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On the mean square of the periodic zeta-function. II

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Abstract. In the paper, the error term in the Atkinson type formula for the periodic zeta-function in the critical strip is considered, and an asymptotic formula for its mean square is obtained. This formula generalizes that proved for the Riemann zeta-function.

Keywords: periodic zeta-function, Riemann zeta-function, the Atkinson formula.

1 Introduction

Let $s = \sigma + \mathrm{i}t$ be a complex variable, and $\lambda \in \mathbb{R}$ be a fixed parameter. The periodic zeta-function $\zeta_\lambda(s)$ is defined for $\sigma > 1$ by the series

$$\zeta_\lambda(s) = \sum_{m=1}^{\infty} \frac{\mathrm{e}^{2\pi\mathrm{i}\lambda m}}{m^s}.$$

For $\lambda \in \mathbb{Z}$, the function $\zeta_\lambda(s)$ reduces to the Riemann zeta-function $\zeta(s)$, thus, can be analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. For $\lambda \notin \mathbb{Z}$, $\zeta_\lambda(s)$ is analytically continued to an entire function.

In view of the periodicity of the coefficients $\mathrm{e}^{2\pi\mathrm{i}\lambda m}$, we may assume that $0 < \lambda \leqslant 1$.

The moment problem for zeta-functions plays an important role in analytic number theory. It consists of finding the asymptotics or estimates for the mean values of powers of the modulus of zeta-functions.

Since $\zeta_\lambda(s) = \mathrm{e}^{2\pi\mathrm{i}\lambda} L(\lambda, 1, s)$, where for $0 < \lambda \leqslant 1$, $L(\lambda, \alpha, s)$ denotes the Lerch zeta-function defined for $\sigma > 1$ by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi\mathrm{i}\lambda m}}{(m+\alpha)^s},$$

and by analytic continuation elsewhere, we have that

$$\int_0^T \bigl|\zeta_\lambda(\sigma + \mathrm{i}t)\bigr|^{2k}\,\mathrm{d}t = \int_0^T \bigl|L(\lambda, 1, \sigma + \mathrm{i}t)\bigr|^{2k}\,\mathrm{d}t.$$

Therefore, the moment problem for $\zeta_\lambda(s)$ reduces to that for the Lerch zeta-function. Let $\zeta(s,\alpha)$ denote the Hurwitz zeta-function,

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s},\quad \sigma>1.$$

Then, for example, if σ is fixed, $1/2<\sigma<1$, by Theorem 4.2.1 of [11], we have that

$$\int_1^T \bigl|\zeta_\lambda(\sigma+\mathrm{i}t)\bigr|^2\,\mathrm{d}t=\zeta(2\sigma)T+\frac{(2\pi)^{2\sigma-1}}{2-2\sigma}\zeta(2-2\sigma,\lambda)T^{2-2\sigma} + O\bigl(T^{1-\sigma}\log T\bigr)+O\bigl(T^{\sigma/2}\bigr).$$

Suppose that λ is a rational number, $\lambda=a/q$ with $a,q\in\mathbb{N}$, $1\leqslant a\leqslant q$. In [8, 9] and [3], the mean square

$$I_\sigma(q,T)\overset{\mathrm{def}}{=}\sum_{a=1}^{q}\int_0^T\bigl|\zeta_{a/q}(\sigma+\mathrm{i}t)\bigr|^2\,\mathrm{d}t$$

with $1/2\leqslant\sigma<1$ has been studied, and the Atkinson-type formula for the error term of the mean square formula has been obtained. Let

$$E(q,T)=I_{1/2}(q,T)-qT\left(\log\frac{qT}{2\pi}-2\gamma_0-1\right),$$

where γ_0 is the Euler constant. Then in [2], the formula for

$$\int_2^T E^2(q,T)\,\mathrm{d}t$$

has been obtained. Let $1/2<\sigma<3/4$ be fixed. The aim of this note is to obtain a formula for the mean square of

$$E_\sigma(q,T)=I_\sigma(q,T)-q\zeta(2\sigma)T-\frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)\sin(\pi\sigma)}{1-\sigma}(qT)^{2-2\sigma},$$

where, as usual, $\Gamma(s)$ denotes the Euler gamma-function. Let

$$\sigma_z(m)=\sum_{d|m}d^z,\quad z\in\mathbb{C}.$$

Theorem 1. *Let σ, $1/2<\sigma<3/4$, be fixed. Then, for $T\to\infty$ and $q\leqslant T^{1-4\sigma/3}$,*

$$\int_2^T E_\sigma^2(q,t)\,\mathrm{d}t=2(5-4\sigma)^{-1}(2\pi)^{2\sigma-3/2}q^{3/2-2\sigma}T^{5/2-2\sigma}\sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}^2(m)}{m^{5/2-2\sigma}} + O\bigl(q^{11/4-2\sigma}T^{7/4-\sigma}\log T\bigr).$$

For $q = 1$, we have the Matsumoto result [12], Theorem 3.

