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On source identification problem for a delay parabolic equation*

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Abstract. In the present study, the inverse problem of a delay parabolic equation with nonlocal conditions is investigated. The stability estimates in Hölder norms for the solution of this problem are established.

Keywords: delay parabolic equation, Banach spaces, positive operators, stability estimates, nonlocal conditions.

1 Introduction

Delay parabolic equations (DPEs) have important applications in a wide range of applications such as physics, chemistry, biology and ecology and other fields. For example, diffusion problems where the current state depends upon an earlier one give rise to parabolic equations with delay. In mathematical modeling, DPEs are used together with boundary conditions specifying the solution on the boundary of the domain. Dirichlet and Neumann conditions are examples of classical boundary conditions (see [1] and the references given therein). In some cases, classical boundary conditions cannot describe process or phenomenon precisely. Therefore, mathematical models of various physical, chemical, biological or environmental processes often involve nonclassical conditions. Such conditions usually are identified as nonlocal boundary conditions and reflect situations when the data on the domain boundary cannot be measured directly, or when the data on the boundary depend on the data inside the domain. The well-posedness of various nonlocal boundary value problems for partial differential and difference equations has been studied extensively by many researchers (see [1–12] and the references given therein).

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Identification problems take an important place in applied sciences and engineering applications and have been studied by many authors (see [13–20] and the references given therein). Solving the direct problem permits the computation of various system outputs of physical interest. On the other hand, when some of the required inputs are not available we may instead be able to determine the missing inputs from outputs that are measured rather than computed by formulating and solving an appropriate inverse problem. In particular, when the missing input is unknown source term in the partial differential equation, the problem is called a source identification problem. The theory and applications of source identification problems for partial differential equations were given in various papers (see [21–23] and the references given therein). The well-posedness of the unknown source identification problem for a parabolic equation has been well-investigated when the unknown function p is dependent on space variable (see [24–29] and the references given therein). Nevertheless when the unknown function p is dependent on t the well-posedness of the source identification problem for a parabolic equation was investigated in [30–34].

The initial-boundary value problems for delay partial differential equations when the delay term is an operator of lower order with respect to other operator term were widely investigated (see [35–38] and the references given therein). In the case where the delay term is an operator of the same order with respect to other operator term is studied mainly if H is a Hilbert space (see, for example, [39] and the references given therein). In fact, there are very few papers which allow E to be a general Banach space (see [40–44]) and in these works, authors look only for partial differential equations under regular data. Moreover, approximate solutions of the delay parabolic equations in the case where the delay term is a simple operator of the same order with respect to other operator term were studied recently in papers [45–49]. However, the well-posedness of the source identification problem for a delay parabolic equation is not well-investigated (see [50]). In this paper, we investigate the source identification problem for a delay parabolic equation with nonlocal conditions

$$\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= a(x)\frac{\partial^2 u(t,x)}{\partial x^2} - \sigma u(t,x) - b\left[a(x)\frac{\partial^2 u(t-\omega,x)}{\partial x^2} - \sigma u(t-\omega,x)\right] \\
&\quad + p(t)q(x) + f(t,x), \quad 0 < x < l,\ 0 < t < \infty, \\
u(t,0) &= u(t,l), \quad u_x(t,0) = u_x(t,l), \quad 0 \leqslant t < \infty, \\
u(t,x) &= \varphi(t,x), \quad 0 \leqslant x \leqslant l,\ -\omega \leqslant t \leqslant 0, \\
u(t,x^*) &= \rho(t), \quad 0 \leqslant x^* \leqslant l,\ 0 \leqslant t < \infty,
\end{aligned} \tag{1}$$

where $u(t,x)$ and $p(t)$ are unknown functions, $\rho(t)$, $\varphi(t,x)$, $a(x)$, and $f(t,x)$ are sufficiently smooth functions, $a(x) \geqslant \delta > 0$, $b \in R^1$ and $\sigma > 0$ is a sufficiently large number with assuming that:

(a) $q(x)$ is a sufficiently smooth function,
(b) $q(x)$ and $q'(x)$ are periodic with length l,
(c) $q(x^*) \neq 0$.

In the present study, the source identification problem (1) for a delay parabolic equation with nonlocal conditions is investigated. The stability estimates in Hölder norms for the solution of this problem are established.

