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Soliton solution and bifurcation analysis of the KP–Benjamin–Bona–Mahoney equation with power law nonlinearity*

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Abstract. This paper studies the Kadomtsev–Petviashvili–Benjamin–Bona–Mahoney equation with power law nonlinearity. The traveling wave solution reveals a non-topological soliton solution with a couple of constraint conditions. Subsequently, the dynamical system approach and the bifurcation analysis also reveals other types of solutions with their corresponding restrictions in place.

Keywords: KP–BBM equation, bifurcation phase portraits, traveling wave solutions.

1 Introduction

The study of nonlinear evolution equations (NLEEs) has been going on for quite a few decades now [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. There has been several improvements that are noticed. For example, various techniques of integrating these NLEEs have been developed. In the past, there were essentially a handful few methods that were available to carry out the integration of these NLEEs. Some of them are Inverse Scattering Transform method, Bäcklund transform, Hirota's bilinear method

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and a couple of others. Today it is a different scenario. There are plentiful integration tools that are available and are being readily applied to integrate and compute several solutions to these equations.

In this paper, there will be one such method of integration that will be applied to extract a non-topological soliton solution of the Kadomtsev–Petviashvili–Benjamin–Bona–Mahoney (KP–BBM) equation. This is the traveling wave solution. Subsequently, the bifurcation analysis will be applied to carry out the qualitative theory of the dynamical system that is going to be obtained by the traveling wave hypothesis. The special cases of the KP–BBM equation and the modified KP–BBM equation has already been studied in the past [9, 14]. This paper generalizes those studies and thus encompasses the previous results, since this paper addresses the power law nonlinearity.

From a historical and practical perspective, KP–BBM equation governs the dynamics of two-dimensional shallow water flow. This is analogous to the regular KP equation. The dynamics of (1+1)-dimensional shallow water waves is governed by the Korteweg-de Vries (KdV) equation. Later a different model was proposed in early 1970s by Benjamin, Bona, Mahoney that was later popularly known as the BBM equation. The difference from KdV equation is that a drifting term is introduced and the spatial dispersion term q_{xxx} is replaced by spatio-temporal dispersion, q_{xxt}. Earlier in 1960s this model was proposed by Peregrine and therefore BBM equation is also alternatively referred to Peregrine equation. Now, KP equation is the two-dimensional analogue of KdV equation that models solitons in 2-dimensions along sea shores. On a similar setting, KP–BBM equation is the two-dimensional analogue of BBM equation that also models solitons in 2-dimensions. In other words KP–BBM equation is analogue of KP equation. The generalization to power law nonlinearity is carried out to get a broader picture from a mathematical perspective in order to secure a deeper mathematical insight.

2 Governing equation

The dimensionless form of the KP–BBM equation that is going to be studied in this paper is given by

$$\left\{q_t + aq_x - b\left(q^n\right)_x + cq_{xxt}\right\}_x + dq_{yy} = 0. \tag{1}$$

In (1), $q(x, y, t)$ is the dependent variable while x and y are the spatial variables and t is the temporal variable. Also, a, b, c and d are constants. The power law nonlinearity parameter is n. The special cases of this equation for $n = 2$ and $n = 3$ are already studied before. It needs to be noted that it is necessary to have $n \neq 0, 1$ since these cases fall outside the category of NLEEs.

2.1 Traveling wave solution

In order to solve (1) by the traveling wave hypothesis, the starting ansatz is

$$q(x, y, t) = g(B_1x + B_2y - vt) = g(s), \tag{2}$$

where B_1 and B_2 are the inverse width of the soliton solution that is being sought, while v is the velocity of the soliton. Also,

$$s = B_1 x + B_2 y - vt. \tag{3}$$

Substituting (2) into (1) and integrating twice, while choosing the integration constant to be zero, since the search is for a soliton solution leads to,

$$cvB_1^3 g'' - \left(aB_1^2 - vB_1 + dB_2^2\right)g + bB_1^2 g^n = 0, \tag{4}$$

where $g' = \mathrm{d}g/\mathrm{d}s$ and $g'' = \mathrm{d}^2 g/\mathrm{d}s^2$. Then, multiplying both sides of (3) by g' and integrating again, after choosing the integration constant to be zero, yields, after simplification

$$\frac{\mathrm{d}g}{\mathrm{d}s} = \sqrt{\frac{2b}{(n+1)cvB_1}}\, g\left(\sqrt{\frac{(n+1)(aB_1^2 - vB_1 + dB_2^2)}{2bB_1^2}} - g^{n-1}\right). \tag{5}$$