The case $3/4 \leqslant \sigma < 1$ will be presented in a subsequent paper.

2 The Atkinson formula

In [1], Atkinson proved a formula for the function

$$E(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + \mathrm{i}t\right)\right|^2 \mathrm{d}t - T \log \frac{T}{2\pi} - (2\gamma_0 - 1)T.$$

This formula is a sum of two finite sums involving some elementary functions and of the error term $O(\log^2 T)$. The proof of the Atkinson formula is also given in [5]. Matsumoto and Meurman [12, 13] gave the Atkinson-type formula for the error term in the mean square formula of the Riemann-zeta function in the critical strip. Modified versions of the Atkinson formula were obtained in [6, 7, 15, 16].

The Atkinson formula is a powerful tool for the investigation of the error term in the mean square formula, and for various allied applications. For example, Heath-Brown [4] applied the Atkinson formula for estimation of the twelfth moment

$$\int_0^T \left|\zeta\left(\frac{1}{2} + \mathrm{i}t\right)\right|^{12} \mathrm{d}t.$$

The Atkinson-type formula is also known for Dirichlet L-functions (see [16]) and for the function $\zeta_{a/q}(s)$ (see [3, 8, 9]). In the latter case, the dependence on q is important. For the proof of the main theorem of this paper, we will use a formula obtained in [3]. For its statement, we need some notation. Let $c_1 T < N < c_2 T$ with some positive constants $c_1 < c_2$,

$$N_1 = N_1(q, T, N) = q\left(\frac{T}{2\pi} + \frac{qN}{2} - \left(\left(\frac{qN}{2}\right)^2 + \frac{qNT}{2\pi}\right)^{1/2}\right)$$

and

$$\operatorname{arsinh}(x) = \log\left(x + \sqrt{1 + x^2}\right).$$

Define the sums

$$\begin{aligned}\sum\nolimits_1(q, T) = 2^{\sigma-1} q^{1-\sigma} \left(\frac{T}{\pi}\right)^{1/2-\sigma} &\sum_{m \leqslant N} \frac{(-1)^{qm} \sigma_{1-2\sigma}(m)}{m^{1-\sigma}} \\ &\times \left(\operatorname{arsinh}\left(\sqrt{\frac{\pi q m}{2T}}\right)\right)^{-1} \left(\frac{T}{2\pi q m} + \frac{1}{4}\right)^{-1/4} \\ &\times \cos\left(2T \operatorname{arsinh}\left(\sqrt{\frac{\pi q m}{2T}}\right) + \sqrt{2\pi q m T + T^2 q^2 m^2} - \frac{\pi}{4}\right)\end{aligned}$$

and

$$\sum\nolimits_2(q,T) = -2q^{1-\sigma}\left(\frac{T}{2\pi}\right)^{1/2-\sigma}\sum_{m\leqslant N_1}\frac{\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}\left(\log\left(\frac{qT}{2\pi m}\right)\right)^{-1}$$
$$\times\cos\left(T\log\left(\frac{qT}{2\pi m}\right) - T + \frac{\pi}{4}\right).$$

Then we have the following result [3].

Lemma 1. *Suppose that* $1/2 < \sigma < 3/4$. *Then, for* $q \leqslant T$,

$$E_\sigma(q,T) = \sum\nolimits_1(q,T) + \sum\nolimits_2(q,T) + R(q,T),$$

where $R(q,T) = O(q^{7/4-\sigma}\log T)$ *with the* O *– constant depending only on* σ.

3 Some estimates

In this section, we present some known estimates that will be applied in the proof of Theorem 1.

Lemma 2. *Let* $g_j(t)$, $j = 1,\dots,k$, *and* $f(t)$ *be real-valued continuous monotonic functions on* $[a,b]$, *and let* $f(t)$ *have a continuous monotonic derivative on* $[a,b]$. *If* $|g_j(t)| \leqslant M_j$, $j = 1,\dots,k$, *and* $|f'(t)| \geqslant M_0^{-1}$ *on* $[a,b]$, *then*

$$\left|\int\limits_a^b \prod_{j=1}^k g_j(t)\mathrm{e}^{2\pi \mathrm{i} f(t)}\,\mathrm{d}t\right| \leqslant 2^{k+3}\prod_{j=0}^k M_j.$$

The lemma is Lemma 15.3 of [5].

