $e_{ji}(t)$, $p_{ij}(t)$, $r_{ij}(t)$ are elements of feedback templates at time t and $\delta_{ji}(t)$, $\tilde{\delta}_{ij}(t)$ are elements of feed-forward templates at time t; $\alpha_{ji}(t)$, $\gamma_{ji}(t)$, $\zeta_{ij}(t)$, $\tilde{\gamma}_{ij}(t)$ denote elements of fuzzy feedback MIN templates at time t and $\beta_{ji}(t)$, $\xi_{ji}(t)$, $q_{ij}(t)$, $\tilde{\xi}_{ij}(t)$ are elements of fuzzy feedback MAX templates at time t; $T_{ji}(t)$, $\tilde{T}_{ij}(t)$ are fuzzy feed-forward MIN templates at time t and $H_{ji}(t)$, $\tilde{H}_{ij}(t)$ are fuzzy feed-forward MAX templates at time t, respectively; $\mu_j(t)$, $\tilde{\mu}_i(t)$ denote the input of the ith neuron and the jth neuron at time t; $I_i(t)$, $J_j(t)$ denote biases of the ith neuron and the jth neuron at time t, respectively; υ_{ji}, $\nu_{ij} : (0,+\infty) \to (0,+\infty)$ correspond to the delay kernel functions and satisfy $\int_0^{+\infty} \upsilon_{ji}(s)\,\mathrm{d}s \leqslant \bar{\upsilon}_{ji}$ and $\int_0^{+\infty} \nu_{ij}(s)\,\mathrm{d}s \leqslant \bar{\nu}_{ij}$, where $\bar{\upsilon}_{ji}$ and $\bar{\nu}_{ij}$ are positive constants; $i = 1,2,\dots,n, j = 1,2,\dots,m$; $\wedge$ and $\vee$ denote the fuzzy AND and fuzzy OR operations, respectively.

In view of the proof of Theorem 1 and Theorem 2, since fuzzy Cohen–Grossberg BAM neural network is a special case of fuzzy Cohen–Grossberg neural network, many results of fuzzy Cohen–Grossberg BAM neural networks can be directly obtained from the ones of fuzzy Cohen–Grossberg neural networks, needing no repetitive discussions. We have the following corollaries.

Corollary 1. *Suppose that the following conditions hold*:

(A1) $k_{ji}(t)$, $\alpha_{ji}(t)$, $\tau_{ji}(t)$, $T_{ji}(t)$, $\beta_{ji}(t)$, $H_{ji}(t)$, $\mu_j(t)$, $I_i(t)$, $e_{ji}(t)$, $\gamma_{ji}(t)$, $\xi_{ji}(t)$, $\delta_{ji}(t)$, $p_{ij}(t)$, $\zeta_{ij}(t)$, $\theta_{ij}(t)$, $\tilde{T}_{ij}(t)$, $q_{ij}(t)$, $\tilde{H}_{ij}(t)$, $\tilde{\mu}_i(t)$, $J_j(t)$, $r_{ij}(t)$, $\tilde{\gamma}_{ij}(t)$, $\tilde{\xi}_{ij}(t)$, $\tilde{\delta}_{ij}(t)$ *are $\omega/2$-anti-periodic continuous functions on* $t \in R$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$;

(A2) a_i, $c_j \in C(R, R^+)$, $a_i(-u) = a_i(u)$, $c_j(-u) = c_j(u)$ *and there exist positive constants* a_i^m, a_i^M, c_j^m, c_j^M *such that* $a_i^m \leqslant a_i(u) \leqslant a_i^M$ *and* $c_j^m \leqslant c_j(u) \leqslant c_j^M$ *for all* $u \in R$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$;

(A3) $b_i, d_j \in C(R, R)$ *are differentiable,* $b_i(0) = 0$, $d_j(0) = 0$, $b_i(-u) = -b_i(u)$, $d_j(-u) = -d_j(u)$ *and there exist positive constants* ρ_i, δ_i, ϵ_j, ϑ_j *such that* $0 < \rho_i \leqslant b_i'(u) \leqslant \delta_i$ *and* $0 < \epsilon_j \leqslant d_j'(u) \leqslant \vartheta_j$ *for all* $u \in R$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$;