2 Preliminaries. Main results

To formulate our results, we introduce the Banach space $\mathring{C}^{\alpha}[0,l]$, $\alpha \in (0,1)$, of all continuous functions $\phi(x)$ defined on $[0,l]$ with $\phi(0)=\phi(l)$ satisfying a Hölder condition for which the following norm is finite:

$$\|\phi\|_{\mathring{C}^{\alpha}[0,l]} = \max_{0\leqslant x\leqslant l}\bigl|\phi(x)\bigr| + \sup_{0\leqslant x<x+h\leqslant l}\frac{|\phi(x+h)-\phi(x)|}{h^{\alpha}}.$$

With the help of the positive operator A we introduce the fractional spaces $E_\alpha = E_\alpha(E,A)$, $0<\alpha<1$, consisting of all $v\in E$ for which the following norm is finite [51]:

$$\|v\|_{E_\alpha} = \|v\|_E + \sup_{\lambda>0}\lambda^{1-\alpha}\bigl\|A\mathrm{e}^{-\lambda A}v\bigr\|_E. \tag{2}$$

In the present paper, $C([-\omega,0],E)$ stands for the Banach space of all abstract continuous functions $\varphi(t)$ defined on $[-\omega,0]$ with values in E equipped with the norm

$$\|\varphi\|_{C([-\omega,0],E)} = \max_{-\omega\leqslant t\leqslant 0}\bigl\|\varphi(t)\bigr\|_E$$

and $L_1([0,\infty),E)$ stands for the Banach space of all strongly measurable E-valued functions $v(t)$ defined on $[0,\infty)$ for which the following norm is finite:

$$\|v\|_{L_1([0,\infty),E)} = \int_0^\infty \bigl\|v(t)\bigr\|_E\,\mathrm{d}t.$$

Finally, we introduce a differential operator A^x defined by the formula

$$A^x u = -a(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \sigma u \tag{3}$$

with the domain $D(A^x) = \{u\in C^{(2)}[0,l]\colon\ u(0)=u(l),\ u'(0)=u'(l)\}$.

It is well known that $A = A^x$ is the strongly positive operator in $C[0,l]$ of all continuous functions $\phi(x)$ defined on $[0,l]$ with norm

$$\|\phi\|_{C[0,l]} = \max_{0\leqslant x\leqslant l}\bigl|\phi(x)\bigr|$$

and, for this operator, the following estimates hold:

$$\bigl\|\mathrm{e}^{-tA}\bigr\|_{C[0,l]\to C[0,l]} \leqslant M\mathrm{e}^{-\delta t}, \quad t\geqslant 0, \tag{4}$$

$$\bigl\|A^{\alpha}\mathrm{e}^{-tA}\bigr\|_{C[0,l]\to C[0,l]} \leqslant Mt^{-\alpha}, \quad t>0, \tag{5}$$

where $\delta, M>0$ [43].

Positive constants have different values in time and they will be indicated with M. On the other hand, $M(\alpha, \beta, \ldots)$ is used to focus on the fact that the constant depends only on $\alpha, \beta, \ldots$.

Moreover, we have the following theorem on the structure of the fractional space $E_\alpha = E_\alpha(C[0,l], A^x)$.

Theorem 1. *For $\alpha \in (0, 1/2)$, the norms of the space $E_\alpha(C[0,l], A^x)$ and the Hölder space $\mathring{C}^\alpha[0,l]$ are equivalent* [43, 51].

The main result of present paper is the following theorem on stability of (1) in spaces $C([0,\infty), \mathring{C}^{2\alpha}[0,l])$, $\alpha \in (0, 1/2)$.

Theorem 2. *Assume that*

$$|b| \leqslant \frac{1-\alpha}{M 2^{2-\alpha}}. \tag{6}$$

Let $\varphi(t,x), \varphi_{xx}(t,x) \in C([-\omega,0], \mathring{C}^{2\alpha}[0,l])$, $\varphi_t(t,x) \in C([-\omega,0], \mathring{C}^{2\alpha+2}[0,l])$, $f_t(t,x) \in L_1([0,\infty), \mathring{C}^{2\alpha}[0,l])$, $f(0,x) \in \mathring{C}^{2\alpha}[0,l]$ and $\rho'(t) \in L_1[0,\infty)$. Then, for the solution of problem (1)*, the following stability estimates hold for all $t \geqslant 0$*:

