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Distributional boundary values of analytic functions and positive definite distributions*

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Abstract. We propose necessary and sufficient conditions for a distribution (generalized function) f of several variables to be positive definite. For this purpose, certain analytic extensions of f to tubular domains in complex space $\mathbb{C}^n$ are studied. The main result is given in terms of the Cauchy transform and completely monotonic functions.

Keywords: positive definite functions, positive definite distributions, Cauchy transform, analytic representations of distributions, completely monotonic functions, convex cones, complex tubular domains, Plemelj formulas.

1 Introduction

A complex-valued function f on $\mathbb{R}^n$ is said to be positive definite if

$$\sum_{j,k=1}^{n} f(x_j - x_k) c_j \bar{c}_k \geqslant 0 \tag{1}$$

for any finite sets $x_1, \dots, x_n \in \mathbb{R}^n$ and for any $c_1, \dots, c_n \in \mathbb{C}$. The Bochner theorem (see, e.g., [5, p. 293] and [2, p. 58]) states that continuous $f : \mathbb{R}^n \to \mathbb{C}$ is positive definite if and only if it is the Fourier transform of a positive finite measure μ on $\mathbb{R}^n$, i.e.,

$$f(x) = \hat{\mu}(x) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x,t)}\, \mathrm{d}\mu(t),$$

$x \in \mathbb{R}^n$. Here and later, for z and λ in $\mathbb{R}^n$ or in $\mathbb{C}^n$, we write $(z, \lambda) = z_1\overline{\lambda_1} + \dots + z_n\overline{\lambda_n}$.

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Definition (1) cannot carry over to distributions (to generalized functions). Therefore, it is convenient to replace (1) by

$$\int\limits_{\mathbb{R}^n} f(x)(\varphi * \varphi^\star)(x)\,\mathrm{d}x \geqslant 0, \quad \varphi^\star(x) := \overline{\varphi(-x)}, \tag{2}$$

where φ runs over $L^1(\mathbb{R}^n)$ or φ runs over all continuous functions on $\mathbb{R}^n$ with compact support. Here $u * v$ denotes the convolution

$$u * v(x) = \int\limits_{\mathbb{R}^n} u(x-t)v(t)\,\mathrm{d}t.$$

If f is continuous, then (2) is equivalent to (1) (see, e.g., [19, p. 420]). Property (2) can be taken as a definition for positive definite distributions. Let us recall some notion. We shall follow [21].

The Schwartz space $S(\mathbb{R}^n)$ consists of infinitely differentiable functions ω such that

$$\sup_{x\in\mathbb{R}^n,\,|u|\leqslant k} \bigl|\bigl(1+\|x\|_2\bigr)^s D_x^u \omega(x)\bigr| < \infty$$

for all $k, s \in \mathbb{N}$. Here u is a non-negative integer multi-index, $|u| = \sum_{j=1}^n u_j$,

$$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2},$$

and $D_x^u = D_{x_1}^{u_1} \cdots D_{x_n}^{u_n}$, where

$$D_{x_j} = \frac{\partial}{\partial x_j}.$$

The set of continuous linear functionals on $S(\mathbb{R}^n)$ is denoted by $S'(\mathbb{R}^n)$. Each $f \in S'(\mathbb{R}^n)$ is called a tempered distribution and the action of f on a test function $\omega \in S(\mathbb{R}^n)$ is written as (f, ω).

Let $D(\mathbb{R}^n)$ be the subspace of $S(\mathbb{R}^n)$ consisting of functions with a compact support. The topology on $D(\mathbb{R}^n)$ is introduced as usual (see [21]). The elements of $D'(\mathbb{R}^n)$ are called distributions. Note that $D(\mathbb{R}^n) \subset S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$ are true in the sense of topological spaces.

A distribution $f \in D'(\mathbb{R}^n)$ is said to be positive definite if

$$(f, \varphi * \varphi^\star) \geqslant 0 \tag{3}$$

for all $\varphi \in D(\mathbb{R}^n)$. The Bochner–Schwartz theorem [21, p. 125] states that $f \in D'(\mathbb{R}^n)$ is positive definite if and only if f is the Fourier transform of a non-negative tempered measure on $\mathbb{R}^n$. Recall that a non-negative measure η on $\mathbb{R}^n$ is said to be tempered if there exists α, $0 \leqslant \alpha < \infty$ such that

$$\int\limits_{\mathbb{R}^n} \bigl(1 + \|x\|_2^2\bigr)^{-\alpha}\,\mathrm{d}\eta(x) < \infty.$$

There are many characterizations of positive definite functions (see, e.g., [8, pp. 70–83]). As far as we known, it is perhaps surprising that there are almost no such results for positive definite distributions. We mention only [17], where attention has been paid to positive definite measures on $\mathbb{R}$, i.e., to distributions of order zero, with applications to a Volterra equation. See also [4] and [7].

Tillmann [20] proved that any $f \in S'(\mathbb{R})$ with a compact support has a decomposition into a positive and a negative distributional frequency parts

$$f = f_{(+)} - f_{(-)}. \tag{4}$$

Here $f_{(+)}$ is the boundary value (on $\mathbb{R}$), in the sense of convergence in $S'(\mathbb{R})$, of certain $g_{(+)}$ that is analytic in the open upper half-plane $\mathbb{C}_{(+)}$. Similarly, $f_{(-)}$ is the boundary value of $g_{(-)}$ that is analytic in $\mathbb{C}_{(-)} = -\mathbb{C}_{(+)}$. Note that (4) is a distributional counterpart of the first Plemelj formula (see [12, p. 358], [1, pp. 155–157], and [13, pp. 4–5]). Then $\{g_{(-)}, g_{(+)}\}$ defines a sectionally analytic function on $\mathbb{C} \setminus \mathbb{R}$. It is called an analytic representation of $f \in S'(\mathbb{R})$. Note that an analytic representation of f is not unique and differs from other representations by at most an entire function.

Let $f \in S'(\mathbb{R})$. If, in addition, f has a compact support, then

$$K(f)(z) = \frac{1}{2\pi i}\left(f_t, \frac{1}{t-z}\right) := \frac{1}{2\pi i}\left(f(\cdot), \frac{1}{\cdot - z}\right) \tag{5}$$

is well defined for all $z \in \mathbb{C} \setminus \mathbb{R}$. The function $K(f)(z)$ is called the Cauchy transform of f and gives an analytic representations for f (see, e.g., [3, p. 73]). Unfortunately, $K(f)$ does not exist, in general, for all $f \in S'(\mathbb{R})$ (see [1, p. 156]). Even so, any $f \in S'(\mathbb{R})$ has a finite order ϱ_f (see [21, p. 77]). Therefore, if $m \geqslant \varrho_f$, then the following generalized Cauchy transform $(f, (z-t)^{-(m+1)})$ is well defined. We derived in [9] necessary and sufficient conditions for $f \in S'(\mathbb{R})$ to be a positive definite distribution in terms of this generalized transform and completely monotonic functions. Let us recall that a function $\theta : (a, b) \to \mathbb{R}$, $-\infty \leqslant a < b \leqslant \infty$, is said to be completely monotonic if it is infinitely differentiable and for its nth derivative functions $\theta^{(n)}$

$$(-1)^n \theta^{(n)}(y) \geqslant 0$$

for each $y \in (a, b)$ and all $n = 0, 1, 2, \ldots$. Further, $\theta(y)$ is said to be absolutely monotonic on (a, b) if a $\theta(-y)$ is completely monotonic on $(-b, -a)$.

