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Nonnegative solutions for a system of impulsive BVPs with nonlinear nonlocal BCs

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Abstract. We study the existence of nonnegative solutions for a system of impulsive differential equations subject to nonlinear, nonlocal boundary conditions. The system presents a coupling in the differential equation and in the boundary conditions. The main tool that we use is the theory of fixed point index for compact maps.

Keywords: fixed point index, cone, impulsive equation, system, positive solution.

1 Introduction

The aim of this paper is to study the existence and multiplicity of positive solutions for a class of systems of ordinary impulsive differential equations subject to nonlinear, nonlocal boundary conditions (BCs). The system presents a coupling in the nonlinearities and in the BCs. Problems with a coupling in the BCs often occur in applications, see, for example, [1–11]. On the other hand, impulsive problems have been studied not only because of a theoretical interest, but also because they model several phenomena in engineering, physics and life sciences. For example, Nieto and co-authors [12, 13] contributed to the field of population dynamics. An introduction to the theory of impulsive differential equations and its applications can be found in the books [14–17].

Systems of second order impulsive boundary value problems (BVPs) have been studied in [18–21]. Here we consider the (fairly general) system of second order differential equations of the form

$$\begin{aligned} &u''(t) + g_1(t)f_1\bigl(t, u(t), v(t)\bigr) = 0, \quad t \in (0,1),\ t \neq \tau_1, \\ &v''(t) + g_2(t)f_2\bigl(t, u(t), v(t)\bigr) = 0, \quad t \in (0,1),\ t \neq \tau_2, \end{aligned} \tag{1}$$

with impulsive terms of the type

$$\begin{aligned} &\Delta u|_{t=\tau_1} = I_1\bigl(u(\tau_1)\bigr), \quad \Delta u'|_{t=\tau_1} = N_1\bigl(u(\tau_1)\bigr), \quad \tau_1 \in (0,1), \\ &\Delta v|_{t=\tau_2} = I_2\bigl(v(\tau_2)\bigr), \quad \Delta v'|_{t=\tau_2} = N_2\bigl(v(\tau_2)\bigr), \quad \tau_2 \in (0,1), \end{aligned} \tag{2}$$

and nonlocal nonlinear BCs of "Sturm–Liouville" kind

$$\begin{aligned} a_{11}u(0)-b_{11}u'(0)&=H_1\big(\alpha_1[u]\big), \qquad a_{12}u(1)+b_{12}u'(1)=L_1\big(\beta_1[v]\big),\\ a_{21}v(0)-b_{21}v'(0)&=H_2\big(\alpha_2[v]\big), \qquad a_{22}v(1)+b_{22}v'(1)=L_2\big(\beta_2[u]\big), \end{aligned} \tag{3}$$

where for $i=1,2$, $a_{i1},b_{i1},a_{i2},b_{i2}\in[0,\infty)$, $a_{i1}+b_{i1}\neq 0, a_{i2}+b_{i2}\neq 0$ and $\lambda=0$ is not an eigenvalue of the problem

$$w''(t)=0, \qquad a_{i1}w(0)-b_{i1}w'(0)=0, \qquad a_{i2}w(1)+b_{i2}w'(1)=0.$$

Here $\Delta w|_{t=\tau}$ denotes the "jump" of the function w in $t=\tau$, that is

$$\Delta w|_{t=\tau}=w(\tau^+)-w(\tau^-),$$

where $w(\tau^-)$ and $w(\tau^+)$ are the left and right limits of w in $t=\tau$ and $\alpha_i[\cdot]$, $\beta_i[\cdot]$ are bounded linear functionals given by positive Riemann–Stieltjes integrals, namely

$$\alpha_i[w]=\int_0^1 w(s)\,\mathrm{d}A_i(s), \qquad \beta_i[w]=\int_0^1 w(s)\,\mathrm{d}B_i(s).$$

This type of formulation includes, as special cases, multi-point or integral conditions, namely

$$\alpha_i[w]=\sum_{j=1}^m \alpha_{ij}w(\eta_{ij}) \quad \text{and} \quad \alpha_i[w]=\int_0^1 \alpha_i(s)w(s)\,\mathrm{d}s,$$

studied, for example, [22–33]. In the case of impulsive equations, nonlocal BCs have been studied by many authors, see, for example, [25, 26, 34–41] and references therein. The functions H_i, L_i are continuous functions; for earlier contributions on problems with nonlinear BCs we refer the reader to [4, 5, 42–48] and references therein.

