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On a problem for a system of two second-order differential equations via the theory of vector fields

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Abstract. We consider Dirichlet boundary value problem for systems of two second-order differential equations with nonlinear continuous and bounded functions in right-hand sides. We prove the existence of a nontrivial solution to the problem comparing behaviors of solutions of auxiliary Cauchy problems at zero solution and at infinity.

Keywords: multiple solutions, Dirichlet boundary value problem, planar vector field, critical points, rotation of a vector field, winding number.

1 Introduction

The system

$$x'' = -k^2 x, \qquad y'' = -\ell^2 y \tag{1}$$

describes two independent harmonic oscillators. Each of equations in (1) is elementary solvable and the general solution of (1) is known.

A couple of functions $(x(t), y(t))$, which satisfy (1) when visualized on the xy-plane can form somewhat complicated figures. These figures are known as Lissajeux ones and they can be shown using oscilloscopes.

A system

$$x'' + k^2 x = \varphi(x, y), \qquad y'' + \ell^2 y = \psi(x, y) \tag{2}$$

is much more complicate due to possibly nonlinear terms in the right-hand sides. This system can be interpreted as description of motion of a particle of unit mass in a force field defined by the vector

$$\bigl(-k^2 x + \varphi(x, y), -\ell^2 y + \psi(x, y)\bigr).$$

If we are interested in motions which start and end at the origin, we should consider system (2) together with the boundary conditions

$$x(0) = y(0) = x(1) = y(1) = 0. \tag{3}$$

In this paper, we consider the boundary value problem (2), (3). A vector function $(x(t), y(t))$ with $C^2[0,1]$ components is a solution if it turns (2) to identity and satisfies conditions (3).

Suppose that the following conditions are fulfilled:

(A1) coefficients k, ℓ are positive and $k, \ell \notin \{\pi n\colon\ n \in \mathbb{N}\}$;

(A2) functions $\varphi, \psi \in C(\mathbb{R}^2, \mathbb{R})$ are *bounded*;

(A3) $\varphi(0,0) = \psi(0,0) = 0$;

(A4) $\varphi_x, \varphi_y, \psi_x, \psi_y \in C(\mathbb{R}^2, \mathbb{R})$.

Assumption (A1) implies that the homogeneous problem

$$x'' + k^2 x = 0, \qquad y'' + \ell^2 y = 0,$$
$$x(0) = y(0) = x(1) = y(1) = 0$$

has only the trivial solution. Then in view of the boundedness of the functions φ, ψ (A2), problem (2), (3) is known to be solvable [5, Chap. 2, §2]. Under the assumption (A3), it has the trivial solution. Therefore, conditions for the existence of a nontrivial solution of (2), (3) are desirable. In this paper, we provide the conditions for nontrivial solvability of (2), (3) exploiting properties of two-dimensional vector fields.

The paper is organized as follows. Section 2 contains basic theorems of the theory of vector fields and some propositions concerning the vector field induced by solutions of the linear systems. In Section 3, the homotopy results for vector field of a given nonlinear system and vector fields of the respective limiting systems at zero and at infinity are formulated and proved. In the next section, different cases with respect to the coefficients of the limiting linear systems are considered and a rotation of a vector field (winding number) is calculated. The main result about nontrivial solvability of boundary value problem under consideration is formulated in the third subsection of Section 4. In Section 5, an application of the main result is shown, the corresponding examples are analyzed.

2 Tools

We use the theory of vector fields based on the popular sources [2, 3, 4].

2.1 Vector fields

Given $(\alpha, \beta) \in \mathbb{R}^2$, we denote by $(x(t;\alpha,\beta), y(t;\alpha,\beta))$ the solution of (2) such that

$$x(0) = y(0) = 0, \qquad x'(0) = \alpha, \qquad y'(0) = \beta. \tag{4}$$

Define the mapping

$$\varPhi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \varPhi(\alpha, \beta) = \big(x(1;\alpha,\beta), y(1;\alpha,\beta)\big), \tag{5}$$

which is well defined since the solutions of (2), (4) are uniquely defined by the initial data $(0, 0, \alpha, \beta)$ in view of assumption (A4).

Using Wintner's theorem [1, p. 3] and seeing the assumption (A2), can be proved that a solution of (2), (4) $(x(t;\alpha,\beta), y(t;\alpha,\beta))$ is extendable to the whole interval $[0, +\infty)$ and, therefore, to the interval $[0,1]$ for all $(0,0,\alpha,\beta)$.

Definition 1. (See [3, p. 27].) A critical point of the vector field Φ is an isolated point (α,β) in the domain of Φ, where $\Phi(\alpha,\beta) = (0,0)$.

Any critical point generates a solution to problem (2), (3). In order to look for nontrivial solutions of (2), (3), we investigate $\Phi(\alpha,\beta)$ and show that, under certain conditions, there exist $(\alpha,\beta) \neq (0,0)$ such that $\Phi(\alpha,\beta) = (0,0)$.