(A4) f_j, g_j, h_i, $\tilde{h}_i \in C(R, R)$, $f_j(-u) = -f_j(u)$, $g_j(-u) = -g_j(u)$, $h_i(-u) = -h_i(u)$, $\tilde{h}_i(-u) = -\tilde{h}_i(u)$ *and there exist positive constants* L_j^f, M_j, L_i^h *and* σ_i *such that*

$$\big|f_j(u) - f_j(v)\big| \leqslant L_j^f|u - v|, \qquad \big|g_j(u)\big| \leqslant M_j,$$
$$\big|h_i(u) - h_i(v)\big| \leqslant L_i^h|u - v|, \qquad \big|h_i(u)\big| \leqslant \sigma_i$$

for all u, $v \in R$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$;

(A5) $G := \left(\begin{smallmatrix} G_1 & -G_2 \\ -G_3 & G_4 \end{smallmatrix}\right)$ *is an M-matrix, where* $G_1 = \operatorname{diag}(a_i^m - \theta_i a_i^m a_i^M \omega)_{n\times n}$, $G_2 = (\tilde{\nu}_{ij})_{n\times m}$, $G_3 = (\varrho_{ji})_{m\times n}$, $G_4 = \operatorname{diag}(c_i^m - \vartheta_j c_j^m c_j^M \omega)_{m\times m}$, $\tilde{\nu}_{ij} = a_i^M(1/\rho_i + a_i^m\omega)\sum_{j=1}^m(\bar{k}_{ji} + \bar{\alpha}_{ji} + \bar{\beta}_{ji})L_j^f$, $\varrho_{ji} = c_j^M(1/\epsilon_j + c_j^m\omega)\sum_{i=1}^n(\bar{p}_{ij} + \bar{\zeta}_{ij} + \bar{q}_{ij})L_i^h$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$.

Then system (11) *has one $\omega/2$-anti-periodic solution.*

Corollary 2. *Assume that* (A1)–(A5) *hold. Suppose further that*:

(A6) *There exist positive constants* M_j^f, M_i^h *such that* $|f_j(u)| \leqslant M_j^f$ *and* $|h_i(u)| \leqslant M_i^h$ *for all* $u \in R$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$;

(A7) *There exist positive constants* L_i^a, L_j^c *such that*

$$\big|a_i(u) - a_i(v)\big| \leqslant L_i^a|u - v|, \qquad \big|c_j(u) - c_j(v)\big| \leqslant L_j^c|u - v|$$

for all $u, v \in R$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$;

(A8) *There exist positive constants* l_i^a, l_j^c *such that*

$$\big(a_i(u)b_i(u) - a_i(v)b_i(v)\big)(u - v) \geqslant 0,$$
$$\big|a_i(u)b_i(u) - a_i(v)b_i(v)\big| \geqslant l_i^a|u - v|,$$

$$\big(c_j(u)d_j(u) - c_j(v)d_j(v)\big)(u - v) \geqslant 0,$$
$$\big|c_j(u)d_j(u) - c_j(v)d_j(v)\big| \geqslant l_j^c|u - v|,$$

for all $u, v \in R$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$;

(A9) $\Gamma := \left(\begin{smallmatrix} A & -CF^{\mathrm{T}}L \\ -DK^{\mathrm{T}}H & B \end{smallmatrix}\right)$ *is an* M*-matrix, is an* M*-matrix, where* $A = \mathrm{diag}(\zeta_i)_{n\times n}$, $\zeta_i = l_i^a + L_i^a(\sum_{j=1}^m(\bar{k}_{ji} + \bar{\alpha}_{ji} + \bar{\beta}_{ji})M_j^f + \bar{\hat{I}}_i)$, $B = \mathrm{diag}(\rho_j)_{m\times m}$, $\rho_j = l_j^c + L_j^c(\sum_{i=1}^n(\bar{p}_{ij} + \bar{\zeta}_{ij} + \bar{q}_{ij})M_i^h + \bar{\hat{J}}_j$, $C = \mathrm{diag}(a_1^M, a_2^M, \ldots, a_n^M)_{n\times n}$, $D = \mathrm{diag}(c_1^M, c_2^M, \ldots, c_m^M)_{m\times m}$, $F = (F_{ji})_{m\times n}$, $C_{ji} = \bar{k}_{ji} + \bar{\alpha}_{ji} + \bar{\beta}_{ji}$, $L = \mathrm{diag}(L_1^f, L_2^f, \ldots, L_m^f)_{m\times m}$, $K = (K_{ij})_{n\times m}$, $F_{ij} = \bar{p}_{ij} + \bar{\zeta}_{ij} + \bar{q}_{ij}$, $H = \mathrm{diag}(L_1^h, L_2^h, \ldots, L_n^h)_{n\times n}$, $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$.