Our idea is to start from the results of [6, 7], valid for non-impulsive systems, and to rewrite the system (1)–(3) as a system of perturbed Hammerstein integral equations, namely

$$\begin{aligned} u(t)&=\gamma_1(t)H_1\big(\alpha_1[u]\big)+\delta_1(t)L_1\big(\beta_1[v]\big)+\int_0^1 k_1(t,s)g_1(s)f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s\\ &\quad+G_1(u)(t),\\ v(t)&=\gamma_2(t)H_2\big(\alpha_2[v]\big)+\delta_2(t)L_2\big(\beta_2[u]\big)+\int_0^1 k_2(t,s)g_2(s)f_2\big(s,u(s),v(s)\big)\,\mathrm{d}s\\ &\quad+G_2(v)(t), \end{aligned}$$

where the functions γ_i,δ_i are the unique solutions of

$$\begin{aligned} {\gamma_i}''(t)&=0, \qquad a_{i1}\gamma_i(0)-b_{i1}{\gamma_i}'(0)=1, \qquad a_{i2}\gamma_i(1)+b_{i2}{\gamma_i}'(1)=0,\\ {\delta_i}''(t)&=0, \qquad a_{i1}\delta_i(0)-b_{i1}{\delta_i}'(0)=0, \qquad a_{i2}\delta_i(1)+b_{i2}{\delta_i}'(1)=1, \end{aligned}$$

and the functions G_i, that are construct in natural manner, take care of the impulses.

Systems of perturbed Hammerstein integral equations were studied in [4, 5, 7, 49–53]. Our existence theory for multiple positive solutions of the perturbed Hammerstein integral equations covers system (1)–(3) as a special case and we show in an example that all the constants that occur in our theory can be computed. Here we focus on *positive* measures, because we want our functionals to preserve some inequalities. Our methodology involves the construction of *new* Stieltjes measures that take into account the boundary conditions and the impulsive effect.

We make use of the classical fixed point index theory (see, for example, [54, 55]) and also benefit of ideas from the papers [6, 7, 44, 51, 56–58].

2 The system of integral equations

We begin with the assumptions on the terms that occur in the system of perturbed Hammerstein integral equations

$$\begin{aligned} u(t) &= \gamma_1(t)H_1\big(\alpha_1[u]\big) + \delta_1(t)L_1\big(\beta_1[v]\big) + G_1(u)(t) + F_1\big(u,v\big)(t), \\ v(t) &= \gamma_2(t)H_2\big(\alpha_2[v]\big) + \delta_2(t)L_2\big(\beta_2[u]\big) + G_2(v)(t) + F_2\big(u,v\big)(t), \end{aligned} \tag{4}$$

where

$$F_i(u,v)(t) := \int_0^1 k_i(t,s)g_i(s)f_i\big(s,u(s),v(s)\big)\,\mathrm{d}s. \tag{5}$$

The functions G_i are given, as in [57], by

$$G_i(w)(t) := \gamma_i(t)\chi_{(\tau_i,1]}(d_{i1}I_i+e_{i1}N_i)\big(w(\tau_i)\big) + \delta_i(t)\chi_{[0,\tau_i]}(d_{i2}I_i+e_{i2}N_i)\big(w(\tau_i)\big)$$

with coefficients

$$d_{i1} = \frac{\delta_i'(\tau_i)}{W_i(\tau_i)}, \quad e_{i1} = \frac{-\delta_i(\tau_i)}{W_i(\tau_i)}, \quad d_{i2} = \frac{\gamma_i'(\tau_i)}{W_i(\tau_i)} \quad \text{and} \quad e_{i2} = \frac{-\gamma_i(\tau_i)}{W_i(\tau_i)},$$

where W_i is the Wronskian, $W_i(t) = \gamma_i(t)\delta_i'(t) - \delta_i(t)\gamma_i'(t)$.