It is convenient for our purposes to consider the initial conditions (4) in the form

$$x(0) = y(0) = 0, \qquad x'(0) = R\cos\theta, \quad y'(0) = R\sin\theta, \quad 0 \leqslant \theta < 2\pi. \tag{6}$$

The initial values are located on circles C_R of radius R, where R varies from zero to infinity:

$$C_R = \{(\alpha,\beta) \in \mathbb{R}^2\colon\ \alpha^2 + \beta^2 = R^2\}, \quad R \in (0, +\infty).$$

In what follows, we make use of Brower degree $\deg(\Phi, C_R, (0,0))$, which is identical with $\gamma(\Phi; C_R)$ – the rotation of a vector field Φ on a circle C_R. $\gamma(\Phi; C_R)$ always is an integer, it is the number of revolutions made by a vector $\Phi(\alpha,\beta)$ during point (α,β) runs through a circle C_R.

Since our considerations are based on the theory developed in the book [3], we prefer formulate the results, both auxiliary and main ones, in terms of rotations of the vector fields.

Theorem 1. (*See* [3, *Thm.* 3.1].) *Suppose a continuous vector field Φ does not have critical points in a closed domain Ω. Then a rotation of a vector field $\gamma(\Phi; \mathcal{G})$ on the boundary $\mathcal{G}$ of Ω is zero.*

Corollary 1. *In conditions of the above theorem, if $\gamma(\Phi; \mathcal{G}) \neq 0$, then there is a critical point in Ω.*

Definition 2. (See [3, p. 33].) Two vector fields Φ and Ψ defined in some closed domain $\Omega \subset \mathbb{R}^2$ are called *homotopic* if there exists continuous function $\mathcal{F}(M;\mu)$ $(M \in \Omega$, $0 \leqslant \mu \leqslant 1)$ satisfying $\mathcal{F}(M;0) = \Phi(M)$ and $\mathcal{F}(M;1) = \Psi(M)$ for any $M \in \Omega$.

Theorem 2. (*See* [3, *Thm.* 4.1].) *Let $\mathcal{G}$ be a closed Jordan curve. If the vector fields Φ and Ψ are homotopic on $\mathcal{G}$, then the rotations of the vector fields $\gamma(\Phi;\mathcal{G})$ and $\gamma(\Psi;\mathcal{G})$ are equal.*

Definition 3. (See [3, p. 36].) A vector field Ψ is *the principal part* of the vector field Φ if $\Phi(M) = \Psi(M) + \omega(M)$, where $\|\omega(M)\| < \|\Psi(M)\|$.

Theorem 3. (*See* [3, *Thm.* 4.6].) *A vector field $\Phi(M)$ is homotopic to its principal part $\Psi(M)$.*

In order to count a rotation of the vector field Φ, let us transfer the vectors $\Phi(\alpha,\beta)$ such that their beginning will be at the origin $\mathcal{O}=(0,0)$. Then the ends of these vectors will depict a closed curve

$$\Phi(C_R)=\Gamma_R=\big\{(x,y)\in\mathbb{R}^2\colon\ x=x(1;R\cos\theta,R\sin\theta),\ y=y(1;R\cos\theta,R\sin\theta),\\ 0\leqslant\theta<2\pi\big\}.$$

So, a rotation of the vectors $\Phi(\alpha,\beta)$ is the same as a rotation of the radius-vectors of the points on the curve Γ_R, therefore, it is equal to a winding number of a closed curve Γ_R around the origin $\mathcal{O}$

$$\gamma(\Phi;C_R)=n(\Gamma_R,\mathcal{O}).$$

Theorem 4. (*See* [4, *Thm.* 2.8.13].) *Under a continuous deformation of a vector field and a closed curve, the winding number does not change as long as curve does not pass through a critical point of a field.*

So, in accordance with the general theory of planar vector fields, any change in $\gamma(\Phi;C_R)=n(\Gamma_R,\mathcal{O})$, when R varies from zero to infinity, means that the vector field Φ passes through a critical point which is not the origin and this, in turn, means that a non-trivial solution to problem (2), (3) emerges.

2.2 Linear systems

The vector field $\Phi(\alpha,\beta)=(x(1;\alpha,\beta),y(1;\alpha,\beta))$ for $(\alpha,\beta)\in C_R$ can be computed explicitly for solutions of the linear system

$$x''=ax+by,\qquad y''=cx+dy. \tag{7}$$

Lemma 1. (*See* [6, *Lemma* 3.1].) *Let* $(x(t;R\cos\theta,\ R\sin\theta),y(t;R\cos\theta,\ R\sin\theta))$ *be a solution of problem* (7), (6)*, then*

$$\begin{aligned}x(1;R\cos\theta,\ R\sin\theta)&=A_1R\cos\theta+B_1R\sin\theta,\\ y(1;R\cos\theta,\ R\sin\theta)&=A_2R\cos\theta+B_2R\sin\theta.\end{aligned} \tag{8}$$

Remark. Lemma above was proved in [6] for all particular cases with respect to the coefficients a, b, c, d.