$$\begin{aligned}
&\|u_t(t)\|_{\mathring{C}^{2\alpha}[0,l]} + \|u(t)\|_{\mathring{C}^{2\alpha+2}[0,l]} + |p(t)| \\
&\quad \leqslant M(a,\delta,\sigma,\alpha,x^*,q,l) \mathrm{e}^{M(\alpha,x^*,q,l)t} \big[\|\varphi\|_{C([-\omega,0],\mathring{C}^{2\alpha+2}[0,l])} + \|f(0)\|_{\mathring{C}^{2\alpha}[0,l]} \\
&\qquad + \|\varphi'\|_{C([-\omega,0],\mathring{C}^{2\alpha}[0,l])} + \|f'\|_{L_1([0,\infty),\mathring{C}^{2\alpha}[0,l])} + \|\rho'\|_{L_1[0,\infty)}\big], \\
&M(\alpha,x^*,q,l) = \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}, \quad 0 < \alpha < \frac{1}{2}.
\end{aligned}$$

Here $L_1[0,\infty)$ stands for the Banach space of all strongly measurable functions $v(t)$ defined on $[0,\infty)$ for which the following norm is finite:

$$\|v\|_{L_1[0,\infty)} = \int_0^\infty |v(t)|\,\mathrm{d}t.$$

Proof. Let us seek the substitution for the solution of the inverse problem in the following form:

$$u(t,x) = \eta(t)q(x) + w(t,x), \tag{7}$$

where

$$\eta(t) = \int_0^t p(s)\,\mathrm{d}s.$$

Taking derivatives from (7), we get

$$\frac{\partial u(t,x)}{\partial t} = p(t)q(x) + \frac{\partial w(t,x)}{\partial t}$$

and

$$\frac{\partial^2 u(t,x)}{\partial x^2} = \eta(t)\frac{\mathrm{d}^2 q(x)}{\mathrm{d}x^2} + \frac{\partial^2 w(t,x)}{\partial x^2}.$$

Moreover, if we put $x = x^*$ in equation (1), we obtain

$$u(t,x^*) = \eta(t)q(x^*) + w(t,x^*) = \rho(t)$$

and

$$\eta(t) = \frac{\rho(t) - w(t,x^*)}{q(x^*)}. \tag{8}$$

Taking derivative of both sides of (8) with respect to t, we achieve

$$p(t) = \frac{\rho'(t) - w_t(t,x^*)}{q(x^*)}. \tag{9}$$

Using the triangle inequality and the identity (9), we obtain

$$\begin{aligned} |p(t)| &\leqslant \frac{1}{|q(x^*)|}\bigl[\bigl|\rho'(t)\bigr| + \bigl|w_t(t,x^*)\bigr|\bigr] \leqslant \frac{1}{|q(x^*)|}\Bigl(\bigl|\rho'(t)\bigr| + \max_{0\leqslant x\leqslant l}\bigl|w_t(t,x)\bigr|\Bigr) \\ &\leqslant \frac{1}{|q(x^*)|}\bigl(\bigl|\rho'(t)\bigr| + \bigl\|w_t(t)\bigr\|_{\mathring{C}^{2\alpha}[0,l]}\bigr) \end{aligned} \tag{10}$$

for any t, $t \in [0,\infty)$. Using equations (1), (7), (8) and under the same assumptions on $q(x)$, one can show that $w(t,x)$ is the solution of the following problem:

$$\begin{aligned} \frac{\partial w(t,x)}{\partial t} &= a(x)\frac{\partial^2 w(t,x)}{\partial x^2} - \sigma w(t,x) - b\biggl[a(x)\frac{\partial^2 \varphi(t-\omega,x)}{\partial x^2} - \sigma\varphi(t-\omega,x)\biggr] \\ &\quad + \eta(t)\biggl[a(x)\frac{\partial^2 q(x)}{\partial x^2} - \sigma q(x)\biggr] + f(t,x), \quad 0<x<l,\ 0<t<\omega, \\ \frac{\partial w(t,x)}{\partial t} &= a(x)\frac{\partial^2 w(t,x)}{\partial x^2} - \sigma w(t,x) - b\biggl[a(x)\frac{\partial^2 w(t-\omega,x)}{\partial x^2} - \sigma w(t-\omega,x)\biggr] \\ &\quad + \bigl[\eta(t) - b\eta(t-\omega)\bigr]\biggl[a(x)\frac{\partial^2 q(x)}{\partial x^2} - \sigma q(x)\biggr] + f(t,x), \\ &\quad 0<x<l,\ \omega<t<\infty, \\ w(t,0) &= w(t,l), \quad w_x(t,0) = w_x(t,l), \quad 0\leqslant t<\infty, \\ w(0,x) &= \varphi(0,x), \quad 0\leqslant x\leqslant l. \end{aligned} \tag{11}$$