Now, separating variables in (5) and integrating reveals the non-topological 1-soliton solution as

$$q(x,y,t) = g(B_1 x + B_2 y - vt) = A\,\mathrm{sech}^{1/(n-1)}\left[B(B_1 x + B_2 y - vt)\right], \tag{6}$$

where the amplitude of the soliton is given by

$$A = \left[\frac{(n+1)(aB_1^2 - vB_1 + dB_2^2)}{2bB_1^2}\right]^{1/(2(n-1))} \tag{7}$$

and the parameter B is given by

$$B = (n-1)\sqrt{\frac{2b}{(n+1)cvB_1}}.$$

The amplitude and the parameter pose the constraint conditions given by

$$bcvB_1 > 0 \tag{8}$$

and

$$b\left(aB_1^2 - vB_1 + dB_2^2\right) > 0. \tag{9}$$

Thus, the non-topological 1-soliton of the KP–BBM equation is given by (6), where the amplitude A is given by (7) along with the restrictions given by (8) and (9) that must stay valid in order for the soliton solutions to exist.

3 Bifurcation analysis

This section will carry out the bifurcation analysis of the KP–BBM equation with power law nonlinearity. Initially, the phase portraits will be obtained and the corresponding qualitative analysis will be discussed. Several interesting properties of the solution structure will be obtained based on the parameter regimes. Subsequently, the traveling wave solutions will be discussed from the bifurcation analysis.

3.1 Bifurcation phase portraits and qualitative analysis

Setting $g' = z$, $\alpha = b/(cvB_1)$ and $\beta = (aB_1^2 - vB_1 + dB_2^2)/(cvB_1^3)$, then via (4) we get the following planar system:

$$\frac{\mathrm{d}g}{\mathrm{d}s} = z, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = -\alpha g^n + \beta g. \tag{10}$$

Obviously, the above system (10) is a Hamiltonian system with Hamiltonian function

$$H(g, z) = z^2 + \frac{2\alpha}{n+1} g^{n+1} - \beta g^2.$$

In order to investigate the phase portrait of (10), set

$$f(g) = -\alpha g^n + \beta g.$$

Obviously, when n is odd number and $\alpha\beta > 0$, $f(g)$ has three zero points, g_-, g_0 and g_+, which are given as follows:

$$g_- = -\left(\frac{\beta}{\alpha}\right)^{1/(n-1)}, \qquad g_0 = 0, \qquad g_+ = \left(\frac{\beta}{\alpha}\right)^{1/(n-1)}.$$

When n is even number, $f(g)$ has two zero points, g_0 and g_*, which are given as follows:

$$g_0 = 0, \qquad g_* = \left(\frac{\beta}{\alpha}\right)^{1/(n-1)}.$$

Letting $(g_i, 0)$ be one of the singular points of system (10), then the characteristic values of the linearized system of system (10) at the singular points $(g_i, 0)$ are

$$\lambda_\pm = \pm\sqrt{f'(g_i)}.$$

From the qualitative theory of dynamical systems, we know that:

(1) If $f'(g_i) > 0$, $(g_i, 0)$ is a saddle point.
(2) If $f'(g_i) < 0$, $(g_i, 0)$ is a center point.
(3) If $f'(g_i) = 0$, $(g_i, 0)$ is a degenerate saddle point.

Therefore, we obtain the bifurcation phase portraits of system (10) in Figs. 1 and 2.

Let

$$H(g, z) = h,$$

where h is Hamiltonian. Next, we consider the relations between the orbits of (10) and the Hamiltonian h.

Set

$$h^* = \left|H(g_+, 0)\right| = \left|H(g_-, 0)\right| = \left|H(g_*, 0)\right|.$$

According to Fig. 1, we get the following propositions.