Theorem 1. (See [9, *Thm.* 1.3].) *Let $f \in S'(\mathbb{R})$ and let n be an integer such that $2n \geqslant \varrho_f$. Suppose $a_1, a_2 \in \mathbb{R}$ and $a_1 \neq a_2$. Let*

$$\widetilde{K}(f, j)(z) = (-1)^n \frac{\mathrm{i}}{\pi}\left(\mathrm{e}^{\mathrm{i}a_j t} f_t, \frac{1}{(z-t)^{2n+1}}\right) \tag{6}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$ and $j = 1, 2$. Then f is positive definite if and only if:

(i) $y \to \widetilde{K}(f, j)(\mathrm{i}y)$, $j = 1, 2$, *are completely monotonic functions for* $y \in (0, \infty)$;

(ii) $y \to -\widetilde{K}(f, j)(\mathrm{i}y)$, $j = 1, 2$, *are absolutely monotonic functions for* $y \in (-\infty, 0)$.

Although the Cauchy kernel $(t-z)^{-1} \notin S(\mathbb{R})$, it belongs to another Schwartz test functions spaces $D_{L_p}(\mathbb{R})$ for each $1 < p \leqslant \infty$ (we give a precise definition later). Thus, the usual Cauchy representation (5) seems possible for all $f \in D'_{L_p}(\mathbb{R}) \subset S'(\mathbb{R})$ (see, e.g., [10, p. 457]). For this reason, we investigate in this paper positive definite distributions in $D'_{L_p}(\mathbb{R}^n)$.

Let $D_{L^p}(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$ (see [15, pp. 199–205]), denote the space of complex-valued functions φ on $\mathbb{R}^n$ such that $D_x^u \varphi(x) \in L_p(\mathbb{R}^n)$ for all non-negative integer multi-indexes u. Obviously,

$$D(\mathbb{R}^n) \subset S(\mathbb{R}^n) \subset D_{L^p}(\mathbb{R}^n). \tag{7}$$

The topology of $D_{L^p}(\mathbb{R}^n)$ is given in terms of countably family of seminorms

$$\|\varphi\|_{p,u} = \left\| D_x^u \varphi(x) \right\|_{L^p(\mathbb{R}^n)}. \tag{8}$$

Since $\|\cdot\|_{p,0}$ is a norm, it follows that the family (8) defines on $D_{L^p}(\mathbb{R}^n)$ a sequentially complete locally convex topology.

Suppose $1 < p, q < \infty$, $1/p + 1/q = 1$. According to Schwartz [15, p. 200], we define $D'_{L^p}(\mathbb{R}^n)$ as the dual space of $D_{L^q}(\mathbb{R}^n)$. Note that if $\varphi \in D_{L^p}(\mathbb{R}^n)$ and $1 \leqslant p < \infty$, then

$$\lim_{x \to \infty} D_x^u \varphi(x) = 0 \tag{9}$$

for all u (see [15, p. 200]). Hence, convergence in $D(\mathbb{R}^n)$ or in $S(\mathbb{R}^n)$ implies convergence in $D_{L^p}(\mathbb{R}^n)$, $1 \leqslant p < \infty$. This means that (7) is also true in the sense of topological spaces. Hence, any $f \in D'_{L^p}(\mathbb{R}^n)$ can be identified with a distribution in $S'(\mathbb{R}^n)$. Thus, for any $1 < p < \infty$, we get

$$D'_{L^p}(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \subset D'(\mathbb{R}^n). \tag{10}$$

We wish to study the Cauchy transform of $f \in D'_{L^p}(\mathbb{R}^n)$ as an analytic representation of f. For this purpose, let us define at first the Cauchy kernel of several variables. This definition is related to a notion of convex cone. A set $\Gamma \subset \mathbb{R}^n$ is said to be a cone (with vertex at zero) if $x \in \Gamma$ implies $\alpha x \in \Gamma$ for all $\alpha > 0$. The dual cone of Γ is defined by

$$\Gamma^* = \left\{ t \in \mathbb{R}^n\colon\ (x,t) \geqslant 0 \text{ for all } x \in \Gamma \right\}.$$

Γ^* is always closed convex cone and $(\Gamma^*)^* = \overline{\operatorname{ch} \Gamma}$, where $\operatorname{ch} \Gamma$ denotes the convex hull of Γ. We say that Γ is salient (acute) if $\overline{\operatorname{ch} \Gamma}$ does not contain any line (one-dimension subspace of $\mathbb{R}^n$). This is equivalent to the statement that the interior set of Γ^* is nonempty. A cone Γ is said to be regular if Γ is an open salient convex cone.

Let $\{\Lambda_j\}_1^m$ be a family of regular cones. We say that $\{\Lambda_j\}_1^m$ covers $\mathbb{R}^n$ exactly if

$$\overline{\bigcup_{j=1}^m \Lambda_j} = \mathbb{R}^n \tag{11}$$

and the Lebesgue measure of $\overline{\Lambda_i} \cap \overline{\Lambda_j}$ is equal to zero whenever $i \neq j$. Any $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ whose entries ω_k are -1 or 1 defines the cone $Q_\omega = \{x \in \mathbb{R}^n\colon$

$x_k\omega_k > 0$ for $k = 1, \dots, n\}$. This cone Q_ω is called a quadrant in $\mathbb{R}^n$ and the collection of all 2^n cones $\{Q_\omega\}_\omega$ covers $\mathbb{R}^n$ exactly. Note that $Q_{(1,\dots,1)}$ is called the positive quadrant in $\mathbb{R}^n$ and is denoted by $\mathbb{R}^n_+$.

For an open cone Γ, the set $T_\Gamma = \mathbb{R}^n + \mathrm{i}\Gamma = \{z = x + \mathrm{i}y\colon x \in \mathbb{R}^n,\ y \in \Gamma\}$ is called a tube domain in $\mathbb{C}^n$. If Γ is regular, then the Cauchy kernel of Γ (or with respect to Γ) is defined as

$$K_\Gamma(z) = \int\limits_{\Gamma^*} \mathrm{e}^{\mathrm{i}(z,t)}\,\mathrm{d}t, \quad z \in T_\Gamma. \tag{12}$$

K_Γ is analytic on T_Γ [21, p. 143].

If f is a distribution on $\mathbb{R}^n$, then

$$K_\Gamma(f)(z) = \frac{1}{(2\pi)^n}\big(f(\cdot), K_\Gamma(z - \cdot)\big) = \frac{1}{(2\pi)^n}\big(f_t, K_\Gamma(z - t)\big), \quad z \in T_\Gamma, \tag{13}$$

is called the Cauchy (or Cauchy–Bochner) transform of f. For example, if $n = 1$, then there are only two regular cones $(-\infty, 0)$ and $(0, \infty)$ in $\mathbb{R}$. If $\Gamma = (0, \infty)$, then we see that (13) coincides with the usual definition of the Cauchy transform (5).