We assume that for every $i = 1, 2$,

- $f_i : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfies Carathéodory conditions, that is, $f_i(\cdot,u,v)$ is measurable for each fixed (u,v) and $f_i(t,\cdot,\cdot)$ is continuous for almost every (a.e.) $t \in [0,1]$, and for each $r > 0$ there exists $\phi_{i,r} \in L^\infty[0,1]$ such that

$$f_i(t,u,v) \leqslant \phi_{i,r}(t) \quad \text{for } u,v \in [0,r] \text{ and a.e. } t \in [0,1].$$

- $k_i : [0,1] \times [0,1] \to [0,\infty)$ is measurable, and for every $\tau \in [0,1]$ we have

$$\lim_{t\to\tau} \big|k_i(t,s) - k_i(\tau,s)\big| = 0 \quad \text{for a.e. } s \in [0,1].$$

- There exist a subinterval $[a_i, b_i] \subseteq (\tau_i, 1]$, a function $\varPhi_i \in L^\infty[0,1]$, and a constant $c_{\varPhi_i} \in (0,1]$ such that

$$k_i(t,s) \leqslant \varPhi_i(s) \quad \text{for } t \in [0,1] \text{ and a.e. } s \in [0,1],$$
$$k_i(t,s) \geqslant c_{\varPhi_i}\varPhi_i(s) \quad \text{for } t \in [a_i,b_i] \text{ and a.e. } s \in [0,1].$$

- $g_i\,\varPhi_i \in L^1[0,1]$, $g_i \geqslant 0$ a.e., and $\int_{a_i}^{b_i} \varPhi_i(s) g_i(s)\,\mathrm{d}s > 0$.
- $\alpha_i[\cdot]$ and $\beta_i[\cdot]$ are linear functionals given by

$$\alpha_i[w] = \int_0^1 w(s)\,\mathrm{d}A_i(s), \qquad \beta_i[w] = \int_0^1 w(s)\,\mathrm{d}B_i(s),$$

involving Riemann–Stieltjes integrals; the functions A_i and B_i are non-decreasing and continuous in τ_i.
- $H_i, L_i : [0,\infty) \to [0,\infty)$ are continuous functions such that there exist $h_{i1}, h_{i2}, l_{i2} \in [0,\infty)$ with

$$h_{i1}w \leqslant H_i(w) \leqslant h_{i2}w, \qquad L_i(w) \leqslant l_{i2}w,$$

for every $w \geqslant 0$.
- $\gamma_i, \delta_i \in C[0,1]$, $\gamma_i, \delta_i \geqslant 0$, and there exist $c_{\gamma_i}, c_{\delta_i} \in (0,1]$ such that

$$\gamma_i(t) \geqslant c_{\gamma_i}\|\gamma_i\|_\infty, \quad \delta_i(t) \geqslant c_{\delta_i}\|\delta_i\|_\infty \quad \text{for every } t \in [a_i,b_i],$$

where $\|w\|_\infty := \sup\{|w(t)|,\ t \in [0,1]\}$.
- I_i, $N_i : [0,\infty) \to \mathbb{R}$ are continuous functions and there exist $p_{i11}, p_{i12} > 0$ and $p_{i22} \geqslant 0$ such that for $w \in [0,\infty)$

$$p_{i11}w \leqslant (d_{i1}I_i + e_{i1}N_i)(w) \leqslant p_{i12}w$$

and

$$0 \leqslant (d_{i2}I_i + e_{i2}N_i)(w) \leqslant p_{i22}w.$$

We consider the Banach space

$$PC_\tau[0,1] := \bigl\{w\colon [0,1] \to \mathbb{R},\ w \text{ is continuous in } t \in [0,1]\backslash\{\tau\}, \text{ there exist } w(\tau^-) = w(\tau) \text{ and } \bigl|w(\tau^+)\bigr| < \infty\bigr\},$$

endowed with the supremum norm $\|\cdot\|_\infty$.