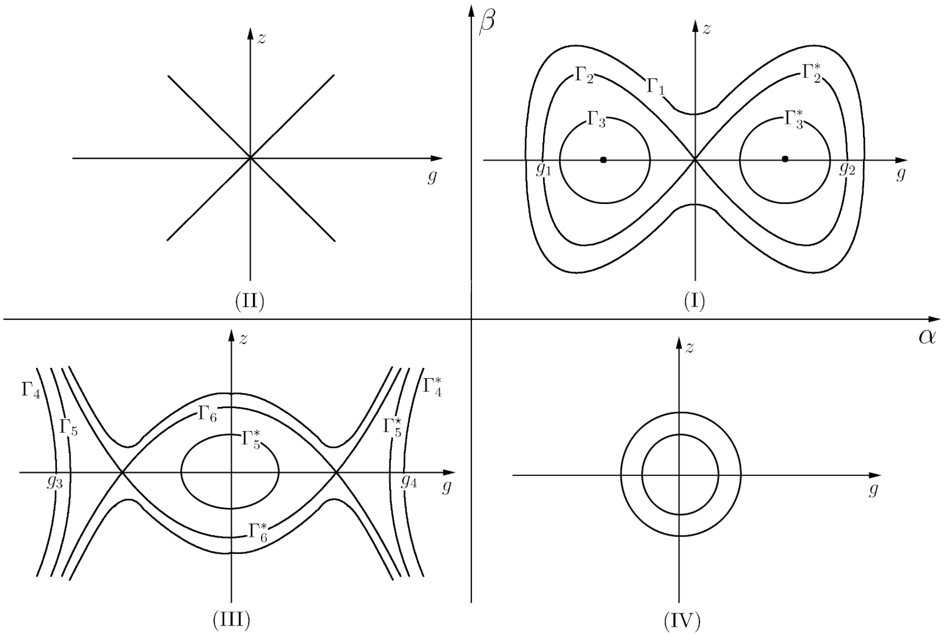

Fig. 1. The bifurcation phase portraits of system (10) when n is odd number: (I) $\alpha > 0$, $\beta > 0$; (II) $\alpha < 0$, $\beta > 0$; (III) $\alpha < 0$, $\beta < 0$; (IV) $\alpha > 0$, $\beta < 0$.

Proposition 1. *When n is odd number, $\alpha > 0$ and $\beta > 0$, we have*:

(1) *When $h > 0$, system* (10) *has a periodic orbit $\mathit{\Gamma}_1$.*
(2) *When $h = 0$, system* (10) *has two homoclinic orbits $\mathit{\Gamma}_2$ and $\mathit{\Gamma}_2^*$.*
(3) *When $-h^* < h < 0$, system* (10) *has two periodic orbits $\mathit{\Gamma}_3$ and $\mathit{\Gamma}_3^*$.*
(4) *When $h \leqslant -h^*$, system* (10) *does not any closed orbit.*

Proposition 2. *When n is odd number, $\alpha < 0$ and $\beta < 0$, we have*:

(1) *When $h = 0$, system* (10) *has two periodic orbits $\mathit{\Gamma}_4$ and $\mathit{\Gamma}_4^*$.*
(2) *When $0 < h < h^*$, system* (10) *has three periodic orbits $\mathit{\Gamma}_5$, $\mathit{\Gamma}_5^*$ and $\mathit{\Gamma}_5^\star$.*
(3) *When $h = h^*$, system* (10) *has two heteroclonic orbits $\mathit{\Gamma}_6$ and $\mathit{\Gamma}_6^*$.*
(4) *When $h < 0$ or $h > h^*$, system* (10) *does not any closed orbit.*

According to Fig. 2, we get the following propositions.

Proposition 3. *When n is even number, $\alpha > 0$ and $\beta > 0$, we have*:

(1) *When $h = 0$, system* (10) *has a homoclinic orbit $\mathit{\Gamma}_7$.*
(2) *When $-h^* < h < 0$, system* (10) *has two periodic orbits $\mathit{\Gamma}_8$ and $\mathit{\Gamma}_8^*$.*
(3) *When $h = -h^*$, system* (10) *has a periodic orbit $\mathit{\Gamma}_9$.*
(4) *When $h < -h^*$ or $h > 0$, system* (10) *does not any closed orbit.*