Lemma 2. (*See* [6, *Prop.* 3.1].) *If* $A_1/A_2\neq B_1/B_2$*, then for* $0\leqslant\theta<2\pi$*, Eqs.* (8) *define an ellipse* K

$$\big(A_2^2+B_2^2\big)x^2-2(A_1A_2+B_1B_2)xy+\big(A_1^2+B_1^2\big)y^2=(A_1B_2-A_2B_1)^2 \tag{9}$$

with a center at the origin.

If $A_1/A_2=B_1/B_2=q$*, then for* $0\leqslant\theta<2\pi$*, Eqs.* (8) *define a segment of a straight line* $y=x/q$ *when* $x\in[-R\sqrt{A_1^2+B_1^2};R\sqrt{A_1^2+B_1^2}]$.

Remark. Using substitutions $\xi = A_2x - A_1y$ and $\eta = B_2x - B_1y$, Eq. (9) can be reduced to the form $\xi^2 + \eta^2 = (A_1B_2 - A_2B_1)^2$.

Corollary 2. *Mapping $\Phi_{lin} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by* (5) *for the linear system* (7) *(whatever the coefficients a, b, c, d be) maps every circle C_R centered at the origin into either an ellipse or a segment of a straight line centered at the origin.*

If a closed Jordan curve Γ in vector field Φ is given by

$$x = x(\theta), \quad y = y(\theta), \quad 0 \leqslant \theta < 2\pi,$$

then a winding number of this curve around the origin may be calculated by Poincaré formula [3], [4]

$$n(\Gamma, \mathcal{O}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{x(\theta)y'(\theta) - y(\theta)x'(\theta)}{x^2(\theta) + y^2(\theta)} \, \mathrm{d}\theta. \tag{10}$$

Proposition 1.

$$\gamma(\Phi_{lin}; C_R) = n(K, \mathcal{O}) = \begin{cases} 1 & \textit{if } A_1B_2 - A_2B_1 > 0, \\ -1 & \textit{if } A_1B_2 - A_2B_1 < 0. \end{cases}$$

Proof. The proof follows from (8), (10) by direct computation. □

3 Comparison of vector fields

We produce our results by comparison of the vector field $\Phi(\alpha, \beta)$ on circles C_R of small and large radiuses R.

3.1 The vector field at zero

We can get information about solutions around the trivial one using the system of equations of variations

$$\begin{aligned} u'' + k^2u &= \varphi_x(0,0)u + \varphi_y(0,0)v, \\ v'' + \ell^2v &= \psi_x(0,0)u + \psi_y(0,0)v. \end{aligned} \tag{11}$$

Consider the equations of variations (11) together with the initial conditions on the unit circle C_1

$$u(0) = v(0) = 0, \qquad u'(0) = \cos\theta, \quad v'(0) = \sin\theta, \quad 0 \leqslant \theta < 2\pi. \tag{12}$$

Since we operate with the initial data given on a circle, we will refer to this problem as a *generalized Cauchy problem* for Eqs. (11).

We can prove the following auxiliary result about solutions of the Cauchy problems (2), (6).

Lemma 3. *Let $(x(t), y(t))$ be a solution of the nonlinear Cauchy problem* (2)*,* (6) *(θ is fixed). If $R \to 0$, then $(x(t), y(t))$ tends to $(Ru(t), Rv(t))$, where $(u(t), v(t))$ is a solution of the linear problem* (11)*,* (12)*.*

Proof. A solution $(x(t),\ y(t))$ of system (2) subject to the initial conditions (6) can be treated as a solution of the system

$$x'' + k^2 x = \varphi_x(0,0)x + \varphi_y(0,0)y + \varepsilon_1(t,x,y,R),$$
$$y'' + \ell^2 y = \psi_x(0,0)x + \psi_y(0,0)y + \varepsilon_2(t,x,y,R),$$

where ε_1 and ε_2 tend to zero as R tends to zero uniformly in (t,x,y), $t \in [0,1]$, $x^2 + y^2 < R^2$.

Functions $\tilde{u}(t) = x(t) - Ru(t)$ and $\tilde{v}(t) = y(t) - Rv(t)$ satisfy

$$\begin{aligned} \tilde{u}'' + k^2\tilde{u} &= \varphi_x(0,0)\tilde{u} + \varphi_y(0,0)\tilde{v} + \varepsilon_1, \\ \tilde{v}'' + \ell^2\tilde{v} &= \psi_x(0,0)\tilde{u} + \psi_y(0,0)\tilde{v} + \varepsilon_2, \end{aligned} \tag{13}$$

$$\tilde{u}(0) = \tilde{v}(0) = 0, \qquad \tilde{u}'(0) = \tilde{v}'(0) = 0. \tag{14}$$

Since ε_1 and ε_2 tend to zero as $R \to 0$, the solution $(\tilde{u}(t), \tilde{v}(t))$ of (13), (14) uniformly tends to the trivial one as a solution of homogeneous linear Cauchy problem. Therefore, $(x(t), y(t)) \to (R\,u(t), R\,v(t))$ as $R \to 0$. □

Consider a vector field Φ defined by solutions of (2), (6) on a circle C_R of fixed radius R. Denote by Φ_0 a vector field defined by solutions of the system of variations (11) and consider this field on the unit circle C_1 (denote by $\Phi_0|_{C_1}$) and on a circle of fixed radius C_R (denote by $\Phi_0|_{C_R}$). Compare the fields Φ and Φ_0 when radius R tends to zero.