The notion of completely monotonic functions on $(0, \infty)$ generalizes also to the case of several variables. Note that cones are the natural domain for these functions. Let Γ be a regular cone in $\mathbb{R}^n$. The directional derivation and the directional difference of a function $\theta : \Gamma \to \mathbb{C}$ along $a = (a_1, \dots, a_n) \in \Gamma$ are defined as follows: $D_a\theta(y) = (a_1 D_{y_1} + \dots + a_n D_{y_n})\theta(y)$, and $\Delta_a\theta(y) = \theta(y + a) - \theta(y)$, respectively. Now θ is called completely monotonic on Γ if

$$(-1)^k \Delta_{\gamma_1}\Delta_{\gamma_2}\dots\Delta_{\gamma_k}\theta(y) \geqslant 0, \quad k = 0, 1, \dots,$$

for each $y \in \Gamma$ and all $\gamma_1, \dots, \gamma_k \in \Gamma$. These conditions are equivalent to that $\theta \in C^\infty(\Gamma)$ and

$$(-1)^k D_{\gamma_1} D_{\gamma_2} \dots D_{\gamma_k}\theta(y) \geqslant 0, \quad y \in \Gamma,\ \gamma_1, \dots, \gamma_k \in \Gamma,\ k = 0, 1, \dots \tag{14}$$

(see [6, p. 172]).

Now we are able to describe positive definite distributions $f \in D'_{L^p}(\mathbb{R}^n)$ in terms of their Cauchy transform $K_\Gamma(f)$. The following theorem is the main result of the present paper. To simplify the proofs, we will do here the case $D'_{L^2}(\mathbb{R}^n)$.

Theorem 2. *Let $f \in D'_{L^2}(\mathbb{R}^n)$. Suppose that $\{\Gamma_j\}_1^m$ is a family of regular cones such that $\{\Gamma_j^*\}_1^m$ covers $\mathbb{R}^n$ exactly. Then f is positive definite if and only if $y \to K_{\Gamma_j}(f)(\mathrm{i}y)$, $y \in \Gamma_j$, is completely monotonic on Γ_j for all $j = 1, 2, \dots, m$.*

We conclude this section with a few examples of positive definite distributions in $D'_{L^p}(\mathbb{R}^n)$. As usual, a function υ (or a measure μ) is identified with a distribution in $D'_{L^p}(\mathbb{R}^n)$ by the formula

$$(\upsilon, \varphi) = \int\limits_{\mathbb{R}^n} \upsilon(x)\varphi(x)\,\mathrm{d}x \quad \left(\text{or} \quad (\mu, \varphi) = \int\limits_{\mathbb{R}^n} \varphi(x)\,\mathrm{d}\mu(x)\right), \quad \varphi \in D_{L^p}\big(\mathbb{R}^n\big). \tag{15}$$

Now obviously, $L^p(\mathbb{R}^n) \subset D'_{L^p}(\mathbb{R}^n)$. Then any positive definite function $v \in L^p(\mathbb{R}^n)$ defines a regular positive definite distribution in $D'_{L^p}(\mathbb{R}^n)$. Further, there exist measures $\mu \in D'_{L^p}(\mathbb{R}^n)$, e.g., distributions of order zero, such that μ are positive definite. Indeed, using (9), we see that any finite measure μ on $\mathbb{R}^n$ with non-negative Fourier transform $\hat{\mu}$ defines by (15) a positive definite distribution in $D'_{L^p}(\mathbb{R}^n)$ for each $1 < p < \infty$. For example, let μ be any finite discrete non-negative symmetric measure on $\mathbb{R}^n$ such that

$$\mu\bigl(\{0\}\bigr) \geqslant \mu\bigl(\mathbb{R}^n \setminus \{0\}\bigr).$$

Obviously, $\hat{\mu} \geqslant 0$ on $\mathbb{R}^n$, so μ is positive definite. Finally, appropriate distributional derivatives of μ give explicit examples of positive definite distributions in $D'_{L^2}(\mathbb{R}^n)$ of any finite order.

2 Preliminaries and proofs

Let us start with some definitions and lemmas. We define the inverse Fourier transform of a finite measure μ as

$$\check{\mu}(\xi) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}(\xi,t)}\, \mathrm{d}\mu(t). \tag{16}$$

In the case if μ has a density φ in $L^1(\mathbb{R}^n)$ or in $S(\mathbb{R}^n)$, then the inverse transform is defined similarly. In addition, the inversion formula $\hat{\check{\varphi}} = \varphi$ holds for suitable φ.

We define the Fourier transform $\mathcal{F}[f]$ of $f \in S'(\mathbb{R}^n)$ by

$$\bigl(\mathcal{F}[f], \psi\bigr) = (f, \hat{\psi}), \tag{17}$$

where ψ is any element of $S(\mathbb{R}^n)$. We can modify slightly definition (3) in the following manner:

Lemma 1. $f \in S'(\mathbb{R}^n)$ *is positive definite if and only if*

$$(f, \omega) \geqslant 0 \tag{18}$$

for every positive definite $\omega \in S(\mathbb{R}^n)$.

Proof. If both $f \in S'(\mathbb{R}^n)$ and $\omega \in S(\mathbb{R}^n)$ are positive definite, then using the Bochner theorem in $S(\mathbb{R}^n)$ and in $S'(\mathbb{R}^n)$, respectively, we get that $\mathcal{F}[f]$ is a nonnegative tempered measure and that $\check{\omega}$ is a nonnegative function in $S(\mathbb{R}^n)$. Hence, $(\mathcal{F}[f], \check{\omega})$ may be defined as usual integral (15). Then (17) implies that $(f, \omega) = (\mathcal{F}[f], \check{\omega}) \geqslant 0$. On the other hand, if $\varphi \in S(\mathbb{R}^n)$, then the Fourier transform of $\varphi * \varphi^\star$ is equal to $|\hat{\varphi}|^2$. Hence, $\varphi * \varphi^\star$ is positive definite. If now $f \in S'(\mathbb{R}^n)$ satisfies (18) for any positive definite $\omega \in S(\mathbb{R}^n)$, then we can set $\omega = \varphi * \varphi^\star$. Thus, (3) holds. □

Remark 1. Since $D(\mathbb{R}^n)$ is dense in $S(\mathbb{R}^n)$, it follows that $f \in S'(\mathbb{R}^n)$ is positive definite if and only if (18) is fulfilled for all $\omega \in D(\mathbb{R}^n)$.

Lemma 2. *Let $\varphi \in D_{L^2}(\mathbb{R}^n)$. If φ is positive definite, then there exists a sequence (ψ_k) of positive definite $\psi_k \in S(\mathbb{R}^n)$, $k = 1, 2, \ldots$, such that $\lim_{k\to\infty} \psi_k = \varphi$ in $D_{L^2}(\mathbb{R}^n)$.*

Proof. Take any non-negative $\sigma \in S(\mathbb{R}^n)$ supported on $[-1, 1]^n \subset \mathbb{R}^n$ and such that

$$\int_{\mathbb{R}^n} \sigma(x)\,\mathrm{d}x = 1. \tag{19}$$

For $a > 0$, we define $\sigma_a(x)$ to be $a^n\sigma(ax)$. Then $\hat{\sigma}_a$ is positive definite. Set

$$\psi_k(x) = \hat{\sigma}_k(x)\varphi(x), \tag{20}$$

$k = 1, 2, \ldots$. The product of positive definite functions is positive definite. Hence, ψ_k is positive definite. Using that $\hat{\sigma}_a \in S(\mathbb{R}^n)$ and that $\varphi \in D_{L^2}(\mathbb{R}^n)$ satisfies (9), we see that $\psi_k \in S(\mathbb{R}^n)$, $k = 1, 2, \ldots$.