We work in the space $PC_{\tau_1}[0,1] \times PC_{\tau_2}[0,1]$ endowed with the norm

$$\bigl\|(u,v)\bigr\| := \max\bigl\{\|u\|_\infty, \|v\|_\infty\bigr\}.$$

Let

$$\tilde{K}_i := \Bigl\{w \in PC_{\tau_i}[0,1]\colon w(t) \geqslant 0 \text{ for } t \in [0,1] \text{ and } \min_{t\in[a_i,b_i]} w(t) \geqslant c_i\|w\|_\infty\Bigr\},$$

where
$$c_i = \min\left\{c_{\Phi_i}, c_{\gamma_i}, c_{\delta_i}, \frac{c_{\gamma_i}\|\gamma_i\|_\infty p_{i11}}{\max\{\|\gamma_i\|_\infty p_{i12}, \|\delta_i\|_\infty p_{i22}\}}\right\},$$
and consider the cone K in $PC_{\tau_1}[0,1] \times PC_{\tau_2}[0,1]$ defined by
$$K := \{(u,v) \in \tilde{K_1} \times \tilde{K_2}\}.$$

For a *positive* solution of system (4) we mean a solution $(u,v) \in K$ of (4) such that $\|(u,v)\| > 0$.

We now show that the integral operator
$$\begin{aligned} T(u,v)(t) &:= \begin{pmatrix} \gamma_1(t)H_1(\alpha_1[u]) + \delta_1(t)L_1(\beta_1[v]) + G_1(u)(t) + F_1(u,v)(t) \\ \gamma_2(t)H_2(\alpha_2[v]) + \delta_2(t)L_2(\beta_2[u]) + G_2(v)(t) + F_2(u,v)(t) \end{pmatrix} \\ &:= \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix} \end{aligned} \tag{6}$$
leaves the cone K invariant and is compact. In order to do this, we use the following compactness criterion, which can be found, for example, in [16] and is an extension of the classical Ascoli–Arzelà theorem.

Lemma 1. *A set $S \subseteq PC_\tau[0,1]$ is relatively compact in $PC_\tau[0,1]$ if and only if S is bounded and quasi-equicontinuous (i.e. for all $u \in S$ and all $\varepsilon > 0$, exists $\beta > 0$ such that $t_1, t_2 \in [0,\tau]$ (or $t_1, t_2 \in (\tau, 1]$) and $|t_1 - t_2| < \beta$ implies $|u(t_1) - u(t_2)| < \varepsilon$).*

Lemma 2. *Operator (6) maps K into K and is compact.*

Proof. Take $(u,v) \in K$ such that $\|(u,v)\| \leqslant r$. Then we have, for $t \in [0,1]$,
$$\begin{aligned} &\Lambda_1(u,v)(t) \\ &\quad := \gamma_1(t)H_1(\alpha_1[u]) + \delta_1(t)L_1(\beta_1[v]) + \int_0^1 k_1(t,s)g_1(s)f_1(s,u(s),v(s))\,\mathrm{d}s \end{aligned}$$
and therefore
$$\begin{aligned} &\|\Lambda_1(u,v)\|_\infty \\ &\quad \leqslant \|\gamma_1\|_\infty H_1(\alpha_1[u]) + \|\delta_1\|_\infty L_1(\beta_1[v]) + \int_0^1 \Phi_1(s)g_1(s)f_1(s,u(s),v(s))\,\mathrm{d}s. \end{aligned}$$

We obtain, as in Lemma 1 of [7],
$$\begin{aligned} \min_{t\in[a_1,b_1]} \Lambda_1(u,v)(t) &\geqslant c_{\gamma_1}\|\gamma_1\|_\infty H_1(\alpha_1[u]) + c_{\delta_1}\|\delta_1\|_\infty L_1(\beta_1[v]) \\ &\quad + c_{\Phi_1}\int_0^1 \Phi_1(s)g_1(s)f_1(s,u(s),v(s))\,\mathrm{d}s \\ &\geqslant \min\{c_{\Phi_i}, c_{\gamma_i}, c_{\delta_i}\}\|\Lambda_1(u,v)\|_\infty. \end{aligned}$$