Theorem 5. *A vector field Φ_0 is a principal part of Φ for sufficiently small R and*

$$\gamma(\Phi; C_R) = \gamma(\Phi_0; C_1).$$

Proof. Let $(x(t;\theta), y(t;\theta)) = (x(t; R\cos\theta, R\sin\theta), y(t; R\cos\theta, R\sin\theta))$ be a solution of problem (2), (6), then on a circle of fixed radius R,

$$\Phi = \bigl(x(1;\theta), y(1;\theta)\bigr).$$

If $(u(t;\theta), v(t;\theta)) = (u(t;\cos\theta, \sin\theta), v(t;\cos\theta, \sin\theta))$ is a solution of equations of variations (11), which satisfies normalized initial conditions (12), then on the unit circle C_1,

$$\Phi_0|_{C_1} = \bigl(u(1;\theta), v(1;\theta)\bigr).$$

Since system (11) is linear and Lemma 1 is valid,

$$\Phi_0|_{C_R} = \bigl(Ru(1;\theta), Rv(1;\theta)\bigr).$$

For any point M on a circle C_R, a vector field Φ can be represented as

$$\Phi(M) = \Phi_0|_{C_R}(M) + \sigma_1(M).$$

Since

$$\|\sigma_1(M)\| = \big\|\Phi(M) - \Phi_0|_{C_R}(M)\big\|$$
$$= \sqrt{\big(x(1;\theta) - Ru(1;\theta)\big)^2 + \big(y(1;\theta) - Rv(1;\theta)\big)^2}.$$

In accordance with Lemma 3,

$$\big\|\sigma_1(M)\big\| \underset{R\to 0}{\longrightarrow} 0,$$

then for sufficiently small R, $\|\sigma_1(M)\| < \|\Phi_0|_{C_R}(M)\|$, therefore, Φ_0 is a principal part of Φ and the rotations of the vector fields $\gamma(\Phi; C_R)$ and $\gamma(\Phi_0; C_R)$ are equal (see Theorems 3 and 2). Notice that $\gamma(\Phi_0; C_R) = \gamma(\Phi_0; C_1)$, thus, $\gamma(\Phi; C_R) = \gamma(\Phi_0; C_1)$ for sufficiently small R. □

3.2 The vector field at infinity

A linear system

$$z'' + k^2 z = 0, \qquad w'' + \ell^2 w = 0 \tag{15}$$

describes the behavior of solutions of nonlinear system (2) at infinity.

Consider Eqs. (15) together with the initial conditions on the unit circle C_1

$$z(0) = w(0) = 0, \qquad z'(0) = \cos\theta, \quad w'(0) = \sin\theta, \quad 0 \leqslant \theta < 2\pi. \tag{16}$$

Lemma 4. *Let $(x(t), y(t))$ be a solution of the nonlinear Cauchy problem* (2)*,* (6) *(θ is fixed). If $R \to \infty$, then $(x(t)/R, y(t)/R)$ tends to $(z(t), w(t))$, where $(z(t), w(t))$ is a solution of the linear problem* (15)*,* (16)*.*

Proof. The functions $\tilde{z}(t) = x(t)/R - z(t)$ and $\tilde{w}(t) = y(t)/R - w(t)$ satisfy

$$\tilde{z}'' + k^2\tilde{z} = \frac{\varphi(x,y)}{R}, \qquad \tilde{w}'' + \ell^2\tilde{w} = \frac{\psi(x,y)}{R}, \tag{17}$$

$$\tilde{z}(0) = \tilde{w}(0) = \tilde{z}'(0) = \tilde{w}'(0) = 0. \tag{18}$$

Under assumption (A2), the right-hand sides of the equations in (17) tend to zero as $R \to \infty$, then solution $(\tilde{z}(t), \tilde{w}(t))$ of (17), (18) uniformly tends to the trivial one as a solution of homogeneous linear Cauchy problem. Therefore, $(x(t)/R, y(t)/R) \to (z(t), w(t))$ as $R \to \infty$. □

Denote by Φ_∞ a vector field defined by solutions of system (15) and consider this field on the unit circle C_1 (denote by $\Phi_\infty|_{C_1}$) and on a circle of fixed radius C_R (denote by $\Phi_\infty|_{C_R}$). Compare the fields Φ and Φ_∞ when radius R tends to infinity.