Now we shall show that $\lim_{k\to\infty} \psi_k = \varphi$ in $D_{L^2}(\mathbb{R}^n)$. Recall that (ψ_k), $\psi_k \in D_{L^2}(\mathbb{R}^n)$, converges to $\varphi \in D_{L^2}(\mathbb{R}^n)$ as $k \to \infty$ if

$$\lim_{k\to\infty} \big\|D_x^u(\psi_k - \varphi)\big\|_{L^2(\mathbb{R}^n)} = 0 \tag{21}$$

for every nonnegative multi-index $u \in \mathbb{R}^n$. To do this, first we will estimate the function $1 - \hat{\sigma}_k(x)$ and its derivatives.

Let $\varepsilon > 0$. The definition of σ_k, conjugate with (19), implies that

$$1 - \hat{\sigma}_k(x) = \hat{\sigma}(0) - \hat{\sigma}\left(\frac{x}{k}\right).$$

Since $\hat{\sigma}$ is a characteristic function, it follows that

$$\big|1 - \hat{\sigma}_k(x)\big| \leqslant 2 \quad \text{for all } x \in \mathbb{R}^n. \tag{22}$$

Moreover, for any $0 < M < \infty$, there exists $0 < K = K(M, \varepsilon) < \infty$ such that

$$\big|1 - \hat{\sigma}_k(x)\big| \leqslant \varepsilon \quad \text{for all } k > K,\ x \in \mathbb{R}^n,\ \|x\|_2 \leqslant M. \tag{23}$$

Let s be a non-negative multi-index such that $|s| \geqslant 1$. Then by (19), we have

$$\big|D_x^s\big(1 - \hat{\sigma}_k(x)\big)\big| = \left|D_x^s \int_{\mathbb{R}^n} \sigma(t)\mathrm{e}^{\mathrm{i}(x,t)/k}\,\mathrm{d}t\right| \leqslant \frac{1}{k^{|s|}} \leqslant \frac{1}{k} \quad \text{for all } x \in \mathbb{R}^n. \tag{24}$$

If $u \in \mathbb{R}^n$ is an arbitrary non-negative multi-index, then it is easily seen that there exists a finite collection $V = \{v\}$ of not necessarily different nonnegative multi-indexes v such that

$$\begin{aligned} D_x^u\big(\varphi(x) - \psi_k(x)\big) &= D_x^u\big(\varphi(x)\big[1 - \hat{\sigma}_k(x)\big]\big) \\ &= \big(1 - \hat{\sigma}_k(x)\big)D_x^u\varphi(x) + \sum_{\substack{v\in V\\ |u-v|>0}} \big(D_x^v\varphi(x)D_x^{u-v}\big[1 - \hat{\sigma}_k(x)\big]\big). \end{aligned} \tag{25}$$

Since $\varphi \in D_{L^2}(\mathbb{R}^n)$, we have that for $\varepsilon > 0$, there exists $0 < M = M(\varepsilon) < \infty$ such that

$$\left(\int\limits_{\|x\|_2 \geqslant M} \left| D_x^s \varphi(x) \right|^2 \mathrm{d}x \right)^{1/2} < \varepsilon \quad \text{for all } s \in \{u, V\}. \tag{26}$$

Now fix any multi-index $u \in \mathbb{R}^n$ and any $\varepsilon > 0$. Then take $0 < M = M(\varepsilon) < \infty$ so that (26) holds. Finally, choose $0 < K = K(M, \varepsilon) < \infty$ such that $K > 1/\varepsilon$ and (23) holds. If $k > K$, then combining (25) with (22), (23), (24), and (26), we have

$$\begin{aligned}
&\left\| D_x^u (\varphi - \psi_k) \right\|_{L^2(\mathbb{R}^n)} \\
&\quad \leqslant \left\| (1 - \hat{\sigma}_k) D_x^u \varphi \right\|_{L^2(\mathbb{R}^n)} + \sum_{\substack{v \in V \\ |u - v| > 0}} \left\| D_x^v \varphi D_x^{u-v} [1 - \hat{\sigma}_k] \right\|_{L^2(\mathbb{R}^n)} \\
&\quad \leqslant \left(\int\limits_{\|x\|_2 \leqslant M} \left| (1 - \hat{\sigma}_k) \right|^2 \left| D_x^u \varphi(x) \right|^2 \mathrm{d}x \right)^{1/2} + \left(\int\limits_{\|x\|_2 \geqslant M} \left| (1 - \hat{\sigma}_k) \right|^2 \left| D_x^u \varphi(x) \right|^2 \mathrm{d}x \right)^{1/2} \\
&\qquad + \frac{1}{k} \sum_{\substack{v \in V \\ |u - v| > 0}} \left\| D_x^v \varphi \right\|_{L^2(\mathbb{R}^n)} \\
&\quad \leqslant \varepsilon \left(\left\| D_x^u \varphi \right\|_{L^2(\mathbb{R}^n)} + 2 + \sum_{\substack{v \in V \\ |u - v| > 0}} \left\| D_x^v \varphi \right\|_{L^2(\mathbb{R}^n)} \right).
\end{aligned} \tag{27}$$

Since V is finite and depends only on v, (27) implies that $\|D_x^u(\varphi - \psi_k)\|_{L^2(\mathbb{R}^n)} \leqslant \mathrm{Const}(u)\varepsilon$ for all $k > K$. This proves (21) and Lemma 2. □

We recall the definition of the Laplace transform. Suppose that Λ is a closed convex salient cone in $\mathbb{R}^n$. Let $S'(\Lambda)$ denote the set of all $f \in S'(\mathbb{R}^n)$ supported on Λ. Then $S'(\Lambda)$ is simultaneously a closed subspace of $S'(\mathbb{R}^n)$ and a commutative convolution algebra [21, p. 64]. For $y \in \mathbb{R}^n$, the Laplace transform of $F \in S'(\Lambda)$ is defined by

$$L_y(F)(x) = \mathcal{F}\left[F(\cdot)\mathrm{e}^{-(y,\cdot)}\right](x) = \mathcal{F}_\xi\left[F(\xi)\mathrm{e}^{-(y,\xi)}\right](x), \quad x \in \mathbb{R}^n. \tag{28}$$

If $y \in \mathrm{int}\Lambda^*$, then $F(\cdot)\mathrm{e}^{-(y,\cdot)}$ belongs to $S'(\mathbb{R}^n)$ (see, e.g., [21, p. 127]). Hence, $L_y(F)(x)$ is well defined for all $y \in \mathrm{int}\Lambda^*$. Further, $L_y(F)(x)$ is analytic on the tube domain $T_{\mathrm{int}\Lambda^*}$ as a function of $z = x + \mathrm{i}y$, and

$$\frac{\partial^{|u|}}{\partial z_1^{u_1} \ldots \partial z_n^{u_n}} L_y(F)(x) = \mathrm{i}^{|u|} \mathcal{F}_\xi\left[\left(\xi_1^{u_1} \cdots \xi_n^{u_n}\right) F(\xi) \mathrm{e}^{-(y,\xi)}\right](x) \tag{29}$$

for any non-negative integer multi-index $u = (u_1, \ldots, u_n)$ [21, p. 128].