Lemma 3. *The estimates*

$$\sum_{m\leqslant x}\frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} \ll x^{2\sigma-1} \quad \textit{and} \quad \sum_{m\leqslant x}\frac{\sigma_{1-2\sigma}(m)}{m^{2-2\sigma}} \ll x^{2\sigma-1}$$

are true.

Proof. Since $1-2\sigma < 0$, the series over primes

$$\sum_p \frac{|\sigma_{1-2\sigma}^2(p) - 1|\log p}{p}$$

is convergent. Therefore, from general mean values theorems for multiplicative functions (see, for example, [10]) we obtain that

$$\sum_{m\leqslant x}\sigma_{1-2\sigma}^2(m) \ll x.$$

This and summing by parts give the estimate of the lemma.

The second estimate of the lemma is obtained similarly. □

Lemma 4. *For every $\varepsilon > 0$,*

$$\sum_{\substack{m\leqslant x \\ m\neq n}} \sum_{n\leqslant x} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{5/4-\sigma}} |\sqrt{m}-\sqrt{n}|^{-1} \ll x^{2\sigma-1+\varepsilon}.$$

Proof. We have that

$$\sum_{\substack{m\leqslant x \\ m\neq n}} \sum_{n\leqslant x} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{5/4-\sigma}} |\sqrt{m}-\sqrt{n}|^{-1}$$
$$\ll \sum_{n<m\leqslant x} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{5/4-\sigma}} (\sqrt{m}-\sqrt{n})^{-1} \ll \sum_{n\leqslant m/2} + \sum_{n>m/2}.$$

We observe that, for $n \leqslant m/2$, we have that $m-n \geqslant m/2$, thus, $(m-n)^{-1} \ll m^{-1}$ for $n \leqslant m/2$. Therefore, using Lemma 3 and partial summation, we find

$$\sum_{n\leqslant m/2} \ll \sum_{m\leqslant x} \frac{\sigma_{1-2\sigma}(m)}{m^{5/4-\sigma}} \sum_{n\leqslant m/2} \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} (\sqrt{m}+\sqrt{n})(m-n)^{-1}$$
$$\ll \sum_{m\leqslant x} \frac{\sigma_{1-2\sigma}(m)}{m^{3/4-\sigma}} \sum_{n\leqslant m/2} \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} (m-n)^{-1}$$
$$\ll \sum_{m\leqslant x} \frac{\sigma_{1-2\sigma}(m)}{m^{7/4-\sigma}} \sum_{n\leqslant m/2} \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}}$$
$$\ll \sum_{m\leqslant x} \frac{\sigma_{1-2\sigma}(m)}{m^{7/4-\sigma}} m^{\sigma-1/4} = \sum_{m\leqslant x} \frac{\sigma_{1-2\sigma}(m)}{m^{2-2\sigma}} \ll x^{2\sigma-1},$$

and, for every $\varepsilon > 0$,

$$\sum_{n>m/2} \ll \sum_{m\leqslant x} \frac{\sigma_{1-2\sigma}(m)}{m^{2-2\sigma}} \sum_{m/2<n<m} \frac{\sigma_{1-2\sigma}(n)}{m-n} \ll x^{2\sigma-1+\varepsilon}.$$ □

Lemma 5. *We have*

$$\sum_{\substack{m\leqslant x \\ m\neq n}} \sum_{n\leqslant x} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}} \left|\log\frac{m}{n}\right|^{-1} \ll x^{2\sigma}\log x.$$

Proof. In view of Lemma 3 and the estimate [5],

$$\sum_{m\leqslant x} \left|\log\frac{m}{n}\right|^{-1} \ll x + n\log x,$$

we find that the considered sum is estimated as

$$\ll \sum_{\substack{m\leqslant x\\ m\neq n}} \sum_{n\leqslant x} \left(\frac{\sigma^2_{1-2\sigma}(m)}{m^{2-2\sigma}} + \frac{\sigma^2_{1-2\sigma}(n)}{n^{2-2\sigma}} \right) \left| \log \frac{m}{n} \right|^{-1}$$

$$\ll \sum_{\substack{m\leqslant x\\ m\neq n}} \sum_{n\leqslant x} \frac{\sigma^2_{1-2\sigma}(m)}{m^{2-2\sigma}} \left| \log \frac{m}{n} \right|^{-1} \ll \sum_{m\leqslant x} \frac{\sigma^2_{1-2\sigma}(m)}{m^{2-2\sigma}} \sum_{n\leqslant x} \left| \log \frac{m}{n} \right|^{-1}$$

$$\ll x^{2\sigma} + \log x \sum_{m\leqslant x} \frac{\sigma^2_{1-2\sigma}(m)}{m^{1-2\sigma}} \ll x^{2\sigma} \log x. \qquad \square$$