On the other hand, for $t \in [0, \tau_1]$ we have

$$G_1(u)(t) \leqslant \|\delta_1\|_\infty p_{122} u(\tau_1)$$

and for $t \in (\tau_1, 1]$

$$G_1(u)(t) \leqslant \|\gamma_1\|_\infty p_{112} u(\tau_1).$$

Therefore for $t \in [0, 1]$ we obtain

$$G_1(u)(t) \leqslant u(\tau_1) \max\bigl\{\|\gamma_1\|_\infty p_{112}, \|\delta_1\|_\infty p_{122}\bigr\}$$

and thus

$$\bigl\|G_1(u)\bigr\| \leqslant u(\tau_1) \max\bigl\{\|\gamma_1\|_\infty p_{112}, \|\delta_1\|_\infty p_{122}\bigr\}.$$

For $t \in [a_1, b_1]$, we get

$$\begin{aligned} G_1(u)(t) &= \gamma_1(t)(d_{11} I_1 + e_{11} N_1)\bigl(u(\tau_1)\bigr) \\ &\geqslant \frac{c_{\gamma_1} \|\gamma_1\|_\infty p_{111}}{\max\{\|\gamma_1\|_\infty p_{112}, \|\delta_1\|_\infty p_{122}\}} u(\tau_1) \max\bigl\{\|\gamma_1\|_\infty p_{112}, \|\delta_1\|_\infty p_{122}\bigr\}. \end{aligned}$$

Thus we obtain

$$\min_{t \in [a_1, b_1]} T_1(u, v)(t) \geqslant c_1 \bigl\|T_1(u, v)\bigr\|_\infty.$$

Moreover, we have $T_1(u, v)(t) \geqslant 0$. Hence we have $T_1(u, v) \in \tilde{K_1}$. In a similar manner we proceed for $T_2(u, v)$.

Furthermore, the map T is compact since the components T_i are sum of compact maps: the compactness of F_i is well-known; the compactness of the term G_i follows, in a similar way as in [57], from Lemma 1; since $\gamma_i, \delta_i, H_i, L_i$ are continuous, the remaining terms map bounded sets into bounded subsets of a finite dimensional space. □

3 Fixed point index calculations

3.1 Preliminaries and notations

We recall some basic facts regarding the classical fixed point index for compact maps, see, for example, [54, 55].

Let K be a cone in a Banach space X. If Ω is a bounded open subset of K (in the relative topology) we denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and the boundary relative to K. When Ω is an open bounded subset of X, we write $\Omega_K = \Omega \cap K$, an open subset of K.

Theorem 1. *Let K be a cone in a Banach space X and let Ω be an open bounded set with $0 \in \Omega_K$ and $\overline{\Omega}_K \neq K$. Assume that $T : \overline{\Omega}_K \to K$ is a compact map such that $x \neq Tx$ for $x \in \partial\Omega_K$. Then the fixed point index $i_K(T, \Omega_K)$ has the following properties*:

(i) *If there exists $e \in K \setminus \{0\}$ such that $x \neq Tx + \mu e$ for all $x \in \partial\Omega_K$ and all $\mu \geqslant 0$, then $i_K(T, \Omega_K) = 0$.*

(ii) *If* $Tx \neq \mu x$ *for all* $x \in \partial\Omega_K$ *and all* $\mu \geqslant 1$*, then* $i_K(T, \Omega_K) = 1$.

(iii) *Let* Ω^1 *be open in* X *with* $\overline{\Omega_K^1} \subset \Omega_K$. *If* $i_K(T, \Omega_K) = 1$ *and* $i_K(T, \Omega_K^1) = 0$, *then* T *has a fixed point in* $\Omega_K \setminus \overline{\Omega_K^1}$. *The same result holds if* $i_K(T, \Omega_K) = 0$ *and* $i_K(T, \Omega_K^1) = 1$.

For our index calculations, we use the following (relative) open bounded sets in K:

$$K_\rho = \{(u, v) \in K\colon \|(u, v)\| < \rho\},$$

and

$$V_\rho = \Big\{(u, v) \in K\colon \min_{t\in[a_1,b_1]} u(t) < \rho \text{ and } \min_{t\in[a_2,b_2]} v(t) < \rho\Big\}.$$

The set V_ρ (in the context of systems) was introduced by the authors in [50] and is equal to the set called $\Omega^{\rho/c}$ in [49]. From now on we set

$$c = \min\{c_1, c_2\}.$$

We utilize the following lemma, the proof is similar to Lemma 5 of [49] and is omitted.