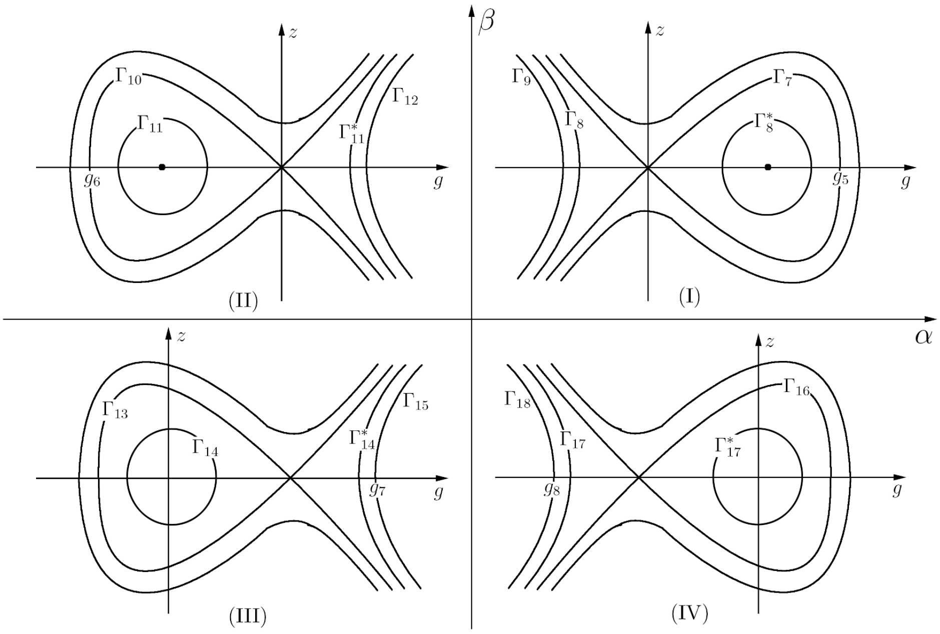

Fig. 2. The bifurcation phase portraits of system (10) when n is even number: (I) $\alpha > 0$, $\beta > 0$; (II) $\alpha < 0$, $\beta > 0$; (III) $\alpha < 0$, $\beta < 0$; (IV) $\alpha > 0$, $\beta < 0$.

Proposition 4. *When n is even number, $\alpha < 0$ and $\beta > 0$, we have*:

(1) *When $h = 0$, system* (10) *has a homoclinic orbit $\mathit{\Gamma}_{10}$.*
(2) *When $-h^* < h < 0$, system* (10) *has two periodic orbits $\mathit{\Gamma}_{11}$ and $\mathit{\Gamma}^*_{11}$.*
(3) *When $h = -h^*$, system* (10) *has a periodic orbit $\mathit{\Gamma}_{12}$.*
(4) *When $h < -h^*$ or $h > 0$, system* (10) *does not any closed orbit.*

Proposition 5. *When n is even number, $\alpha < 0$ and $\beta < 0$, we have*:

(1) *When $h = h^*$, system* (10) *has a homoclinic orbit $\mathit{\Gamma}_{13}$.*
(2) *When $0 < h < h^*$, system* (10) *has two periodic orbits $\mathit{\Gamma}_{14}$ and $\mathit{\Gamma}^*_{14}$.*
(3) *When $h = 0$, system* (10) *has a periodic orbits $\mathit{\Gamma}_{15}$.*
(4) *When $h < 0$ or $h > h^*$, system* (10) *does not any closed orbit.*

Proposition 6. *When n is even number, $\alpha > 0$ and $\beta < 0$, we have*:

(1) *When $h = h^*$, system* (10) *has a homoclinic orbit $\mathit{\Gamma}_{16}$.*
(2) *When $0 < h < h^*$, system* (10) *has two periodic orbits $\mathit{\Gamma}_{17}$ and $\mathit{\Gamma}^*_{17}$.*
(3) *When $h = 0$, system* (10) *has a periodic orbits $\mathit{\Gamma}_{18}$.*
(4) *When $h < 0$ or $h > h^*$, system* (10) *does not any closed orbit.*

From the qualitative theory of dynamical systems, we know that a smooth solitary wave solution of a partial differential system corresponds to a smooth homoclinic orbit of a traveling wave equation. A smooth kink wave solution or a unbounded wave solution corresponds to a smooth heteroclinic orbit of a traveling wave equation. Similarly, a periodic orbit of a traveling wave equation corresponds to a periodic traveling wave solution of a partial differential system. According to above analysis, we have the following propositions.

Proposition 7. *When n is odd number, $\alpha > 0$ and $\beta > 0$, we have*:

(1) *When $h > 0$,* (1) *has two periodic wave solutions* (*corresponding to the periodic orbits Γ_1 in Fig.* 1).
(2) *When $h = 0$,* (1) *has two solitary wave solutions* (*corresponding to the homoclinic orbits Γ_2 and Γ_2^* in Fig.* 1).
(3) *When $-h^* < h < 0$,* (1) *has two periodic wave solutions* (*corresponding to the periodic orbits Γ_3 and Γ_3^* in Fig.* 1).