So, the end of proof of Theorem is based on estimate (10) and the following theorem. □

Theorem 3. *For the solution of problem* (11)*, the following stability estimate holds for any* $t, t \geqslant 0$:

$$\begin{aligned} \|w_t\| &\leqslant M(a,\delta,\sigma,\alpha,x^*,q,l)\mathrm{e}^{M(\alpha,x^*,q,l)t}\bigl[\|\varphi\|_{C([-\omega,0],\mathring{C}^{2\alpha+2}[0,l])} + \bigl\|f(0)\bigr\|_{\mathring{C}^{2\alpha}[0,l]} \\ &\quad + \|\varphi'\|_{C([-\omega,0],\mathring{C}^{2\alpha}[0,l])} + \|f'\|_{L_1([0,\infty),\mathring{C}^{2\alpha}[0,l])} + \|\rho'\|_{L_1[0,\infty)}\bigr], \quad 0<\alpha<\frac{1}{2}. \end{aligned}$$

Proof. We can rewrite the problem (11) in the following abstract form:

$$\begin{aligned}
&w_t(t) + Aw(t) = B\varphi(t-\omega) + (aq'' - \sigma q)\eta(t) + f(t), \quad 0 < t < \omega, \\
&w_t(t) + Aw(t) = Bw(t-\omega) + (aq'' - \sigma q)\big[\eta(t) - b\eta(t-\omega)\big] + f(t), \quad \omega < t < \infty, \\
&w(0) = \varphi(0)
\end{aligned}$$

in a Banach space $E = C[0,l]$ with the positive operator $A = A^x$ defined by formula (3) and the unbounded operator $B = bA^x$. Here $f(t) = f(t,x)$ and $w(t) = w(t,x)$ are, respectively, known and unknown abstract functions defined on $(0,\infty)$ with values in $E = C[0,l]$, $w(t,x^*)$ is unknown scalar function defined on $(0,\infty)$, $q = q(x)$, $q'' = q''(x)$, $\varphi = \varphi(x)$ and $a = a(x)$ are elements of $E = C[0,l]$ and $q(x^*)$ is a real number. Finally, we can rewrite condition (6) in the following form:

$$\left\|BA^{-1}\right\|_{C[0,l]\to C[0,l]} \leqslant \frac{1-\alpha}{M2^{2-\alpha}} \tag{12}$$

for any t, $t \in [0,\infty)$. Let us $0 \leqslant t \leqslant \omega$. Then, using the Cauchy formula, we establish

$$\begin{aligned}
w(t) = {}& \mathrm{e}^{-tA}\varphi(0) + \int_0^t \mathrm{e}^{-(t-s)A}B\varphi(s-\omega)\,\mathrm{d}s \\
& + \int_0^t \mathrm{e}^{-(t-s)A}(aq'' - \sigma q)\eta(s)\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A}f(s)\,\mathrm{d}s.
\end{aligned}$$

Taking the derivative of both sides, we obtain that

$$\begin{aligned}
w_t(t) = {}& -A\mathrm{e}^{-tA}\varphi(0) - \int_0^t A\mathrm{e}^{-(t-s)A}B\varphi(s-\omega)\,\mathrm{d}s \\
& - \int_0^t A\mathrm{e}^{-(t-s)A}(aq'' - \sigma q)\eta(s)\,\mathrm{d}s - \int_0^t A\mathrm{e}^{-(t-s)A}f(s)\,\mathrm{d}s \\
& + B\varphi(t-\omega) + (aq'' - \sigma q)\eta(t) + f(t).
\end{aligned}$$