4 Proof of Theorem 1

By Lemma 1, we have that

$$\int_T^{2T} E^2_\sigma(q,t)\,\mathrm{d}t = \int_T^{2T} \sum\nolimits_1^2(q,T)\,\mathrm{d}t + 2\int_T^{2T} \sum\nolimits_1(q,T)\Big(\sum\nolimits_2(q,T) + R(q,t)\Big)\,\mathrm{d}t$$
$$+ \int_T^{2T} \Big(\sum\nolimits_2(q,T) + R(q,t)\Big)^2\,\mathrm{d}t. \tag{1}$$

For brevity, we write

$$g_1(t,qm) = \left(\operatorname{arsinh}\left(\sqrt{\frac{\pi qm}{2t}}\right)\right)^{-1}, \qquad g_2(t,qm) = \left(\frac{t}{2\pi qm} + \frac{1}{4}\right)^{-1/4}$$

and

$$f(t,qm) = 2t\operatorname{arsinh}\left(\sqrt{\frac{\pi qm}{2t}}\right) + \sqrt{2\pi qmt + t^2q^2m^2} - \frac{\pi}{4}.$$

Then, taking $N = T$ in Lemma 1, we have that

$$\sum\nolimits_1(q,t) = 2^{\sigma-1}q^{1-\sigma}\left(\frac{t}{\pi}\right)^{1/2-\sigma} \sum_{m\leqslant N} \frac{(-1)^{qm}\sigma_{1-2\sigma}(m)}{m^{1-\sigma}}$$
$$\times g_1(t,qm)g_2(t,qm)\cos\big(f(t,qm)\big).$$

Hence,

$$\sum\nolimits_1^2(q,t) = 2^{2\sigma-2}q^{2-2\sigma}\left(\frac{t}{\pi}\right)^{1-2\sigma} \sum_{m\leqslant T}\sum_{n\leqslant T} \frac{(-1)^{qm+qn}\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}}$$
$$\times g_1(t,qm)g_1(t,qn)g_2(t,qm)g_2(t,qn)\cos\big(f(t,qm)\big)\cos\big(f(t,qn)\big)$$

$$= 2^{2\sigma-3}q^{2-2\sigma}\left(\frac{t}{\pi}\right)^{1-2\sigma}\sum_{m\leqslant T}\sum_{n\leqslant T}\frac{(-1)^{qm+qn}\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}}$$
$$\times g_1(t,qm)g_1(t,qn)g_2(t,qm)g_2(t,qn)$$
$$\times \bigl(\cos\bigl(f(t,qm)+f(t,qn)\bigr)+\cos\bigl(f(t,qm)-f(t,qn)\bigr)\bigr). \tag{2}$$

Let $S_1(q,t)$ be the part of $\sum_1^2(q,t)$ in (2) with $m=n$. Then

$$\int_T^{2T} S_1(q,t)\,\mathrm{d}t$$
$$= 2^{2\sigma-3}q^{2-2\sigma}\pi^{2\sigma-1}\mathrm{Re}\sum_{m\leqslant T}\frac{\sigma^2_{1-2\sigma}(m)}{m^{2-2\sigma}}\int_T^{2T} t^{1-2\sigma}g_1^2(t,qm)g_2^2(t,qm)\mathrm{e}^{2\mathrm{i}f(t,qm)}\,\mathrm{d}t$$
$$+ 2^{2\sigma-3}q^{2-2\sigma}\pi^{2\sigma-1}\sum_{m\leqslant T}\frac{\sigma^2_{1-2\sigma}(m)}{m^{2-2\sigma}}\int_T^{2T} t^{1-2\sigma}g_1^2(t,qm)g_2^2(t,qm)\,\mathrm{d}t. \tag{3}$$

It is not difficult to see that, for $t\in[T,2T]$,

$$g_1(t,qm)\ll\begin{cases}\sqrt{T/(mq)} & \text{if } m\leqslant T/(4q),\\ 1 & \text{if } m>T/(4q)\end{cases} \tag{4}$$

and

$$g_2(t,qm)\ll\begin{cases}(mq/T)^{1/4} & \text{if } m\leqslant T/(4q),\\ 1 & \text{if } m>T/(4q).\end{cases} \tag{5}$$