Proposition 8. *When n is odd number, $\alpha < 0$ and $\beta < 0$, we have*:

(1) *When $h = 0$,* (1) *has four periodic blow-up wave solutions* (*corresponding to the periodic orbit Γ_4 and Γ_4^* in Fig.* 1).
(2) *When $0 < h < h^*$,* (1) *has four periodic blow-up wave solutions and a periodic wave solution* (*corresponding to the periodic orbits Γ_5, Γ_5^* and $\Gamma_5^\star$ in Fig.* 1).
(3) *When $h = h^*$,* (1) *has two kink profile solitary wave solutions and two unbounded wave solutions* (*corresponding to the heteroclinic orbits Γ_6 and Γ_6^* in Fig.* 1).

Proposition 9. *When n is even number, $\alpha > 0$ and $\beta > 0$, we have*:

(1) *When $h = 0$,* (1) *has a solitary wave solution* (*corresponding to the homoclinic orbit Γ_7 in Fig.* 2).
(2) *When $-h^* < h < 0$,* (1) *has a periodic wave solution and two periodic blow-up wave solutions* (*corresponding to the periodic orbits Γ_8 and Γ_8^* in Fig.* 2).
(3) *When $h = -h^*$,* (1) *has two periodic blow-up wave solutions* (*corresponding to the periodic orbit Γ_9 in Fig.* 2).

Proposition 10. *When n is even number, $\alpha < 0$ and $\beta > 0$, we have*:

(1) *When $h = 0$,* (1) *has a solitary wave solution* (*corresponding to the homoclinic orbit Γ_{10} in Fig.* 2).
(2) *When $-h^* < h < 0$,* (1) *has a periodic wave solution and two periodic blow-up wave solutions* (*corresponding to the periodic orbits Γ_{11} and Γ_{11}^* in Fig.* 2).
(3) *When $h = -h^*$,* (1) *has two periodic blow-up wave solutions* (*corresponding to the periodic orbit Γ_{12} in Fig.* 2).

Proposition 11. *When n is even number, $\alpha < 0$ and $\beta < 0$, we have*:

(1) *When $h = h^*$,* (1) *has a solitary wave solution* (*corresponding to the homoclinic orbit Γ_{13} in Fig.* 2).
(2) *When $0 < h < h^*$,* (1) *has a periodic wave solution and two periodic blow-up wave solutions* (*corresponding to the periodic orbits Γ_{14} and Γ_{14}^* in Fig.* 2).

(3) *When $h = 0$, (1) has two periodic blow-up wave solutions (corresponding to the periodic orbit Γ_{15} in Fig. 2).*

Proposition 12. *When n is even number, $\alpha > 0$ and $\beta < 0$, we have:*

(1) *When $h = h^*$, (1) has a solitary wave solution (corresponding to the homoclinic orbit Γ_{16} in Fig. 2).*
(2) *When $0 < h < h^*$, (1) has a periodic wave solution and two periodic blow-up wave solutions (corresponding to the periodic orbits Γ_{17} and Γ_{17}^* in Fig. 2).*
(3) *When $h = 0$, (1) has two periodic blow-up wave solutions (corresponding to the periodic orbit Γ_{18} in Fig. 2).*

3.2 Traveling wave solutions

Firstly, we will obtain the explicit expressions of traveling wave solutions for the (1) when n is odd number, $\alpha > 0$ and $\beta > 0$. From the phase portrait, we see that there are two symmetric homoclinic orbits Γ_2 and Γ_2^* connected at the saddle point $(0, 0)$. In (g, z)-plane, the expressions of the homoclinic orbits are given as

$$z = \pm\sqrt{-\frac{2\alpha}{n+1}g^{n+1} + \beta g^2}. \tag{11}$$

Substituting (11) into $\mathrm{d}g/(\mathrm{d}s) = z$ and integrating them along the orbits Γ_{10} and Γ_{11}, we have

$$\pm\int_{g_1}^{g} \frac{1}{\sqrt{-\frac{2\alpha}{n+1}\xi^{n+1} + \beta\xi^2}}\,\mathrm{d}\xi = \int_0^s \mathrm{d}\xi,$$

$$\pm\int_{g_2}^{g} \frac{1}{\sqrt{-\frac{2\alpha}{n+1}\xi^{n+1} + \beta\xi^2}}\,\mathrm{d}\xi = \int_0^s \mathrm{d}\xi,$$

Completing above integrals, we obtain

$$g = \pm\left(\frac{(n+1)\beta}{\alpha(1 - \cosh((n-1)\sqrt{\beta}s))}\right)^{1/(n-1)}.$$

Using the notations of (2) and (3), we get the following singular solitary wave solutions

$$q_{1\pm}(x, y, t) = \pm\left(\frac{(n+1)\beta}{\alpha(1 - \cosh((n-1)\sqrt{\beta}s))}\right)^{1/(n-1)},$$

where $s = B_1x + B_2y - vt$, $\alpha = b/(cvB_1)$ and $\beta = (aB_1^2 - vB_1 + dB_2^2)/(cvB_1^3)$.