Then system (11) *has one* $\omega/2$*-anti-periodic solution, which is globally exponentially stable.*

6 An example

In this section, we present an example to illustrate the feasibility of our results obtained in Sections 3 and 4.

Example. Let $n = 2$. Consider the following fuzzy Cohen–Grossberg neural network

$$\begin{aligned}
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(x_i(t))\Bigg[& b_i\big(x_i(t)\big) - \sum_{j=1}^{2} k_{ij}(t)f_j\big(x_j\big(t - \tau_{ij}(t)\big)\big) \\
& - \bigwedge_{j=1}^{n} \alpha_{ij}(t)f_j\big(x_j\big(t - \tau_{ij}(t)\big)\big) - \bigvee_{j=1}^{2} \beta_{ij}(t)f_j\big(x_j\big(t - \tau_{ij}(t)\big)\big) \\
& - \sum_{j=1}^{2} e_{ij}(t) \int_0^{+\infty} v_{ij}(s)g_j\big(x_j(t - s)\big)\,\mathrm{d}s \\
& - \bigwedge_{j=1}^{2} \gamma_{ij}(t) \int_0^{+\infty} v_{ij}(s)g_j\big(x_j(t - s)v)\,\mathrm{d}s \\
& - \bigvee_{j=1}^{2} \xi_{ij}(t) \int_0^{+\infty} v_{ij}(s)g_j\big(x_j(t - s)\big)\,\mathrm{d}s \\
& - \sum_{j=1}^{2} \delta_{ij}(t)\mu_j(t) - \bigwedge_{j=1}^{2} T_{ij}(t)\mu_j(t) - \bigvee_{j=1}^{2} H_{ij}(t)\mu_j(t) - I_i(t)\Bigg], \qquad (12)
\end{aligned}$$

where

$$(a_i)_{2\times 1} = \begin{pmatrix} 1.2 + \frac{2}{\pi}\arctan|u| \\ 1.1 + \frac{2}{\pi}\arctan|u| \end{pmatrix}, \qquad (b_i)_{2\times 1} = \frac{1}{20\pi}\begin{pmatrix} u \\ u \end{pmatrix}, \qquad v_{ij}(s) = \mathrm{e}^{-2s},$$

$$(f_j)_{2\times1} = (g_j)_{2\times1} = \begin{pmatrix} 0.5\sin u \\ 0.4\sin u \end{pmatrix}, \qquad (k_{ij})_{2\times2} = \frac{1}{20\pi}\begin{pmatrix} \sin t & \cos t \\ \cos t & \sin t \end{pmatrix},$$

$$\big(\alpha_{ij}(t)\big)_{2\times2} = \begin{pmatrix} \frac{1}{10}\cos t & \frac{1}{20}\sin t \\ 0 & \frac{1}{5}\cos t \end{pmatrix}, \quad \big(\beta_{ij}(t)\big)_{2\times2} = \begin{pmatrix} -\frac{1}{10}\sin t & 0 \\ \frac{1}{5}\cos t & \frac{1}{20}\sin t \end{pmatrix},$$

$$\big(\delta_{ij}(t)\big)_{2\times2} = \begin{pmatrix} \frac{1}{5}\cos t & 0 \\ 0 & \frac{1}{10}\cos t \end{pmatrix}, \quad \big(T_{ij}(t)\big)_{2\times2} = \begin{pmatrix} 0 & \frac{1}{10}\cos t \\ \frac{1}{4}\cos t & 0 \end{pmatrix},$$