Moreover, for $t\in[T,2T]$,

$$\bigl|f'(t,qm)\bigr|\gg\begin{cases}\sqrt{mq/T} & \text{if } m\leqslant T/(4q),\\ 1 & \text{if } m>T/(4q).\end{cases} \tag{6}$$

In view of estimates (4)–(6), using Lemmas 2 and 3, we find that

$$2^{2\sigma-3}q^{2-2\sigma}\pi^{2\sigma-1}\mathrm{Re}\sum_{m\leqslant T}\frac{\sigma^2_{1-2\sigma}(m)}{m^{2-2\sigma}}\int_T^{2T} t^{1-2\sigma}g_1^2(t,qm)g_2^2(t,qm)\mathrm{e}^{2\mathrm{i}f(t,qm)}\,\mathrm{d}t$$
$$\ll q^{1-2\sigma}T^{2-2\sigma}\sum_{m\leqslant T/4q}\frac{\sigma^2_{1-2\sigma}(m)}{m^{3-2\sigma}}+q^{2-2\sigma}T^{1-2\sigma}\sum_{T/4q<m\leqslant T}\frac{\sigma^2_{1-2\sigma}(m)}{m^{2-2\sigma}}$$
$$\leqslant q^{1-2\sigma}T^{2-2\sigma}+q^{2-2\sigma}. \tag{7}$$

For the second term in the right-hand side of (3), we use more precise estimates for the functions $g_1(t,qm)$ and $g_2(t,qm)$, namely, for $t\in[T,2T]$ (see [2]),

$$g_1^2(t,qm)=\begin{cases}2t/(\pi mq)+O(1) & \text{if } m\leqslant T/(4q),\\ O(1) & \text{if } m>T/(4q),\end{cases}$$

and

$$g_2^2(t,qm)=\begin{cases}(2\pi qm/t)^{1/2}+O((qm/t)^{3/2}) & \text{if } m\leqslant T/(4q),\\ O(1) & \text{if } m>T/(4q).\end{cases}$$

Hence, for $t\in[T,2T]$,

$$\begin{aligned}&t^{1-2\sigma}g_1(t,qm)g_2(t,qm)\\ &\quad=\begin{cases}2^{2/3}t^{3/2-2\sigma}(\pi mq)^{-1/2}+O(t^{1/2-2\sigma}(qm)^{1/2}) & \text{if } m\leqslant T/(4q),\\ O(t^{1-2\sigma}) & \text{if } m>T/(4q).\end{cases}\end{aligned}$$

Therefore, by application of Lemma 3,

$$\begin{aligned}&2^{2\sigma-3}q^{2-2\sigma}\pi^{2\sigma-1}\sum_{m\leqslant T}\frac{\sigma^2_{1-2\sigma}(m)}{m^{2-2\sigma}}\int_T^{2T}t^{1-2\sigma}g_1^2(t,qm)g_2^2(t,qm)\,\mathrm{d}t\\
&\quad=2^{2\sigma-3/2}\pi^{2\sigma-3/2}q^{3/2-2\sigma}\sum_{m\leqslant T/(4q)}\frac{\sigma^2_{1-2\sigma}(m)}{m^{5/2-2\sigma}}\int_T^{2T}t^{3/2-2\sigma}\,\mathrm{d}t\\
&\qquad+O\biggl(q^{5/2-2\sigma}\sum_{m\leqslant T/(4q)}\frac{\sigma^2_{1-2\sigma}(m)}{m^{3/2-2\sigma}}\int_T^{2T}t^{1/2-2\sigma}\,\mathrm{d}t\biggr)\\
&\qquad+O\biggl(q^{2-2\sigma}\sum_{T/(4q)<m\leqslant T}\frac{\sigma^2_{1-2\sigma}(m)}{m^{2-2\sigma}}\int_T^{2T}t^{1-2\sigma}\,\mathrm{d}t\biggr)\\
&\quad=\frac{2^{2\sigma-3/2}\pi^{2\sigma-3/2}q^{3/2-2\sigma}}{5/2-2\sigma}+\sum_{m\leqslant T/(4q)}\frac{\sigma^2_{1-2\sigma}(m)}{m^{5/2-2\sigma}}\bigl((2T)^{5/2-2\sigma}-T^{5/2-2\sigma}\bigr)\\
&\qquad+O\biggl(q^{5/2-2\sigma}T^{3/2-2\sigma}\sum_{m\leqslant T/(4q)}\frac{\sigma^2_{1-2\sigma}(m)}{m^{3/2-2\sigma}}\biggr)\\
&\qquad+O\biggl(q^{2-2\sigma}T^{2-2\sigma}\sum_{T/(4q)<m\leqslant T}\frac{\sigma^2_{1-2\sigma}(m)}{m^{2-2\sigma}}\biggr)\\
&\quad=\frac{2^{2\sigma-3/2}\pi^{2\sigma-3/2}q^{3/2-2\sigma}}{5/2-2\sigma}+\sum_{m=1}^{\infty}\frac{\sigma^2_{1-2\sigma}(m)}{m^{5/2-2\sigma}}\bigl((2T)^{5/2-2\sigma}-T^{5/2-2\sigma}\bigr)\\
&\qquad+O\bigl(q^{2-2\sigma}T\bigr).\end{aligned}\tag{8}$$