Similarly, when n is even number, substituting (11) into $\mathrm{d}g/(\mathrm{d}s) = z$ and integrating them along the orbits Γ_7 (or Γ_{11}), we get the solitary wave solution $q_{1+}(x, y, t)$ (or $q_{1-}(x, y, t)$).

Secondly, we will obtain the explicit expressions of traveling wave solutions for (1) when n is odd number, $\alpha < 0$ and $\beta < 0$. From the phase portrait, we note that there are two special orbits Γ_4 and Γ_4^*, which have the same hamiltonian with that of the center point $(0,0)$. In (g,z)-plane the expressions of the orbits are given as

$$z = \pm\sqrt{-\frac{2\alpha}{n+1}g^{n+1} + \beta g^2}. \tag{12}$$

Substituting (12) into $\mathrm{d}g/(\mathrm{d}s) = z$ and integrating them along the two orbits Γ_4 and Γ_5, it follows that

$$\pm\int_{g_3}^{g} \frac{1}{\sqrt{-\frac{2\alpha}{n+1}\xi^{n+1} + \beta\xi^2}}\,\mathrm{d}\xi = \int_0^s \mathrm{d}\xi,$$

$$\pm\int_{g_4}^{g} \frac{1}{\sqrt{-\frac{2\alpha}{n+1}\xi^{n+1} + \beta\xi^2}}\,\mathrm{d}\xi = \int_0^s \mathrm{d}\xi,$$

$$\pm\int_{g}^{\infty} \frac{1}{\sqrt{-\frac{2\alpha}{n+1}\xi^{n+1} + \beta\xi^2}}\,\mathrm{d}\xi = \int_0^s \mathrm{d}\xi.$$

Completing above integrals, we obtain

$$g = \pm\left(\frac{(n+1)\beta}{2\alpha}\sec^2\left(\frac{(n-1)\sqrt{-\beta}}{2}s\right)\right)^{1/(n-1)},$$

$$g = \pm\left(\frac{(n+1)\beta}{2\alpha}\csc^2\left(\frac{(n-1)\sqrt{-\beta}}{2}s\right)\right)^{1/(n-1)}.$$

From the notations of (2) and (3), we get the following periodic blow-up wave solutions:

$$q_{2\pm}(x,y,t) = \pm\left(\frac{(n+1)\beta}{2\alpha}\sec^2\left(\frac{(n-1)\sqrt{-\beta}}{2}s\right)\right)^{1/(n-1)},$$

$$q_{3\pm}(x,y,t) = \pm\left(\frac{(n+1)\beta}{2\alpha}\csc^2\left(\frac{(n-1)\sqrt{-\beta}}{2}s\right)\right)^{1/(n-1)},$$

where $s = B_1x + B_2y - vt$, $\alpha = b/(cvB_1)$ and $\beta = (aB_1^2 - vB_1 + dB_2^2)/(cvB_1^3)$.

Similarly, when n is even number, substituting (11) into $\mathrm{d}g/(\mathrm{d}s) = z$ and integrating them along the orbits Γ_{15}(or Γ_{18}), we get the periodic blow-up wave solutions $q_{2+}(x,y,t)$ (or $q_{2-}(x,y,t)$) and $q_{3+}(x,y,t)$ (or $q_{3-}(x,y,t)$).

4 Conclutions

This paper studies the KP–BBM equation with power law nonlinearity. The traveling wave hypothesis is applied to obtain a direct 1-soliton solution to this equation. Subsequently, the bifurcation analysis is carried out to study this equation from a dynamical system point of view. The phase portraits are obtained corresponding to the appropriate parameter regimes. Finally, this bifurcation analysis also extracts a few more traveling wave solutions to this equation. This analysis is very useful in the future study of this equation. In future, the perturbed KP–BBM equation will be considered. Several mathematical analysis will be carried out in presence of perturbation terms and those results will be reported for publication, in future elsewhere.

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