Applying formulas

$$\begin{aligned}
& -\int_0^t A\mathrm{e}^{-(t-s)A}(aq'' - \sigma q)\eta(s)\,\mathrm{d}s \\
& = -(aq'' - \sigma q)\eta(t) + \mathrm{e}^{-tA}(aq'' - \sigma q)\eta(0) + \int_0^t \mathrm{e}^{-(t-s)A}(aq'' - \sigma q)\eta'(s)\,\mathrm{d}s \\
& = -(aq'' - \sigma q)\eta(t) + \int_0^t \mathrm{e}^{-(t-s)A}(aq'' - \sigma q)p(s)\,\mathrm{d}s - \int_0^t A\mathrm{e}^{-(t-s)A}B\varphi(s-\omega)\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= -B\varphi(t-\omega) + \mathrm{e}^{-tA}B\varphi(-\omega) + \int_0^t \mathrm{e}^{-(t-s)A}B\varphi'(s-\omega)\,\mathrm{d}s \\
&\quad - \int_0^t A\mathrm{e}^{-(t-s)A}f(s)\,\mathrm{d}s \\
&= -f(t) + \mathrm{e}^{-tA}f(0) + \int_0^t \mathrm{e}^{-(t-s)A}f(s)\,\mathrm{d}s,
\end{aligned}
$$

we get

$$
\begin{aligned}
w_t(t) = \mathrm{e}^{-tA}w_t(0) &+ \int_0^t \mathrm{e}^{-(t-s)A}B\varphi'(s-\omega)\,\mathrm{d}s \\
&+ \int_0^t \mathrm{e}^{-(t-s)A}(aq''-\sigma q)p(s)\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A}f'(s)\,\mathrm{d}s.
\end{aligned}
$$

Here

$$
w_t(0) = -A\varphi(0) + B\varphi(-\omega) + f(0).
$$

Applying this formula and the semigroup property, the condition (12) and the estimates (4), (5), we obtain

$$
\begin{aligned}
&\lambda^{1-\alpha}\left\|A\mathrm{e}^{-\lambda A}w_t(t)\right\|_E \\
&\quad\leqslant \lambda^{1-\alpha}\left\|A\mathrm{e}^{-(\lambda+t)A}w_t(0)\right\|_E + \lambda^{1-\alpha}\int_0^t \left\|A\mathrm{e}^{-((\lambda+t-s)/2)A}\right\|_{E\to E}\left\|BA^{-1}\right\|_{E\to E} \\
&\qquad\times \left\|A\mathrm{e}^{-((\lambda+t-s)/2)A}\varphi'(s-\omega)\right\|_E\,\mathrm{d}s \\
&\qquad+ \lambda^{1-\alpha}\int_0^t \left\|A\mathrm{e}^{-(\lambda+t-s)A}Aq\right\|_E\left|p(s)\right|\mathrm{d}s + \lambda^{1-\alpha}\int_0^t \left\|A\mathrm{e}^{-(\lambda+t-s)A}f'(s)\right\|_E\,\mathrm{d}s \\
&\quad\leqslant \frac{\lambda^{1-\alpha}}{(\lambda+t)^{1-\alpha}}\left\|w_t(0)\right\|_{E_\alpha} + \frac{1-\alpha}{M2^{2-\alpha}}\int_0^t \frac{M\lambda^{1-\alpha}2^{2-\alpha}}{(\lambda+t-s)^{2-\alpha}}\,\mathrm{d}s \max_{0\leqslant t\leqslant\omega}\left\|\varphi'(t-\omega)\right\|_{E_\alpha} \\
&\qquad+ \int_0^t \frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{1-\alpha}}\frac{1}{|q(x^*)|}\left(\left|\rho'(s)\right| + \left\|w_s(s)\right\|_{\mathring{C}^{2\alpha}[0,l]}\right)\mathrm{d}s\|Aq\|_{E_\alpha} \\
&\qquad+ \int_0^t \frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{1-\alpha}}\left\|f'(s)\right\|_{E_\alpha}\,\mathrm{d}s
\end{aligned}
$$

$$\leqslant \left\|w_t(0)\right\|_{E_\alpha} + \max_{-\omega\leqslant t\leqslant 0}\left\|\varphi'(t)\right\|_{E_\alpha} + \int_0^t \left\|f'(s)\right\|_{E_\alpha} \mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int_0^t \left|\rho'(s)\right| \mathrm{d}s$$
$$+ \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int_0^t \left\|w_s(s)\right\|_{\mathring{C}^{2\alpha}[0,l]} \mathrm{d}s$$
$$\leqslant \left(1+|b|\right) \max_{-\omega\leqslant t\leqslant 0}\left\|A\varphi(t)\right\|_{E_\alpha} + \left\|f(0)\right\|_{E_\alpha} + \max_{-\omega\leqslant t\leqslant 0}\left\|\varphi'(t)\right\|_{E_\alpha}$$
$$+ \int_0^t \left\|f'(s)\right\|_{E_\alpha} \mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int_0^t \left|\rho'(s)v\right| \mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int_0^t \left\|w_s(s)\right\|_{\mathring{C}^{2\alpha}[0,l]} \mathrm{d}s$$