Lemma 3. *The sets* K_ρ *and* V_ρ *have the following properties*:

- $K_\rho \subset V_\rho \subset K_{\rho/c}$.
- $(w_1, w_2) \in \partial V_\rho$ *iff* $(w_1, w_2) \in K$ *and* $\min_{t\in[a_i,b_i]} w_i(t) = \rho$ *for some* $i \in \{1, 2\}$ *and* $\min_{t\in[a_i,b_i]} w_i(t) \leqslant \rho$ *for each* $i \in \{1, 2\}$.
- *If* $(w_1, w_2) \in \partial V_\rho$, *then for some* $i \in \{1, 2\}$ $\rho \leqslant w_i(t) \leqslant \rho/c$ *for each* $t \in [a_i, b_i]$ *and for each* $i \in \{1, 2\}$ *we have* $0 \leqslant w_i(t) \leqslant \rho/c$ *for each* $t \in [a_i, b_i]$ *and* $\|w_i\|_\infty \leqslant \rho/c$.

We introduce, in a similar way as in [59], the linear functionals

$$\tilde{\alpha}_i[w] := h_{i2}\alpha_i[w] + p_{i12}w(\tau_i) := \int_0^1 w(s)\,\mathrm{d}\tilde{A}_i(s), \quad i = 1, 2,$$

$$\bar{\alpha}_i[w] := h_{i1}\alpha_i[w] + p_{i11}w(\tau_i) := \int_0^1 w(s)\,\mathrm{d}\bar{A}_i(s), \quad i = 1, 2,$$

and, for a measure $\mathrm{d}C$, we use the notation

$$\mathcal{K}_C^i(s) := \int_0^1 k_i(t, s)\,\mathrm{d}C(t).$$

We assume from now on that

- $\tilde{\alpha}_1[\gamma_1] < 1$, and $\tilde{\alpha}_2[\gamma_2] < 1$.

3.2 Index on the set K_ρ

We prove a result concerning the fixed point index on the set K_ρ.

Lemma 4. *Assume that*

(I^1_ρ) *there exists $\rho > 0$ such that for every $i = 1, 2$*

$$\left(\frac{\|\gamma_i\|_\infty \tilde{\alpha}_i[\delta_i]}{1-\tilde{\alpha}_i[\gamma_i]} + \|\delta_i\|_\infty\right)\left(l_{i2}\beta_i[1] + p_{i22}\right) + f_i^{0,\rho}\left(\frac{1}{m_i} + \frac{\|\gamma_i\|_\infty}{1-\tilde{\alpha}_i[\gamma_i]}\int_0^1 \mathcal{K}^i_{\tilde{A}_i}(s) g_i(s)\,\mathrm{d}s\right) < 1, \tag{7}$$

where

$$f_i^{0,\rho} = \sup\left\{\frac{f_i(t,u,v)}{\rho}\colon (t,u,v) \in [0,1]\times[0,\rho]\times[0,\rho]\right\}$$

and

$$\frac{1}{m_i} = \sup_{t\in[0,1]} \int_0^1 k_i(t,s) g_i(s)\,\mathrm{d}s.$$

Then $i_K(T, K_\rho) = 1$.

Proof. We show that $T(u,v) \neq \mu(u,v)$ for all $\mu \geqslant 1$ when $(u,v) \in \partial K_\rho$; this ensures, that the index is 1 on K_ρ. In fact, if this is not so, then there exist $(u,v) \in K$ with $\|(u,v)\| = \rho$ and $\mu \geqslant 1$ such that $\mu(u,v)(t) = T(u,v)(t)$. Assume, without loss of generality, that $\|u\|_\infty = \rho$ and $\|v\|_\infty \leqslant \rho$. We have for $t \in [0,1]$

$$\begin{aligned}\mu u(t) &= \gamma_1(t)\bigl(H_1\bigl(\alpha_1[u]\bigr) + \chi_{(\tau_1,1]}(d_{11}I_1 + e_{11}N_1)\bigl(u(\tau_1)\bigr)\bigr) \\ &\quad + \delta_1(t)\bigl(L_1\bigl(\beta_1[v]\bigr) + \chi_{[0,\tau_1]}(d_{12}I_1 + e_{12}N_1)\bigl(u(\tau_1)\bigr)\bigr) + F_1(u,v)(t).\end{aligned}$$