Now we briefly touch upon the problem whether the Cauchy transform is well defined on $D'_{L^2}(\mathbb{R}^n)$. The following simple lemma contains a precise statement. For completeness, we also give its proof.

Lemma 3. *Let Γ be a regular cone in $\mathbb{R}^n$. If $f \in D'_{L^2}(\mathbb{R}^n)$, then the Cauchy transform* (28) *is well defined on T_Γ. Moreover, it is analytic on T_Γ and*

$$\frac{\partial^{|u|}}{\partial z_1^{u_1} \ldots \partial z_n^{u_n}} K_\Gamma(f)(z) = \frac{1}{(2\pi)^n} \left(f(\cdot), \frac{\partial^{|u|}}{\partial z_1^{u_1} \ldots \partial z_n^{u_n}} K_\Gamma(z - \cdot) \right), \quad z \in T_\Gamma, \tag{30}$$

for each non-negative multi-index $u = (u_1, \ldots, u_n)$.

Proof. Fix any $y \in \Gamma$ and set

$$E_{y,u}(\xi) = \xi^{u_1} \cdots \xi^{u_n} \mathrm{e}^{-(y,\xi)}, \tag{31}$$

$\xi \in \Gamma^*$. Since Γ is open, then it is easy to see that there exists $\delta = \delta(y) > 0$ such that $(y, \xi) \geqslant \delta \|\xi\|_2$ for all $\xi \in \Gamma^*$ (see also [18, p. 104]). Then

$$\left|E_{y,u}(\xi)\right| \leqslant |\xi_1|^{u_1} \cdots |\xi|^{u_n} \mathrm{e}^{-\delta\|\xi\|_2} \leqslant \prod_{k=1}^{n} \left(|\xi_k|^{u_k} \mathrm{e}^{-\delta|\xi_k|}\right)$$

for $\xi \in \Gamma^*$. Let χ_{Γ^*} denote the indicator function of Γ^*. Then we see that

$$E_{y,u}(\xi)\chi_{\Gamma^*}(\xi) \in L^s(\mathbb{R}^n) \quad \text{for all } 1 \leqslant s \leqslant \infty. \tag{32}$$

Clearly, $\chi_{\Gamma^*} \in S'(\mathbb{R}^n)$. Hence, if we take in (28) $F = \chi_{\Gamma^*}$, then have for any $t \in \mathbb{R}^n$ that

$$\begin{aligned} L_y(\chi_{\Gamma^*})(x - t) &= \mathcal{F}_\xi\left[\chi_{\Gamma^*}(\xi)\mathrm{e}^{-(y,\xi)}\right](x - t) = \mathcal{F}_\xi\left[\chi_{\Gamma^*}(\xi)E_{y,0}(\xi)\right](x - t) \\ &= \int\limits_{\mathbb{R}^n} \chi_{\Gamma^*}(\xi)E_{y,0}(\xi)\mathrm{e}^{\mathrm{i}(x-t)}\,\mathrm{d}\xi = \int\limits_{\Gamma^*} \mathrm{e}^{\mathrm{i}(z-t)}\,\mathrm{d}\xi = K_\Gamma(z - t), \end{aligned} \tag{33}$$

where $z = x + \mathrm{i}y \in T_\Gamma$. Now (32), together with the Plancherel theorem in $L^2(\mathbb{R}^n)$, implies that for any $z \in T_\Gamma$, the function $t \to K_\Gamma(z - t)$ belongs to $L^2(\mathbb{R}^n)$. Using (31) and (32) with a general non-negative multi-index $u = (u_1, \ldots, u_n)$, we find in a similar way that

$$D_t^u K_\Gamma(z - t) = (-\mathrm{i})^{|u|} \int\limits_{\mathbb{R}^n} \chi_{\Gamma^*}(\xi)E_{y,u}(\xi)\mathrm{e}^{\mathrm{i}(x-t)}\,\mathrm{d}\xi,$$

$z = x + \mathrm{i}y \in T_\Gamma$. Hence, again by (32), we obtain that $t \to D_t^u K_\Gamma(z - t))$ belongs to $L^2(\mathbb{R}^n)$ for all non-negative multi-indexes u, e.g., $t \to K_\Gamma(z - t)$ belongs to $D_{L^2}(\mathbb{R}^n)$. Thus, (13) is well defined on $D'_{L^2}(\mathbb{R}^n)$ for all $z \in T_\Gamma$. Finally, using (29) and properties (given above) of the Laplace transform (28), we have that $K_\Gamma(f)(z)$ is analytic on T_Γ and (30) is fulfilled. This finishes the proof of Lemma 3. □

We are now in a position to prove the main theorem. For the sake of clarity, we divide the proof into two parts.

Proof of Theorem 2 (*Necessity*). Let $f \in D'_{L^2}(\mathbb{R}^n)$ and suppose that Γ is an arbitrary regular cone in $\mathbb{R}^n$. By Lemma 3, the Cauchy transform (13) is well defined and (30) holds for $z \in T_\Gamma$. In particular, if $z = \mathrm{i}y$ with $y \in \Gamma$, then

$$\frac{\partial^{|u|}}{\partial y_1^{u_1} \dots \partial y_n^{u_n}} K_\Gamma(f)(\mathrm{i}y) = \frac{1}{(2\pi)^n}\left(f(\cdot), \frac{\partial^{|u|}}{\partial y_1^{u_1} \dots \partial y_n^{u_n}} K_\Gamma(\mathrm{i}y - \cdot)\right) \tag{34}$$

for each multi-index u. Combining (29) and (33), we get

$$\frac{\partial^{|u|}}{\partial y_1^{u_1} \dots \partial y_n^{u_n}} K_\Gamma(\mathrm{i}y) = \mathrm{i}^{2|u|} \int\limits_{\Gamma^*} \left(\xi_1^{u_1} \cdots \xi_n^{u_n}\right) \mathrm{e}^{-(y,\, \xi)}\, \mathrm{d}\xi. \tag{35}$$

In particular, for the directional derivative $D_\gamma K_\Gamma(\mathrm{i}y - t)$ with $\gamma \in \Gamma$, we have

$$\begin{aligned} D_\gamma K_\Gamma(\mathrm{i}y - t) &= \sum_{s=1}^{n} \gamma_s \frac{\partial}{\partial y_s} K_\Gamma(\mathrm{i}y - t) = (\gamma, D_y) K_\Gamma(\mathrm{i}y - t) \\ &= -\int\limits_{\Gamma^*} (\gamma, \xi) \mathrm{e}^{-(y,\xi)} \mathrm{e}^{-\mathrm{i}(t,\xi)}\, \mathrm{d}\xi. \end{aligned} \tag{36}$$

Iterating (36), we obtain

$$D_{\gamma_1} D_{\gamma_2} \dots D_{\gamma_k} K_\Gamma(\mathrm{i}y - t) = (-1)^k \int\limits_{\Gamma^*} \prod_{j=1}^{k} (\gamma_j, \xi) \mathrm{e}^{-(y,\xi)} \mathrm{e}^{-\mathrm{i}(t,\xi)}\, \mathrm{d}\xi \tag{37}$$

for any choice $\gamma_1, \dots, \gamma_k \in \Gamma$.