for every t, $0 \leqslant t \leqslant \omega$, and λ, $\lambda > 0$. This shows that

$$\left\|w_t(t)\right\|_{E_\alpha} \leqslant \left(1+|b|\right) \max_{-\omega\leqslant t\leqslant 0}\left\|A\varphi(t)\right\|_{E_\alpha} + \left\|f(0)\right\|_{E_\alpha} + \max_{-\omega\leqslant t\leqslant 0}\left\|\varphi'(t)\right\|_{E_\alpha}$$
$$+ \int_0^t \left\|f'(s)\right\|_{E_\alpha} \mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int_0^t \left|\rho'(s)\right| \mathrm{d}s$$
$$+ \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int_0^t \left\|w_s(s)\right\|_{\mathring{C}^{2\alpha}[0,l]} \mathrm{d}s \tag{13}$$

for every t, $0 \leqslant t \leqslant \omega$. Applying the integral inequality, we obtain

$$\left\|w_t(t)\right\|_{E_\alpha} \leqslant \Bigg[\left(1+|b|\right) \max_{-\omega\leqslant t\leqslant 0}\left\|A\varphi(t)\right\|_{E_\alpha} + \left\|f(0)\right\|_{E_\alpha} + \max_{-\omega\leqslant t\leqslant 0}\left\|\varphi'(t)\right\|_{E_\alpha}$$
$$+ \int_0^\omega \left\|f'(s)\right\|_{E_\alpha} \mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\Bigg]\int_0^\omega \left|\rho'(s)\right| \mathrm{d}s\, \mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)t} \tag{14}$$

for every t, $0 \leqslant t \leqslant \omega$. Now we consider the case $\omega \leqslant t < \infty$. Applying the mathematical induction, one can easily show that it is true for every t. Namely, assume that the inequality

$$\left\|w_t(t)\right\|_{E_\alpha} \leqslant \Bigg[\left(1+|b|\right) \max_{-\omega\leqslant t\leqslant 0}\left\|A\varphi(t)\right\|_{E_\alpha} + \left\|f(0)\right\|_{E_\alpha} + \max_{-\omega\leqslant t\leqslant 0}\left\|\varphi'(t)\right\|_{E_\alpha}$$
$$+ \int_0^{n\omega} \left\|f'(s)\right\|_{E_\alpha} \mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int_0^{n\omega} \left|\rho'(s)\right| \mathrm{d}s\Bigg]\mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)t} \tag{15}$$

is true for t, $(n-1)\omega \leqslant t \leqslant n\omega$, $n = 1, 2, 3, \ldots$, for some n. Using the Cauchy formula, we establish

$$w(t) = \mathrm{e}^{-(t-n\omega)A} w(n\omega) + \int_{n\omega}^{t} \mathrm{e}^{-(t-s)A} B w(s-\omega)\,\mathrm{d}s$$
$$+ \int_{n\omega}^{t} \mathrm{e}^{-(t-s)A} (aq'' - \sigma q)\eta(s)\,\mathrm{d}s + \int_{n\omega}^{t} \mathrm{e}^{-(t-s)A} f(s)\,\mathrm{d}s. \tag{16}$$

Taking the derivative of both sides with respect to t, we obtain

$$w_t(t) = -A\mathrm{e}^{-(t-n\omega)A} w(n\omega) - \int_{n\omega}^{t} A\mathrm{e}^{-(t-s)A} B w(s-\omega)\,\mathrm{d}s$$
$$- \int_{n\omega}^{t} A\mathrm{e}^{-(t-s)A} (aq'' - \sigma q)\eta(s)\,\mathrm{d}s - A\int_{n\omega}^{t} \mathrm{e}^{-(t-s)A} f(s)\,\mathrm{d}s$$
$$+ Bw(t-\omega) + (aq'' - \sigma q)\eta(t) + f(t).$$