Theorem 6. *A vector field Φ_∞ is a principal part of Φ for large enough R and*

$$\gamma(\Phi; C_R) = \gamma(\Phi_\infty; C_1).$$

Proof. Let $(x(t;\theta), y(t;\theta)) = (x(t; R\cos\theta, R\sin\theta), y(t; R\cos\theta, R\sin\theta))$ be a solution of problem (2), (6), then on a circle of fixed radius R,

$$\Phi = \big(x(1;\theta), y(1;\theta)\big).$$

If $(z(t;\theta), w(t;\theta)) = (z(t;\cos\theta,\sin\theta), w(t;\cos\theta,\sin\theta))$ is a solution of system (15), which satisfies normalized initial conditions (16), then on the unit circle C_1,

$$\Phi_\infty|_{C_1} = \big(z(1;\theta), w(1;\theta)\big).$$

Since system (15) is linear and Lemma 1 is valid,

$$\Phi_\infty|_{C_R} = \big(Rz(1;\theta), Rw(1;\theta)\big).$$

For any point M on a circle C_R, a vector field Φ can be represented as

$$\Phi(M) = \Phi_\infty|_{C_R}(M) + \sigma_2(M).$$

Since

$$\begin{aligned}\|\sigma_2(M)\| &= \big\|\Phi(M) - \Phi_\infty|_{C_R}(M)\big\| \\ &= \sqrt{\big(x(1;\theta) - Rz(1;\theta)\big)^2 + \big(y(1;\theta) - Rw(1;\theta)\big)^2}.\end{aligned}$$

In accordance with Lemma 4, a solution $(x(t), y(t))$ tends to $(Rz(t), Rw(t))$ as R tends to infinity, thus,

$$\big\|\sigma_2(M)\big\| \underset{R\to\infty}{\longrightarrow} 0$$

and, for large enough R, $\|\sigma_2(M)\| < \|\Phi_\infty|_{C_R}(M)\|$, therefore, Φ_∞ is a principal part of Φ and the rotations of the vector fields $\gamma(\Phi; C_R)$ and $\gamma(\Phi_\infty; C_R)$ are equal (see Theorems 3 and 2). Notice that $\gamma(\Phi_\infty; C_R) = \gamma(\Phi_\infty; C_1)$, thus, $\gamma(\Phi; C_R) = \gamma(\Phi_\infty; C_1)$ for large enough R. □

4 Investigation of limiting linear systems

We have two limiting linear systems, namely, system (11), which describes the behavior of solutions of nonlinear system (2) near the trivial solution and uncoupled linear system (15), which describes the behavior of solutions of (2) at infinity.

4.1 Investigation of uncoupled linear system at infinity

The vector field Φ_∞ on the unit circle C_1 can be computed explicitly for solutions of system (15).

Proposition 2. *Let* $(z(t;\theta), w(t;\theta))$ *be a solution of problem* (15), (16)*, then*

$$z(1;\theta) = \frac{\sin k}{k}\cos\theta, \qquad w(1;\theta) = \frac{\sin\ell}{\ell}\sin\theta. \tag{19}$$

For $0 \leqslant \theta < 2\pi$*, Eqs.* (19) *define a closed curve* Γ_∞*, which is an ellipse*

$$\Gamma_\infty\colon \quad \frac{k^2}{\sin^2 k} z^2 + \frac{\ell^2}{\sin^2 \ell} w^2 = 1.$$

Proposition 3.

$$\gamma(\Phi_\infty; C_1) = n(\Gamma_\infty, \mathcal{O}) = \begin{cases} 1 & \text{if } \sin k \sin \ell > 0, \\ -1 & \text{if } \sin k \sin \ell < 0. \end{cases}$$

Proof. The proof follows from (19), (10) by direct computation. □

4.2 Investigation of limiting linear system at zero

The limiting system at zero is given by (11) and after regrouping the terms it looks

$$u'' = \big(\varphi_x(0,0) - k^2\big)u + \varphi_y(0,0)v,$$
$$v'' = \psi_x(0,0)u + (\psi_y(0,0) - \ell^2)v,$$

or

$$u'' = au + bv, \qquad v'' = cu + dv, \tag{20}$$

where

$$\begin{aligned} a &= \varphi_x(0,0) - k^2, & b &= \varphi_y(0,0), \\ c &= \psi_x(0,0), & d &= \psi_y(0,0) - \ell^2. \end{aligned} \tag{21}$$

The characteristic equation for (20) is

$$\lambda^4 - (a+d)\lambda^2 + (ad - bc) = 0. \tag{22}$$

A form of solution of the system (20) and therefore a vector field Φ_0 depends on values of a, b, c, d, but in all cases the following results are valid.

Proposition 4. *Let* $u(t;\theta), v(t;\theta))$ *be a solution of system* (20)*, which satisfy* (12)*, then*

$$\begin{aligned} u(1;\theta) &= A_1 \cos\theta + B_1 \sin\theta, \\ v(1;\theta) &= A_2 \cos\theta + B_2 \sin\theta. \end{aligned} \tag{23}$$

If $A_1/A_2 \neq B_1/B_2$*, then for* $0 \leqslant \theta < 2\pi$*, Eqs.* (23) *define an ellipse* $\Gamma_0 = \Phi_0(C_1)$ *centered at the origin and*

$$\gamma(\Phi_0;\, C_1) = n(\Gamma_0, \mathcal{O}) = \begin{cases} 1 & \text{if } \Delta > 0, \\ -1 & \text{if } \Delta < 0, \end{cases}$$

where $\Delta = A_1B_2 - A_2B_1$.