Denote by $S_2(q,t)$ the part of $\sum_1^2(q,t)$ in (2) with $m \neq n$. In this case, for $t \in [T, 2T]$, we use the estimate

$$f'(t,qm) \pm f'(t,qn) \gg \begin{cases} \sqrt{q/T}|\sqrt{m} \pm \sqrt{n}| & \text{if } m \leqslant T/(4q),\ n \leqslant T/(4q),\\ |\log(m/n)| & \text{if } m > T/(4q),\ n > T/(4q),\\ 1 & \text{if } m \leqslant T/(4q),\ n > T/(4q) \text{ or } m > T/(4q),\ n \leqslant T/(4q). \end{cases}$$

This, Lemmas 2–5 and estimates (4), (5) give the estimate

$$\begin{aligned} \int_T^{2T} S_2(q,t)\,\mathrm{d}t &\ll q^{1-2\sigma}T^{2-2\sigma} \sum_{\substack{m\leqslant T/(4q)}} \sum_{\substack{n\leqslant T/(4q)\\ m\neq n}} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(m)}{(mn)^{5/4-\sigma}}|\sqrt{m}-\sqrt{n}|^{-1} \\ &\quad + q^{2-2\sigma}T^{1-2\sigma} \sum_{T/(4q)<m\leqslant T} \sum_{\substack{T/(4q)<n\leqslant T\\ m\neq n}} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}|\log(m/n)|} \\ &\quad + q^{7/4-2\sigma}T^{5/4-2\sigma} \sum_{m\leqslant T} \sum_{\substack{n\leqslant T\\ m\neq n}} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}} \\ &\ll q^{2-4\sigma}T^{1+\varepsilon} + q^{2-2\sigma}T\log T + q^{7/4-2\sigma}T^{1/4} \ll q^{2-2\sigma}T^{1+\varepsilon}. \end{aligned}$$

This, (3), (7) and (8) show that

$$\begin{aligned} \int_T^{2T} \sum\nolimits_1^2(q,t)\,\mathrm{d}t &= 2^{2\sigma-3/2}\left(\frac{5}{2}-2\sigma\right)^{-1}\pi^{2\sigma-3/2}q^{3/2-2\sigma} \\ &\quad \times \sum_{m=1}^{\infty} \frac{\sigma_{1-2\sigma}^2(m)}{m^{5/2-2\sigma}}\big((2T)^{5/2-2\sigma} - T^{5/2-2\sigma}\big) + O\big(q^{2-2\sigma}T^{1+\varepsilon}\big). \qquad (9) \end{aligned}$$

Now we will estimate

$$\int_T^{2T} \sum\nolimits_2^2(q,t)\,\mathrm{d}t.$$

By the definition of $\sum_2(q,t)$, we have that

$$\begin{aligned} \sum\nolimits_2^2(q,t) &= 4q^{2-2\sigma}\left(\frac{t}{2\pi}\right)^{1-2\sigma} \sum_{m\leqslant N_1}\sum_{n\leqslant N_1} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}} \\ &\quad \times \left(\log\left(\frac{qt}{2\pi m}\right)\right)^{-1}\left(\log\left(\frac{qt}{2\pi n}\right)\right)^{-1}\cos\big(g(t,qm)\big)\cos\big(g(t,qn)\big) \end{aligned}$$