Since

$$\tilde{\alpha}_1[u] \geqslant H_1\bigl(\alpha_1[u]\bigr) + (d_{11}I_1 + e_{11}N_1)\bigl(u(\tau_1)\bigr),$$

we obtain

$$\mu u(t) \leqslant \gamma_1(t)\tilde{\alpha}_1[u] + \delta_1(t)\bigl(l_{12}\beta_1[v] + \bigl(d_{12}I_1 + e_{12}N_1\bigr)\bigl(u(\tau_1)\bigr)\bigr) + F_1(u,v)(t),$$

and moreover, since $v(t) \leqslant \rho$ and $u(t) \leqslant \rho$ for all $t \in [0,1]$, we obtain

$$\begin{aligned}\mu u(t) &\leqslant \gamma_1(t)\tilde{\alpha}_1[u] + \delta_1(t)\bigl(l_{12}\beta_1[\rho] + p_{122}u(\tau_1)\bigr) + F_1(u,v)(t) \\ &\leqslant \gamma_1(t)\tilde{\alpha}_1[u] + \delta_1(t)\rho\bigl(l_{12}\beta_1[1] + p_{122}\bigr) + F_1(u,v)(t).\end{aligned} \tag{8}$$

Applying $\tilde{\alpha}_1$ to both sides of (8) gives

$$\mu\tilde{\alpha}_1[u] \leqslant \tilde{\alpha}_1[\gamma_1]\tilde{\alpha}_1[u] + \tilde{\alpha}_1[\delta_1]\rho\bigl(l_{12}\beta_1[1] + p_{122}\bigr) + \tilde{\alpha}_1\bigl[F_1(u,v)\bigr].$$

Thus we have

$$\big(\mu - \tilde{\alpha}_1[\gamma_1]\big)\tilde{\alpha}_1[u] \leqslant \tilde{\alpha}_1[\delta_1]\rho\big(l_{12}\beta_1[1] + p_{122}\big) + \tilde{\alpha}_1\big[F_1(u,v)\big],$$

that is

$$\tilde{\alpha}_1[u] \leqslant \rho\frac{\tilde{\alpha}_1[\delta_1](l_{12}\beta_1[1] + p_{122})}{\mu - \tilde{\alpha}_1[\gamma_1]} + \frac{\tilde{\alpha}_1[F_1(u,v)]}{\mu - \tilde{\alpha}_1[\gamma_1]}.$$

Substituting into (8) gives

$$\begin{aligned}\mu u(t) &\leqslant \gamma_1(t)\left(\rho\frac{\tilde{\alpha}_1[\delta_1](l_{12}\beta_1[1] + p_{122})}{\mu - \tilde{\alpha}_1[\gamma_1]} + \frac{\tilde{\alpha}_1[F_1(u,v)]}{\mu - \tilde{\alpha}_1[\gamma_1]}\right)\\ &\quad + \delta_1(t)\rho\big(l_{12}\beta_1[1] + p_{122}\big) + F_1(u,v)(t)\\ &= \rho\frac{\gamma_1(t)\tilde{\alpha}_1[\delta_1](l_{12}\beta_1[1] + p_{122})}{\mu - \tilde{\alpha}_1[\gamma_1]} + \rho\delta_1(t)\big(l_{12}\beta_1[1] + p_{122}\big)\\ &\quad + \frac{\gamma_1(t)}{\mu - \tilde{\alpha}_1[\gamma_1]}\int_0^1 \mathcal{K}^1_{\tilde{A}_1}(s)g_1(s)f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s + F_1(u,v)(t).\end{aligned}$$

Since $\mu \geqslant 1$, we have $1/(\mu - \tilde{\alpha}_1[\gamma_1]) \leqslant 1/(1 - \tilde{\alpha}_1[\gamma_1])$ and therefore

$$\begin{aligned}\mu u(t) &\leqslant \rho\frac{\gamma_1(t)\tilde{\alpha}_1[\delta_1](l_{12}\beta_1[1] + p_{122})}{1 - \tilde{\alpha}_1[\gamma_1]} + \rho\delta_1(t)\big(l_{12}\beta_1[1] + p_{122}\big)\\ &\quad + \frac{\gamma_1(t)}{1 - \tilde{\alpha}_1[\gamma_1]}\int_0^1 \mathcal{K}^1_{\tilde{A}_1}(s)g_1(s)f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s + F_1(u,v)(t).\end{aligned}$$