Calculations of a winding number of an ellipse Γ_0 around the origin (or calculations of a rotation of a vector field Φ_0 on the unit circle C_1) in particular cases was conducted in the work [6]. Therefore, we omit the proofs for the cases below and provide only formulations.

Case 1. If the conditions

$$\text{(C1)} \qquad (a-d)^2 + 4bc > 0, \qquad ad - bc > 0, \qquad a + d < 0$$

hold, then the characteristic equation (22) has two pairs of conjugate purely imaginary roots

$$\lambda_{1,2} = \pm m_1 \mathrm{i}, \quad m_1 = \sqrt{\frac{-a-d+\sqrt{(a-d)^2+4bc}}{2}},$$
$$\lambda_{3,4} = \pm m_2 \mathrm{i}, \quad m_2 = \sqrt{\frac{-a-d-\sqrt{(a-d)^2+4bc}}{2}}.$$

Proposition 5. *If conditions* (C1) *are fulfilled, then*

$$\gamma(\Phi_0; C_1) = n(\Gamma_0, \mathcal{O}) = \begin{cases} 1 & \textit{if } \sin m_1 \sin m_2 > 0, \\ -1 & \textit{if } \sin m_1 \sin m_2 < 0. \end{cases}$$

Case 2. If the conditions

$$\text{(C2)} \qquad (a-d)^2 + 4bc > 0, \qquad ad - bc < 0$$

hold, then the characteristic equation (22) has a pair of opposite real roots and a pair of conjugate purely imaginary roots

$$\lambda_{1,2} = \pm s, \qquad s = \sqrt{\frac{a+d+\sqrt{(a-d)^2+4bc}}{2}},$$
$$\lambda_{3,4} = \pm m\mathrm{i}, \quad m = \sqrt{\frac{-a-d+\sqrt{(a-d)^2+4bc}}{2}}.$$

Proposition 6. *If conditions* (C2) *are fulfilled, then*

$$\gamma(\Phi_0; C_1) = n(\Gamma_0, \mathcal{O}) = \begin{cases} 1 & \textit{if } \sin m > 0, \\ -1 & \textit{if } \sin m < 0. \end{cases}$$

Case 3. If the conditions

$$\text{(C3)} \qquad ad - bc = 0, \qquad a + d < 0$$

hold, then the characteristic equation (22) has a pair of conjugate purely imaginary roots and a double zero

$$\lambda_{1,2} = 0, \qquad \lambda_{3,4} = \pm n\mathrm{i}, \quad n = \sqrt{-(a+d)}.$$

Proposition 7. *If conditions* (C3) *are fulfilled, then*

$$\gamma(\Phi_0;\, C_1) = n(\Gamma_0, \mathcal{O}) = \begin{cases} 1 & \textit{if } \sin n > 0, \\ -1 & \textit{if } \sin n < 0. \end{cases}$$

Cases 4–9. If the conditions

(C4) $$(a-d)^2+4bc>0, \qquad ad-bc>0, \qquad a+d>0$$

hold, then the characteristic equation (22) has two pairs of opposite real roots

$$\lambda_{1,2}=\pm s_1, \quad s_1=\sqrt{\frac{a+d+\sqrt{(a-d)^2+4bc}}{2}},$$
$$\lambda_{3,4}=\pm s_2, \quad s_2=\sqrt{\frac{a+d-\sqrt{(a-d)^2+4bc}}{2}}.$$

If the conditions

(C5) $$(a-d)^2+4bc=0, \qquad a+d>0$$

hold, then the characteristic equation (22) has multiple real roots

$$\lambda_{1,2}=\sqrt{\frac{a+d}{2}}=r, \qquad \lambda_{3,4}=-\sqrt{\frac{a+d}{2}}=-r.$$

If the conditions

(C6) $$(a-d)^2+4bc=0, \qquad a+d<0$$

hold, then the characteristic equation (23) has multiple imaginary roots

$$\lambda_{1,2}=p\mathrm{i}, \quad \lambda_{3,4}=-p\mathrm{i}, \quad p=\sqrt{-\frac{a+d}{2}}.$$

If the condition

(C7) $$(a-d)^2+4bc<0$$

holds, then the characteristic equation (22) has the complex roots

$$\lambda_{1,2,3,4}=\pm\delta\pm\mu\mathrm{i}, \quad \text{where } \delta\neq 0,\ \mu^2-\delta^2=-\frac{a+d}{2}.$$

If the conditions

(C8) $$ad-bc=0, \qquad a+d>0$$

hold, then the characteristic equation (22) has a pair of opposite real roots and a double zero

$$\lambda_{1,2}=0, \qquad \lambda_{3,4}=\pm q, \quad q=\sqrt{a+d}.$$

If the conditions

(C9) $$ad-bc=0, \qquad a+d=0$$

hold, then only zero is multiple root of the characteristic equation (22)

$$\lambda_{1,2,3,4}=0.$$

Proposition 8. *If one of conditions* (C4)–(C9) *is fulfilled, then*

$$\gamma(\Phi_0; C_1) = n(\Gamma_0, \mathcal{O}) = 1.$$

4.3 Main result

It follows from the above considerations that the following result is true.