$$= 2q^{2-2\sigma}\left(\frac{t}{2\pi}\right)^{1-2\sigma}\sum_{m\leqslant N_1}\sum_{n\leqslant N_1}\frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-\sigma}}$$
$$\times\left(\log\left(\frac{qt}{2\pi m}\right)\right)^{-1}\left(\log\left(\frac{qt}{2\pi n}\right)\right)^{-1}$$
$$\times\left(\cos\bigl(g(t,qm)+g(t,qn)\bigr)+\cos\bigl(g(t,qm)-g(t,qn)\bigr)\right), \tag{10}$$

where

$$g(t,qm) = t\log\left(\frac{qt}{2\pi m}\right) - t + \frac{\pi}{4},$$

and

$$N_1 = N_1(q,t,T) = q\left(\frac{t}{2\pi} + \frac{qT}{2} - \left(\left(\frac{qT}{2}\right)^2 + \left(\frac{qTt}{2\pi}\right)^{1/2}\right)\right).$$

It is not difficult to see, that

$$N_1 \leqslant \frac{t^2}{4\pi^2 T}.$$

Therefore, for $m \leqslant N_1$ and $t \in [T, 2T]$,

$$\frac{qt}{2\pi m} \geqslant \pi q > 3.$$

This implies the estimate

$$\left(\log\frac{qt}{2\pi m}\right)^{-1} \ll 1. \tag{11}$$

Moreover, by the definition of $g(t,qm)$, we find that

$$\bigl(g(t,qm) \pm g(t,qn)\bigr)' \gg \left|\log\frac{m}{n}\right|. \tag{12}$$

Denote by $Z_1(q,t)$ the part of $\sum_1^2(q,t)$ in (10) with $m \neq n$, and let $T_1 \geqslant T$ be such that $N_1(q,t,T) \geqslant \max(m,n)$ for $t \geqslant T_1$. Then we deduce from Lemmas 2 and 5 and from estimates (10)–(12) that

$$\int_T^{2T} Z_1(q,t)\,\mathrm{d}t = 2q^{2-2\sigma}(2\pi)^{2\sigma-1}\sum_{m\leqslant N_1(q,t,T)}\sum_{n\leqslant N_1(q,t,T)}\frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-2\sigma}}$$
$$\times t^{1-2\sigma}\left(\log\left(\frac{qt}{2\pi m}\right)\right)^{-1}\left(\log\left(\frac{qt}{2\pi n}\right)\right)^{-1}$$
$$\times\left(\cos\bigl(g(t,qm)+g(t,qn)\bigr)+\cos\bigl(g(t,qm)-g(t,qn)\bigr)\right)\mathrm{d}t$$

$$= 2q^{2-2\sigma}(2\pi)^{2\sigma-1} \sum_{m\leqslant N_1(q,2T,T)} \sum_{n\leqslant N_1(q,2T,T)} \frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-2\sigma}}$$

$$\times \int_{T_1}^{2T} t^{1-2\sigma}\left(\log\left(\frac{qt}{2\pi m}\right)\right)^{-1}\left(\log\left(\frac{qt}{2\pi n}\right)\right)^{-1}$$

$$\times \left(\cos\bigl(g(t,qm)+g(t,qn)\bigr)+\cos\bigl(g(t,qm)-g(t,qn)\bigr)\right)\mathrm{d}t$$

$$\ll q^{2-2\sigma}T^{1-2\sigma}\sum_{m\leqslant T}\sum_{n\leqslant T}\frac{\sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)}{(mn)^{1-2\sigma}}\left|\log\frac{m}{n}\right|^{-1}$$

$$\ll q^{2-2\sigma}T\log T. \tag{13}$$

Now let $Z_2(q,t)$ be the part of $\sum_2^2(q,t)$ in (10) with $m=n$. Then by Lemma 3, we find that

$$\int_T^{2T} Z_2(q,t)\,\mathrm{d}t \ll q^{2-2\sigma}T^{1-2\sigma}\sum_{m\leqslant T}\frac{\sigma_{1-2\sigma}^2(m)}{m^{2-2\sigma}} \ll q^{2-2\sigma}T.$$

This, (10) and (3) show that

$$\int_T^{2T} \sum\nolimits_2^2(q,t)\,\mathrm{d}t \ll q^{2-2\sigma}T\log T. \tag{14}$$