Applying formulas

$$-\int_{n\omega}^{t} A\mathrm{e}^{-(t-s)A} (aq'' - \sigma q)\eta(s)\,\mathrm{d}s$$
$$= -(aq'' - \sigma q)\eta(t) + \mathrm{e}^{-(t-n\omega)A}(aq'' - \sigma q)\eta(n\omega) + \int_{n\omega}^{t} \mathrm{e}^{-(t-s)A}(aq'' - \sigma q)\eta'(s)\,\mathrm{d}s$$
$$= -(aq'' - \sigma q)\eta(t) + \mathrm{e}^{-(t-n\omega)A}(aq'' - \sigma q)\eta(n\omega) + \int_{n\omega}^{t} \mathrm{e}^{-(t-s)A}(aq'' - \sigma q)p(s)\,\mathrm{d}s,$$

$$-\int_{n\omega}^{t} A\mathrm{e}^{-(t-s)A} B w(s-\omega)\,\mathrm{d}s$$
$$= -Bw(t-n\omega) + \mathrm{e}^{-(t-n\omega)A} B w(n\omega - \omega) + \int_{n\omega}^{t} \mathrm{e}^{-(t-s)A} B w'(s-\omega)\,\mathrm{d}s$$
$$-\int_{n\omega}^{t} A\mathrm{e}^{-(t-s)} f(s)\,\mathrm{d}s$$
$$= -f(t) + \mathrm{e}^{-(t-n\omega)A} f(n\omega) + \int_{n\omega}^{t} \mathrm{e}^{-(t-s)A} f'(s)\,\mathrm{d}s,$$

we obtain

$$w_t(t) = \mathrm{e}^{-(t-n\omega)A} w_t(n\omega) + \int_{n\omega}^{t} \mathrm{e}^{-(t-s)} B w'(s-\omega)\,\mathrm{d}s + \int_{n\omega}^{t} \mathrm{e}^{-(t-s)} (aq'' - \sigma q) p(s)\,\mathrm{d}s + \int_{n\omega}^{t} \mathrm{e}^{-(t-s)A} f'(s)\,\mathrm{d}s.$$

Using this formula and the semigroup property, the condition (12) and the estimates (4), (5), we obtain

$$\begin{aligned}
&\lambda^{1-\alpha} \|A\mathrm{e}^{-\lambda A} w_t(t)\|_E \\
&\leqslant \lambda^{1-\alpha} \|A\mathrm{e}^{-(\lambda+t-n\omega)A} w_t(n\omega)\|_E + \lambda^{1-\alpha} \int_{n\omega}^{t} \|A\mathrm{e}^{-((\lambda+t-s)/2)A}\|_{E\to E} \|BA^{-1}\|_{E\to E} \\
&\quad \times \|A\mathrm{e}^{-((\lambda+t-s)/2)A} w'(s-\omega)\|_E \,\mathrm{d}s \\
&\quad + \lambda^{1-\alpha} \int_{n\omega}^{t} \|A\mathrm{e}^{-(\lambda+t-s)A} Aq\|_E |p(s)|\,\mathrm{d}s + \lambda^{1-\alpha} \int_{n\omega}^{t} \|A\mathrm{e}^{-(\lambda+t-s)A} f(s)\|_E \,\mathrm{d}s \\
&\leqslant \frac{\lambda^{1-\alpha}}{(\lambda+t)^{1-\alpha}} \|w_t(n\omega)\|_{E_\alpha} + \frac{1-\alpha}{M2^{2-\alpha}} \int_{n\omega}^{t} \frac{M\lambda^{1-\alpha}2^{2-\alpha}}{(\lambda+t-s)^{2-\alpha}}\,\mathrm{d}s \max_{n\omega\leqslant s\leqslant t} \|w'(s-\omega)\|_{E_\alpha} \\
&\quad + \int_{n\omega}^{t} \frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{1-\alpha}} \frac{1}{|q(x^*)|} \left(|\rho'(s)| + \|w_s(s)\|_{\mathring{C}^{2\alpha}[0,l]}\right) \mathrm{d}s \|Aq\|_{E_\alpha} \\
&\quad + \int_{n\omega}^{t} \frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{1-\alpha}} \|f'(s)\|_{E_\alpha} \,\mathrm{d}s \\
&\leqslant \max_{n\omega-\omega\leqslant t\leqslant n\omega} \|w_t(t)\|_{E_\alpha} + \int_{n\omega}^{t} \|f'(s)\|_{E_\alpha}\,\mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|} \int_{n\omega}^{t} |\rho'(s)|\,\mathrm{d}s \\
&\quad + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|} \int_{n\omega}^{t} \|w_s(s)\|_{\mathring{C}^{2\alpha}[0,l]}\,\mathrm{d}s
\end{aligned}$$