$$\big(H_{ij}(t)\big)_{2\times2} = \begin{pmatrix} 0 & \frac{1}{20}\sin t \\ \frac{1}{5}\sin t & 0 \end{pmatrix}, \quad \big(e_{ij}(t)\big)_{2\times2} = \begin{pmatrix} \frac{1}{5}\cos t & 0 \\ -\frac{1}{10}\sin t & \frac{1}{5}\sin t \end{pmatrix},$$

$$\big(\gamma_{ij}(t)\big)_{2\times2} = \begin{pmatrix} \frac{1}{5}\sin t & \frac{1}{10}\sin t \\ \frac{1}{10}\cos t & \frac{1}{20}\sin t \end{pmatrix}, \quad \big(\xi_{ij}(t)\big)_{2\times2} = \begin{pmatrix} 0 & \frac{1}{5}\sin t \\ \frac{1}{10}\cos t & \frac{1}{10}\cos t \end{pmatrix}.$$

By calculation, we have

$$G = \begin{pmatrix} 0.04 & -0.125 \\ -0.06 & 0.675 \end{pmatrix}, \qquad \varGamma = \begin{pmatrix} -0.3 & -0.42 \\ -0.26 & -0.437 \end{pmatrix}$$

and G, $\varGamma$ are M-matrices. We can verify that all the assumptions in Theorems 1 and 2 are satisfied. Therefore, (12) has a π-anti-periodic solution, which is globally exponentially stable (see Figs. 1 and 2).

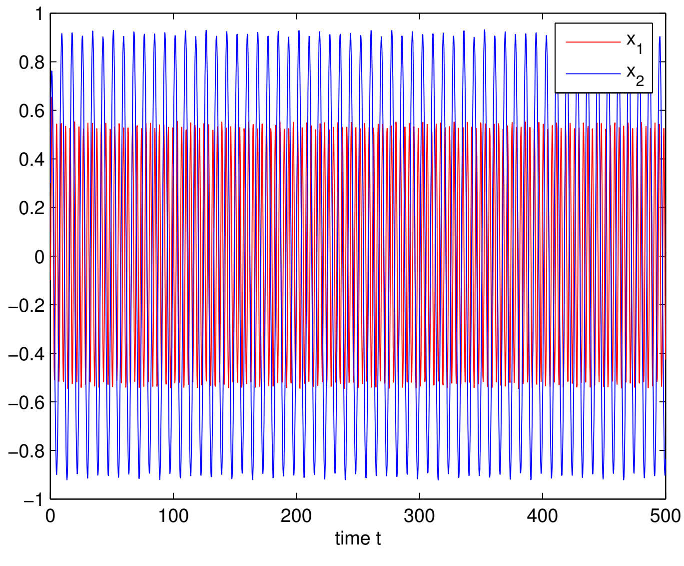

Fig. 1. Responds of x_1, x_2 with time t.

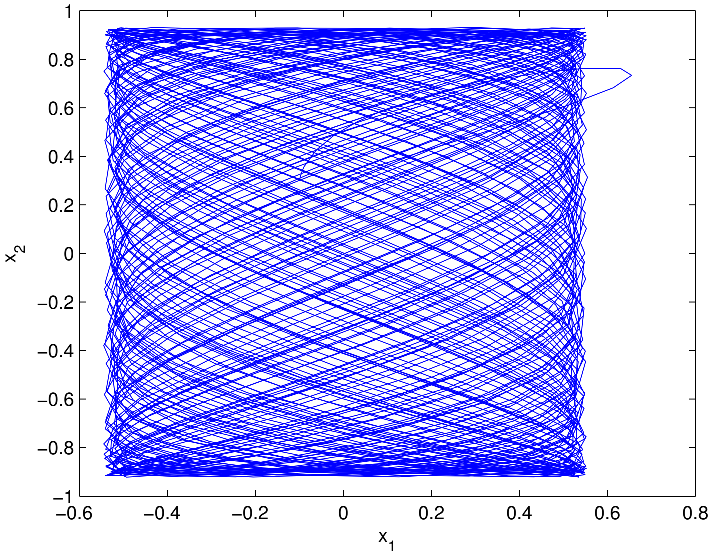

Fig. 2. Responds of x_1, x_2.

Acknowledgments. The authors expresses their gratitude to the referees for their valuable comments and suggestions which have led to a significant improvement on the presentation and quality of this paper.

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