Clearly, in virtue of the estimate $R(q,t)=O(q^{7/4-\sigma}\log T)$,

$$\int_T^{2T} R^2(q,t)\,\mathrm{d}t \ll Tq^{7/2-2\sigma}\log^2 T.$$

This, together with (14), gives

$$\int_T^{2T}\left(\sum\nolimits_2(q,t)+R(q,t)\right)^2\mathrm{d}t \ll \int_T^{2T}\sum\nolimits_2^2(q,t)\,\mathrm{d}t + \int_T^{2T}R^2(q,t)\,\mathrm{d}t$$

$$\ll q^{7/2-2\sigma}T\log^2 T. \tag{15}$$

Moreover, the Cauchy–Schwarz inequality and (9), (15) imply the estimate

$$\int_T^{2T}\sum\nolimits_1(q,t)\left(\sum\nolimits_2(q,t)+R(q,t)\right)\mathrm{d}t$$

$$\ll \left(\int_T^{2T}\sum\nolimits_1^2(q,t)\,\mathrm{d}t\right)^{1/2}\left(\int_T^{2T}\left(\sum\nolimits_2(q,t)+R(q,t)\right)^2\mathrm{d}t\right)^{1/2}$$

$$\ll q^{11/4-2\sigma}T^{7/4-\sigma}\log T.$$

This estimate together with (1), (9) and (15) gives

$$\int_{T}^{2T} E_\sigma^2(q,t)\,\mathrm{d}t = 2^{2\sigma-3/2}\left(\frac{5}{2}-2\sigma\right)^{-1}\pi^{2\sigma-3/2}q^{3/2-2\sigma} \times \sum_{m=1}^{\infty}\frac{\sigma_{1-2\sigma}^2(m)}{m^{5/2-2\sigma}}\big((2T)^{5/2-2\sigma}-T^{5/2-2\sigma}\big) + O\big(q^{7/2-2\sigma}T\log^2 T\big) + O\big(q^{11/4-2\sigma}T^{7/4-\sigma}\log T\big).$$

Now, taking $T/2, T/2^2, \ldots$ in place of T and summing, we obtain the theorem.

References

1. F.V. Atkinson, The mean-value of the Riemann zeta function, *Acta Math.*, **81**:353–376, 1949.
2. S. Černigova, A. Laurinčikas, On the mean square of the periodic zeta-function, in A. Laurinčikas et al. (Eds.), *Analytic and Probabilistic Methods in Number Theory*, TEV, Vilnius, 2002, pp. 91–99.
3. S. Černigova, A. Laurinčikas, The Atkinson type formula for the periodic zeta-function, *Chebyshevskii Sb.*, **14**(2), 180–199, 2013.
4. D.R. Heath-Brown, The twelfth power moment of the Riemann zeta-function, *Q. J. Math. Oxford*, **29**:443–462, 1978.
5. A. Ivič, *The Riemann Zeta-Function: The Theory of the Riemann Zeta-Function with Applications*, Wiley, New York, 1985.
6. M. Jutila, Transformation formulae for Dirichlet polynomials, *J. Number Theory*, **18**:135–156, 1984.
7. M. Jutila, Atkinson's formula revisited, in *Voronoi's Impact on Modern Science. Book 1*, Institute of Mathematics, Kyiv, 1998, pp. 137–154.
8. J. Karaliūnaitė, The Atkinson formula for the periodic zeta-function in the critical strip, in *Voronoi's Impact on Modern Science, Book 4, Vol 1. Proceedings of the 4th International Conference on Analytic Number Theory and Spatial Tessellations*, Institute of Mathematics, Kyiv, 2008, pp. 48–58.
9. J. Karaliūnaitė, A. Laurinčikas, The Atkinson formula for the periodic zeta-function *Lith. Math. J.*, **47**(4), 504–516, 2007.
10. J. Kubilius, The method of generating Dirichlet series in the theory of distribution of arithmetic functions. I, *Litov. Mat. Sb.*, **11**:125–134, 1971 (in Russian).
11. A. Laurinčikas, R. Garunkštis, *The Lerch Zeta-Function*, Kluwer Academic Publisher, Dordrecht, 2002.
12. K. Matsumoto, The mean square of the Riemann zeta-function in the critical strip, *Japan J. Math.*, **15**(1):1–13, 1989.

13. K. Matsumoto, T. Meurman, The mean square of the Riemann zeta-function in the critical strip. III, *Acta Arith.*, **64**(4):357–382, 1993.

14. T. Meurman, A generalization of Atkinson's formula to L-functions, *Acta Arith.* **47**:351–370, 1986.

15. T. Meurman, On the mean square of the Riemann zeta-function, *Q. J. Math., Oxf. II. Ser.*, **38**:337–343, 1987.

16. Y. Motohashi, A note on the mean value of the zeta and L-functions. IV, *Proc. Japan Acad., Ser. A*, **62**:311–313, 1986.