for every t, $n\omega \leqslant t \leqslant n\omega + \omega$, and λ, $\lambda > 0$. This shows that

$$\begin{aligned}
\|w_t(t)\|_{E_\alpha} &\leqslant \max_{n\omega-\omega\leqslant t\leqslant n\omega} \|w_t(t)\|_{E_\alpha} + \int_{n\omega}^{t} \|f'(s)\|_{E_\alpha}\,\mathrm{d}s \\
&\quad + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|} \int_{n\omega}^{t} |\rho'(s)|\,\mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|} \int_{n\omega}^{t} \|w_s(s)\|_{\mathring{C}^{2\alpha}[0,l]}\,\mathrm{d}s \qquad (17)
\end{aligned}$$

for every t, $n\omega \leqslant t \leqslant n\omega + \omega$. Applying the integral inequality, we obtain

$$\begin{aligned}\|w_t(t)\|_{E_\alpha} \leqslant \Bigg[&\max_{n\omega-\omega\leqslant t\leqslant n\omega} \|w'(t)\|_{E_\alpha} + \int\limits_{n\omega}^{n\omega+\omega} \|f'(s)\|_{E_\alpha}\,\mathrm{d}s \\ &+ \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|} \int\limits_{n\omega}^{n\omega+\omega} |\rho'(s)|\,\mathrm{d}s\Bigg] \mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)(t-n\omega)} \end{aligned} \tag{18}$$

for every t, $n\omega \leqslant t \leqslant n\omega + \omega$. Applying estimates (15) and (18), we get

$$\begin{aligned}&\|w_t(t)\|_{E_\alpha} \\ &\quad\leqslant \max_{n\omega-\omega\leqslant t\leqslant n\omega} \|w'(t)\|_{E_\alpha} \mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)(t-n\omega)} \\ &\qquad+ \Bigg[\int\limits_{n\omega}^{n\omega+\omega} \|f'(s)\|_{E_\alpha}\,\mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int\limits_{n\omega}^{n\omega+\omega} |\rho'(s)|\,\mathrm{d}s\Bigg] \mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)(t-n\omega)} \\ &\quad\leqslant \Bigg[\big(1+|b|\big) \max_{-\omega\leqslant t\leqslant 0} \|A\varphi(t)\|_{E_\alpha} + \|f(0)\|_{E_\alpha} \max_{-\omega\leqslant t\leqslant 0} \|\varphi'(t)\|_{E_\alpha} \\ &\qquad+ \int\limits_0^{n\omega} \|f'(s)\|_{E_\alpha}\,\mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int\limits_0^{n\omega} |\rho'(s)|\,\mathrm{d}s\Bigg] \mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)t} \\ &\qquad+ \Bigg[\int\limits_{n\omega}^{n\omega+\omega} \|f'(s)\|_{E_\alpha}\,\mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int\limits_{n\omega}^{n\omega+\omega} |\rho'(s)|\,\mathrm{d}s\Bigg] \mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)(t-n\omega)} \\ &\quad\leqslant \Bigg[\big(1+|b|\big) \max_{-\omega\leqslant t\leqslant 0} \|A\varphi(t)\|_{E_\alpha} + \|f(0)\|_{E_\alpha} + \max_{-\omega\leqslant t\leqslant 0} \|\varphi'(t)\|_{E_\alpha} \\ &\qquad+ \int\limits_0^{(n+1)\omega} \|f'(s)\|_{E_\alpha}\,\mathrm{d}s + \frac{\|Aq\|_{E_\alpha}}{|q(x^*)|}\int\limits_0^{(n+1)\omega} |\rho'(s)|\,\mathrm{d}s\Bigg] \mathrm{e}^{(\|Aq\|_{E_\alpha}/|q(x^*)|)t}\end{aligned}$$

for every t, $n\omega \leqslant t \leqslant n\omega + \omega$. This result and Theorem 1 completes the proof of Theorem 2. □

3 Conclusion

In the present study, the source identification problem (1) for a delay parabolic equation with nonlocal conditions is investigated. The stability estimates in Hölder norms for the solution of this problem are established. Moreover, applying the result of the monograph [43], the high order of accuracy single-step difference schemes for the numerical solution of the source identification problem (1) for a delay parabolic equation with nonlocal conditions can be presented. Of course, the stability estimates for the solution of these difference schemes have been established without any assumptions about the grid steps.

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