Theorem 7. *Let conditions* (A1) *to* (A4) *hold. Let Φ_0 and Φ_∞ are vector fields defined by solutions of the linear problems* (11), (12) *and* (15), (16), *respectively. If*

$$\gamma(\Phi_0; C_1)\gamma(\Phi_\infty; C_1) = -1,$$

then problem (2), (3) *has at least one nontrivial solution.*

For any of the above cases, an appropriate theorem may be formulated about the existence of a nontrivial solution of the boundary value problem (2), (3). For instance, the following theorem is valid.

Theorem 8. *Let conditions* (A1) *to* (A4) *and* (C1) *hold. If $\sin k \sin \ell \sin m_1 \sin m_2 < 0$, where $\pm m_1$i, $\pm m_2$i $(m_1, m_2 > 0)$ are the roots of the characteristic equation for the linear system* (20) *with* (21)*, then the nonlinear problem* (2), (3) *has at least one nontrivial solution.*

5 Examples

Example 1. Consider the problem

$$\begin{aligned} &x'' + 50x = 16\sin(y + 3x^2), \\ &y'' + 22y = -12\arctan x, \\ &x(0) = y(0) = x(1) = y(1) = 0. \end{aligned} \tag{24}$$

In this case, $k^2 = 50$, $\ell^2 = 22$, $\varphi = 16\sin(y + 3x^2)$ and $\psi = -12\arctan x$, conditions (A1) to (A4) hold.

A behavior of solutions of given system at infinity may be described by solutions of the linear problem

$$z'' + 50z = 0, \qquad w'' + 22w = 0.$$

Since $\sin k \sin \ell = \sin\sqrt{50}\sin\sqrt{22} \approx -0.709 < 0$, then $\gamma(\Phi_\infty; C_1) = -1$.

Since $\varphi_x(0,0) = 0$, $\varphi_y(0,0) = 16$, $\psi_x(0,0) = -12$, $\psi_y(0,0) = 0$, then the linear system

$$u'' = -50u + 16v, \qquad v'' = -12u - 22v \tag{25}$$

describes a behavior of solutions of system in (24) near the trivial solution. The coefficients of system (25) satisfy conditions (C1) (see Case 1). Since $\sin m_1 \sin m_2 = \sin\sqrt{38}\sin\sqrt{34} \approx 0.052 > 0$, $\gamma(\Phi_0; C_1) = 1$.

Therefore, applying Theorem 8, there exists at least one nontrivial solution of the given problem (24). We have computed it. Fig. 1a illustrates a solution of (24) with initial data $x'(0) = 3.6834$, $y'(0) = 1.6284$, Fig. 1b shows phase portrait of this solution.

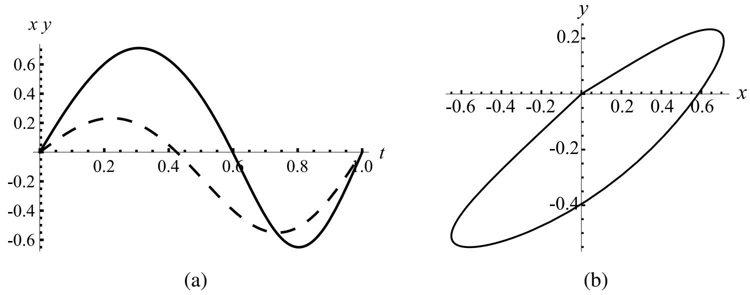

Fig. 1. Nontrivial solution of problem (24).

Example 2. For the problem

$$x'' + \frac{\pi^2}{4}x = 3\arctan x + \arctan y,$$
$$y'' + \frac{9\pi^2}{4}y = \sin y, \tag{26}$$
$$x(0) = y(0) = x(1) = y(1) = 0,$$

the limiting linear systems, which describe a behavior of solutions of system in (26) at infinity and near the trivial solution, are following:

$$\begin{cases} z'' + \frac{\pi^2}{4}z = 0, \\ w'' + \frac{9\pi^2}{4}w = 0, \end{cases} \qquad \begin{cases} u'' = (3 - \frac{\pi^2}{4})u + v, \\ v'' = (1 - \frac{9\pi^2}{4})v, \end{cases}$$

We obtain that $\sin k \sin \ell = \sin(\pi/2)\sin(3\pi/2) = -1 < 0$, then $\gamma(\Phi_\infty; C_1) = -1$.