For fixed y and γ in Γ, set

$$H(\xi) := (\gamma, \xi) \mathrm{e}^{-(y,\xi)} \chi_{\Gamma^*}(\xi),$$

$\xi \in \Gamma^*$. Obviously, H coincides on Γ^* with a finite linear combination of functions (31) with appropriate quotients. This, conjugate with (42), implies that H is integrable on $\mathbb{R}^n$. Moreover, (γ, ξ) is nonnegative for $\xi \in \Gamma^*$. Thus, applying the Bochner theorem (see [5, p. 293] and [12, p. 125]) to the right-hand side of (37), we see that for any fixed $y \in \Gamma$ and all $\gamma_1, \dots, \gamma_k \in \Gamma$,

$$(-1)^k D_{\gamma_1} D_{\gamma_2} \dots D_{\gamma_k} K_\Gamma(\mathrm{i}y - t) \tag{38}$$

is positive definite as a function of $t \in \mathbb{R}^n$.

Suppose, in addition, that $f \in D'_{L^2}(\mathbb{R}^n)$ is positive definite. Then by Lemmas 1 and 2, we have

$$(-1)^k \left(D_{\gamma_1} D_{\gamma_2} \dots D_{\gamma_k} K_\Gamma(\mathrm{i}y - \cdot), f(\cdot)\right) \geqslant 0$$

for $y \in \Gamma$. Combining this with (34), we see that

$$(-1)^k D_{\gamma_1} D_{\gamma_2} \dots D_{\gamma_k} K_\Gamma(f)(\mathrm{i}y) \geqslant 0$$

for all $y \in \Gamma$ and each $\gamma_1, \dots, \gamma_k \in \Gamma$. Finally, this shows that $y \to K_\Gamma(f)(\mathrm{i}y)$ is a completely monotonic function on Γ. Necessity of Theorem 2 is proved. □

Lemma 4. *Suppose that $\{\Gamma_k\}_1^m$ is a family of regular cones such that $\{\Gamma_k^*\}_1^m$ covers exactly $\mathbb{R}^n$. Let $y_k \in \Gamma_k$, $k = 1, \dots, m$. If $\omega \in D(\mathbb{R}^n)$, then*

$$\lim_{\max \|y_k\|_2 \to 0} \sum_{k=1}^m K_{\Gamma_k}(\omega)(x + \mathrm{i}y_k) = \omega(x) \tag{39}$$

in the topology of $D'_{L^2}(\mathbb{R}^n)$.

Proof. Obviously, each $\omega \in D(\mathbb{R}^n)$ defines by

$$(\omega, \varphi) = \int_{\mathbb{R}^n} \omega(x)\varphi(x)\,\mathrm{d}x,$$

$\varphi \in D_{L^2}(\mathbb{R}^n)$, a distribution in $D'_{L^2}(\mathbb{R}^n)$. Therefore, if Γ is a regular cone in $\mathbb{R}^n$, then

$$K_\Gamma(\omega)(z) = \frac{1}{(2\pi)^n}\big(K_\Gamma(z - \cdot), \omega(\cdot)\big) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} K_\Gamma(z - \alpha)\omega(\alpha)\,\mathrm{d}\alpha, \tag{40}$$

where the integral converges absolutely for $z \in T_\Gamma$. Since

$$K_\Gamma(z - \alpha) = \int_{\Gamma^*} \mathrm{e}^{\mathrm{i}(x,t)}\mathrm{e}^{-(y,t)}\mathrm{e}^{-\mathrm{i}(\alpha,t)}\,\mathrm{d}t$$

and this integral converges also absolutely for $\alpha \in \mathbb{R}^n$ and $z \in T_\Gamma$, it follows by the Fubini theorem that

$$\begin{aligned}
K_\Gamma(\omega)(z) &= \frac{1}{(2\pi)^n}\int_{\Gamma^*}\bigg[\int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}(\alpha,t)}\omega(\alpha)\,\mathrm{d}\alpha\bigg]\mathrm{e}^{\mathrm{i}(x,t)}\mathrm{e}^{-(y,t)}\,\mathrm{d}t \\
&= \int_{\Gamma^*} \check{\omega}(t)\mathrm{e}^{\mathrm{i}(x,t)}\mathrm{e}^{-(y,t)}\,\mathrm{d}t = \int_{\Gamma^*} \check{\omega}(t)\mathrm{e}^{\mathrm{i}(x,t)}\mathrm{e}^{-|(y,t)|}\,\mathrm{d}t \\
&= \int_{\mathbb{R}^n} \check{\omega}(t)\mathrm{e}^{\mathrm{i}(x,t)}\mathrm{e}^{-|(y,t)|}\chi_{\Gamma^*}(t)\,\mathrm{d}t.
\end{aligned} \tag{41}$$

For $y_k \in \Gamma_k$, $k = 1, \dots, m$, $Y = \{y_1, \dots, y_m\}$, set

$$\Omega_Y(t) = \sum_{k=1}^m \chi_{\Gamma_k^*}(t)\mathrm{e}^{-|(y_k,\, t)|}, \tag{42}$$

$t \in \mathbb{R}^n$. If u is a non-negative integer multi-index, then using (41), we get

$$\begin{aligned}
D_x^u\Bigg(\sum_{k=1}^m K_{\Gamma_k}(\omega)(x + \mathrm{i}y_k) - \omega(x)\Bigg) &= D_x^u\Bigg(\int_{\mathbb{R}^n}\big[\Omega_Y(t) - 1\big]\check{\omega}(t)\mathrm{e}^{\mathrm{i}(x,t)}\,\mathrm{d}t\Bigg) \\
&= \mathrm{i}^{|u|}\int_{\mathbb{R}^n}\big[\Omega_Y(t) - 1\big]t_1^{u_1}\cdots t_n^{u_n}\check{\omega}(t)\mathrm{e}^{\mathrm{i}(x,t)}\,\mathrm{d}t
\end{aligned}$$

for $x \in \mathbb{R}^n$. Here using the Parseval equality for Fourier transform, we have

$$\left\| D_x^u \left(\sum_{k=1}^m K_{\Gamma_k}(\omega)(x + \mathrm{i}y_k) - \omega(x) \right) \right\|_{L^2(\mathbb{R}^n)}^2 = (2\pi)^n \left\| \left(\Omega_Y(t) - 1 \right) t_1^{u_1} \cdots t_n^{u_n} \check{\omega}(t) \right\|_{L^2(\mathbb{R}^n)}^2. \tag{43}$$