One has that $a = 3 - \pi^2/4$, $b = 1$, $c = 0$, $d = 1 - 9\pi^2/4$, thus, conditions (C2) hold. Since $\sin m = \sin\sqrt{-1 + 9\pi^2/4} \approx -0.9943 < 0$, $\gamma(\Phi_0; C_1) = -1$ and, therefore, no conclusion can be made about solvability of problem (26).

In fact, calculations show that the first and the second zeros $t_1(\alpha)$ and $t_2(\alpha)$ of solutions $y(t;\alpha)$ of the Cauchy problems

$$y'' + \frac{9\pi^2}{4}y = \sin y, \qquad y(0) = 0, \qquad y'(0) = \alpha$$

satisfy the inequalities $t_1(\alpha) < 1$ and $t_2(\alpha) > 1$ for all $\alpha \neq 0$ and never pass through $t = 1$. Therefore, no nontrivial solutions of problem (26).

Example 3. Consider the problem

$$x'' + (\pi + 0.1)^2 x = \arctan(x + y^2),$$
$$y'' + 49y = 10\arctan y, \tag{27}$$
$$x(0) = y(0) = 0 = x(1) = y(1)$$

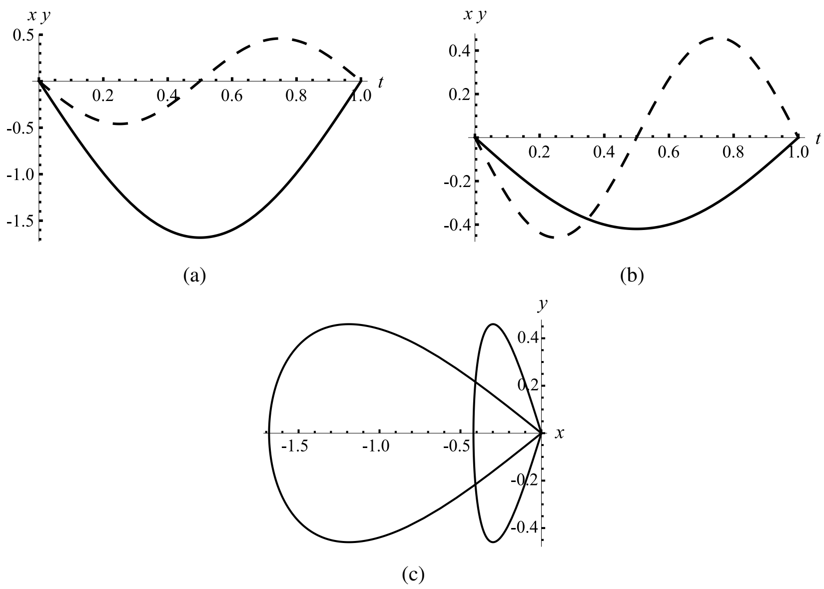

Fig. 2. Nontrivial solutions of problem (27): (a) $x'(0) = -5.27309$, $y'(0) = -2.88183$; (b) $x'(0) = -1.33054$, $y'(0) = -2.88182$.

and two corresponding limiting linear systems at infinity and at zero, respectively, both systems are uncoupled:

$$\begin{cases} z'' + (\pi + 0.1)^2 z = 0, \\ w'' + 49w = 0, \end{cases} \qquad \begin{cases} u'' + ((\pi + 0.1)^2 - 1)u = 0, \\ v'' + 39v = 0. \end{cases}$$

The vector fields Φ_∞ and Φ_0 induced by solutions of the linear systems above have the same rotation, indeed $\sin(\pi + 0.1)\sin\sqrt{49} \approx -0.0656$, thus, $\gamma(\Phi_\infty; C_1) = -1$; $\sin\sqrt{(\pi + 0.1)^2 - 1}\sin\sqrt{39} \approx -0.0022$, thus, $\gamma(\Phi_0; C_1) = -1$.

Theorem 7 establishes the sufficient condition for nontrivial solvability of the problem. In the case of a given problem (27), this condition is not fulfilled, but we have computed two different solutions of (27). In addition, the initial values of y for these solutions differ little from each other. Their graphs and phase portraits are shown in Fig. 2.

6 Conclusions

Solutions of systems of the type (2) define mappings $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$. These mappings map the circles C_R of initial values into closed curves in $x(1)y(1)$-plane. The image of circle C_R is simply ellipse Γ_R in case system (2) is linear. A winding number of an ellipse Γ_R around the origin $\mathcal{O}$ is equal to 1 or -1.

A behavior of solutions of nonlinear system (2) near the trivial solution and at infinity can be described by two limiting linear systems (11) and (15), respectively. Under some additional conditions with respect to the coefficients of limiting linear systems, the winding numbers of ellipses $\mathit{\Gamma}_0$ and $\mathit{\Gamma}_\infty$ are different. This means that the boundary value problem (2), (3) has at least one nontrivial solution.

In a case the winding numbers of $\mathit{\Gamma}_0$ and $\mathit{\Gamma}_\infty$ around the origin are equal, no conclusion can be made about nontrivial solvability of problem (2), (3), further investigation is needed.

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