Since $\{\Gamma_k^*\}_1^m$ covers exactly $\mathbb{R}^n$, it follows easily from (42) that

$$\Omega_Y(t) = 1 + \theta(t) + \sum_{k=1}^m \left(\mathrm{e}^{-(y_k,t)} - 1 \right) \chi_{\Gamma_k^*}(t),$$

where $\theta(t) = 0$ almost everywhere on $\mathbb{R}^n$ and

$$1 - \sum_{k=1}^m \mathrm{e}^{-|(y_k,t)|} \to 0, \quad \text{as } \max_k \|y_k\|_2 \to 0,$$

uniformly on compact subsets of $\mathbb{R}^n$. On the other hand, $\check{\omega}(t)$ as well as $t_1^{u_1} \cdots t_n^{u_n} \check{\omega}(t)$ belong to $S(\mathbb{R}^n)$. Thus, the norm in the right-hand side of (43) tends to zero as $\max_k \|y_k\|_2 \to 0$. This proves (39) and the lemma. □

Proof of Theorem 2 (*Sufficiency*). Suppose that Γ is any regular cone such that $y \to K_\Gamma(f)(\mathrm{i}y)$ is completely monotonic on Γ. Fix $\gamma \in \Gamma$. Since Γ is convex, it follows that Γ is also an additive semigroup. Because $\gamma + \overline{\Gamma} \subset \Gamma$, the function

$$F_\gamma(y) = K_\Gamma(f)\left(\mathrm{i}(\gamma + y)\right) \tag{44}$$

is well defined for all $y \in \overline{\Gamma}$. Moreover, F_γ is continuous and completely monotonic on $\overline{\Gamma}$. Then (see [6, p. 172] and [2, p. 89]) there exists a non-negative measure μ_γ on $(\overline{\Gamma})^*$ such that

$$F_\gamma(y) = \int_{(\overline{\Gamma})^*} \mathrm{e}^{-(y,\zeta)} \, \mathrm{d}\mu_\gamma(\zeta)$$

for all $y \in \overline{\Gamma}$. Clearly, $(\overline{\Gamma})^* = \Gamma^*$. Since F_γ is continuous on $\overline{\Gamma}$, we see that μ_γ is a finite measure on Γ^*. Therefore, F_γ can be continued analytically on the tube domain T_Γ as the Laplace transform of μ_γ, i.e., for $z = x + \mathrm{i}y \in T_\Gamma$,

$$F_\gamma(z) = \int_{\Gamma^*} \mathrm{e}^{\mathrm{i}(z,\zeta)} \, \mathrm{d}\mu_\gamma(\zeta). \tag{45}$$

By (44), $F_\gamma(z)$ coincides with $K_\Gamma(f)(\mathrm{i}\gamma + z)$ for $z = \mathrm{i}y$, $y \in \overline{\Gamma}$. We claim that this is true on the whole tube domain T_Γ. To this end, we use the following identity theorem (see e.g., [16, p. 21]): if h is an analytic function on an open domain D on $\mathbb{C}^n$ such that h vanishes on a real neighborhood of a point $z_0 = x_0 + \mathrm{i}y_0 \in D$, i.e., h vanishes on

$$\left\{ z = x + \mathrm{i}y \in D \colon |x - x_0| < r, \ y = y_0 \right\},$$

then $h \equiv 0$ on D. Of course, this statement is valid also in the case if we replace this real neighborhood by an imaginary neighborhood of z_0, i.e., on the set $\{z = x + \mathrm{i}y \in D\colon\ x = x_0,\ |y - y_0| < r\}$. Take any $z_0 = \mathrm{i}y_0 \in T_\Gamma$. By (45), analytic functions $F_\gamma(z)$ and $K_\Gamma(f)(i\gamma + z)$ coincide on any image neighborhood $I_{z_0} = \{z = x + \mathrm{i}y \in \mathbb{C}^n\colon\ |y - y_0| < r,\ x = x_0\}$ of z_0 such that $I_{z_0} \subset T_\Gamma$. This yields the claim that

$$K_\Gamma(f)(\mathrm{i}\gamma + z) = F_\gamma(z) = \int\limits_{\Gamma^*} \mathrm{e}^{\mathrm{i}(z,\zeta)}\,\mathrm{d}\mu_\gamma(\zeta) = \int\limits_{\Gamma^*} \mathrm{e}^{\mathrm{i}(x,\zeta)}\mathrm{e}^{-(y,\zeta)}\,\mathrm{d}\mu_\gamma(\zeta) \tag{46}$$

for $z = x + \mathrm{i}y \in T_\Gamma$.

Using the representation (46) and having the Bochner theorem, we see that for any $y \in \Gamma$, the function $x \to F_\gamma(x + \mathrm{i}y)$ is continuous and positive definite on $\mathbb{R}^n$. This is also true for all $\gamma \in \Gamma$. Thus, since Γ is an open cone and $F_\gamma(z) = K_\Gamma(f)(\mathrm{i}\gamma + z)$ on T_Γ, we obtain that for any fixed $y \in \Gamma$, the function

$$x \to K_\Gamma(f)(x + \mathrm{i}y) \tag{47}$$

is continuous and positive definite for $x \in \mathbb{R}^n$.

Suppose now that $\{\Gamma_k\}_1^m$ is a family of regular cones such that $\{\Gamma_k^*\}_1^m$ covers $\mathbb{R}^n$ exactly. Next, take any collection $y_k \in \Gamma_k$ for $k = 1, \dots, m$. Let $\omega \in D(\mathbb{R}^n)$. Since f is a linear functional on $D_{L^2}(\mathbb{R}^n)$, we get

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} \left(\sum_{k=1}^m K_{\Gamma_k}(f)(x + \mathrm{i}y_k)\right)\omega(x)\,\mathrm{d}x \\
&\quad = \frac{1}{(2\pi)^n}\sum_{k=1}^m \int\limits_{\mathbb{R}^n} \big(f(\cdot), \omega(x)K_{\Gamma_k}(x + \mathrm{i}y_k - \cdot)\big)\,\mathrm{d}x \\
&\quad = \frac{1}{(2\pi)^n}\sum_{k=1}^m \int\limits_{\mathbb{R}^n} \big(f_t, \omega(x)K_{\Gamma_k}(x + \mathrm{i}y_k - t)\big)\,\mathrm{d}x \\
&\quad = \frac{1}{(2\pi)^n}\sum_{k=1}^m \int\limits_{\mathbb{R}^n} f_t\big(\omega(x)K_{\Gamma_k}(x + \mathrm{i}y_k - t)\big)\,\mathrm{d}x.
\end{aligned} \tag{48}$$

We claim that

$$\sum_{k=1}^m \int\limits_{\mathbb{R}^n} f_t\big(\omega(x)K_{\Gamma_k}(x + \mathrm{i}y_k - t)\big)\,\mathrm{d}x = \sum_{k=1}^m f_t\left(\int\limits_{\mathbb{R}^n} \omega(x)K_{\Gamma_k}(x + \mathrm{i}y_k - t)\,\mathrm{d}x\right). \tag{49}$$

To verify the claim, let us recall from the proof of Lemma 3 that for fixed $x \in \mathbb{R}^n$ and $y_k \in \Gamma_k$, the map

$$t \to K_{\Gamma_k}(x + \mathrm{i}y_k - t) \tag{50}$$

is an element of $D_{L^2}(\mathbb{R}^n)$. Therefore, the map defined by

$$\Psi_{k,t}(x) := \omega(x)K_{\Gamma_k}(x + \mathrm{i}y_k - t), \tag{51}$$

$x \in \operatorname{supp}(\omega)$, is a vector-valued function

$$\Psi_{k,t}\colon\ \operatorname{supp}(\omega) \to D_{L^2}(\mathbb{R}^n).$$

Therefore, (49) is equivalent to the condition that these functions $\Psi_{k,t}$ are Pettis integrable over $\operatorname{supp}(\omega)$ (see, e.g., [11, p. 164]). Since $D_{L^2}(\mathbb{R}^n)$ is a Frechet space, $\operatorname{supp}(\omega)$ is a compact subset of $\mathbb{R}^n$, and the dual space $D'_{L^2}(\mathbb{R}^n)$ separates $D_{L^2}(\mathbb{R}^n)$ elements (indeed, it is easy to see that already regular distributions from $L^2(\mathbb{R}^n)$ separate points of $D_{L^2}(\mathbb{R}^n)$), it follows (see, e.g., [14, pp. 77–78]) that if $\Psi_{k,t}$ is continuous, then

$$\begin{aligned}\int\limits_{\mathbb{R}^n} f\bigl(\omega(x)K_{\Gamma_k}(x+\mathrm{i}y_k-t)\bigr)\,\mathrm{d}x &= \int\limits_{\mathbb{R}^n} f\bigl(\Psi_{k,t}(x)\bigr)\,\mathrm{d}x = f\Biggl(\,\int\limits_{\mathbb{R}^n}\Psi_{k,t}(x)\,\mathrm{d}x\Biggr)\\ &= f\Biggl(\,\int\limits_{\mathbb{R}^n}\omega(x)K_{\Gamma_k}(x+\mathrm{i}y_k-t)\,\mathrm{d}x\Biggr) \qquad (52)\end{aligned}$$

for all $f \in D'_{L^2}(\mathbb{R}^n)$. Now, by comparing (49) and (52), we see that it remains to show that $\Psi_{k,t}$, $k = 1,\dots,m$, are continuous. This means that for each $x \in \operatorname{supp}(\omega)$ and any non-negative multi-index u, it should be true that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\bigl\|D_t^u\bigl(\Psi_{k,t}(x+\varepsilon)-\Psi_{k,t}(x)\bigr)\bigr\|_{L^2(\mathbb{R}^n)}\\ &\quad= \lim_{\varepsilon\to 0}\Biggl(\,\int\limits_{\mathbb{R}^n}\bigl|D_t^u\bigl[\omega(x+\varepsilon)K_{\Gamma_k}(x+\varepsilon+\mathrm{i}y_k-t)-\omega(x)K_{\Gamma_k}(x+\mathrm{i}y_k-t)\bigr]\bigr|^2\,\mathrm{d}t\Biggr)^{1/2}\\ &\quad= 0. \qquad (53)\end{aligned}$$

Obviously,

$$\begin{aligned}&\bigl\|D_t^u\bigl(\Psi_{k,t}(x+\varepsilon)-\Psi_{k,t}(x)\bigr)\bigr\|_{L^2(\mathbb{R}^n)}\\ &\quad\leqslant \max_{x\in\mathbb{R}^n}\bigl|\omega(x+\varepsilon)-\omega(x)\bigr|\Biggl(\,\int\limits_{\mathbb{R}^n}\bigl|D_t^uK_{\Gamma_k}(x+\mathrm{i}y_k-t)\bigr|^2\,\mathrm{d}t\Biggr)^{1/2}+\max_{x\in\mathbb{R}^n}\bigl|\omega(x+\varepsilon)\bigr|\\ &\qquad\times\Biggl(\,\int\limits_{\mathbb{R}^n}\bigl|D_t^uK_{\Gamma_k}(x+\mathrm{i}\varepsilon+\mathrm{i}y_k-t)-D_t^uK_{\Gamma_k}(x+\mathrm{i}y_k-t)\bigr|^2\,\mathrm{d}t\Biggr)^{1/2}. \qquad (54)\end{aligned}$$

Since all functions (50) and their derivatives in t are in $L^2(\mathbb{R}^n)$, it follows that they are L^2-continuous. This means that if a function g belongs to $L^2(\mathbb{R}^n)$, then

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^n}\bigl|g(v+\varepsilon)-g(v)\bigr|^2\,\mathrm{d}v = 0.$$

Then (53) is an immediate consequence of (54). Thus, our claim (49) is proved.

By (50), we have

$$\frac{1}{(2\pi)^n}\sum_{k=1}^{m}\int\limits_{\mathbb{R}^n} K_{\Gamma_k}(x+\mathrm{i}y_k-t)\omega(x)\,\mathrm{d}x=\sum_{k=1}^{m}K_{\Gamma_k}(\omega)(-t+\mathrm{i}y_k), \tag{55}$$

$t\in\mathbb{R}^n$. This, together with (48) and (49), gives that

$$\begin{aligned}&\int\limits_{\mathbb{R}^n}\left(\sum_{k=1}^{m}K_{\Gamma_k}(f)(-x+\mathrm{i}y_k)\right)\omega(x)\,\mathrm{d}x\\&\quad=\frac{1}{(2\pi)^n}f_t\left(\sum_{k=1}^{m}\int\limits_{\mathbb{R}^n}\omega(x)K_{\Gamma_k}(-x+\mathrm{i}y_k-t)\,\mathrm{d}x\right)\\&\quad=f_t\left(\sum_{k=1}^{m}K_{\Gamma_k}(\omega)(-t+\mathrm{i}y_k)\right)\end{aligned} \tag{56}$$

Clearly, a function $\zeta:\mathbb{R}^n\to\mathbb{C}$ is positive definite if and only if $\zeta_{(-)}(x):=\zeta(-x)$, $x\in\mathbb{R}^n$, is positive definite. Since (47) is continuous and positive definite, it follows that

$$x\to\sum_{k=1}^{m}K_{\Gamma_k}(f)(-x+\mathrm{i}y_k)$$

is also continuous and positive definite for $x\in\mathbb{R}^n$. Suppose now, in addition, that $\omega\in D(\mathbb{R}^n)$ is positive definite. Then by Remark 1, we have

$$\int\limits_{\mathbb{R}^n}\left(\sum_{k=1}^{m}K_{\Gamma_k}(f)(-x+\mathrm{i}y_k)\right)\omega(x)\,\mathrm{d}x\geqslant 0.$$

This, conjugate with (56), implies that

$$f_t\left(\sum_{k=1}^{m}K_{\Gamma_k}(\omega)(-t+\mathrm{i}y_k)\right)\geqslant 0.$$

Thus, by Lemma 4, we have

$$(f,\omega_{(-)})\geqslant 0.$$

Since ω was an arbitrary positive definite function in $D(\mathbb{R}^n)$, it follows from Remark 1 that f is a positive definite distribution. This completes the proof. □

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