Taking the supremum of t on $[0,1]$ gives

$$\begin{aligned}\mu\rho &\leqslant \rho\frac{\|\gamma_1\|_\infty\tilde{\alpha}_1[\delta_1](l_{12}\beta_1[1] + p_{122})}{1 - \tilde{\alpha}_1[\gamma_1]} + \rho\|\delta_1\|_\infty\big(l_{12}\beta_1[1] + p_{122}\big)\\ &\quad + \rho\frac{\|\gamma_1\|_\infty}{1 - \tilde{\alpha}_1[\gamma_1]}f_i^{0,\rho}\int_0^1 \mathcal{K}^1_{\tilde{A}_1}(s)g_1(s)\,\mathrm{d}s + \rho f_i^{0,\rho}\frac{1}{m_1}.\end{aligned}$$

Using the hypothesis (7) we obtain $\mu\rho < \rho$. This contradicts the fact that $\mu \geqslant 1$ and proves the result. □

3.3 Index on the set V_ρ

We give two lemma about the index on a set V_ρ. In Lemma 5, we assume that the nonlinearities f_1, f_2 have the same growth. The idea in Lemma 6 is similar to the one in Lemma 4 of [51]: we control the growth of one nonlinearity f_i, at the cost of having to deal with a larger domain. For other results on the existence of solutions with different growth on the nonlinearities see [53, 60].

Lemma 5. *Assume that*:

(I^0_ρ) *there exist* $\rho > 0$ *such that for every* $i = 1, 2$

$$f_{i,(\rho,\rho/c)}\left(\frac{c_{\gamma_i}\|\gamma_i\|_\infty}{1-\bar{\alpha}_i[\gamma_i]}\int_{a_i}^{b_i}\mathcal{K}^i_{\bar{A}_i}(s)g_i(s)\,\mathrm{d}s + \frac{1}{M_i}\right) > 1, \tag{9}$$

where

$$f_{1,(\rho,\rho/c)} = \inf\left\{\frac{f_1(t,u,v)}{\rho}\colon\ (t,u,v) \in [a_1,b_1]\times\left[\rho,\frac{\rho}{c}\right]\times\left[0,\frac{\rho}{c}\right]\right\},$$
$$f_{2,(\rho,\rho/c)} = \inf\left\{\frac{f_2(t,u,v)}{\rho}\colon\ (t,u,v) \in [a_2,b_2]\times\left[0,\frac{\rho}{c}\right]\times\left[\rho,\frac{\rho}{c}\right]\right\}$$

and

$$\frac{1}{M_i} = \inf_{t\in[a_i,b_i]}\int_{a_i}^{b_i}k_i(t,s)g_i(s)\,\mathrm{d}s.$$

Then $i_K(T, V_\rho) = 0$.

Proof. Let $e(t) \equiv 1$ for $t \in [0,1]$. Then $(e,e) \in K$. We prove that

$$(u,v) \neq T(u,v) + \mu(e,e) \quad \text{for } (u,v) \in \partial V_\rho \text{ and } \mu \geqslant 0.$$

In fact, if this does not happen, there exist $(u,v) \in \partial V_\rho$ and $\mu \geqslant 0$ such that $(u,v) = T(u,v) + \mu(e,e)$. Without loss of generality, we can assume that for all $t \in [a_1,b_1]$ we have

$$\rho \leqslant u(t) \leqslant \frac{\rho}{c}, \quad \min u(t) = \rho \quad \text{and} \quad 0 \leqslant v(t) \leqslant \frac{\rho}{c}.$$

For $t \in [a_1,b_1]$, we have

$$\begin{aligned} u(t) &= \gamma_1(t)\big(H_1\big(\alpha_1[u]\big) + (d_{11}I_1 + e_{11}N_1)\big(u(\tau_1)\big)\big) + \delta_1(t)L_1\big(\beta_1[v]\big) \\ &\quad + F_1(u,v)(t) + \mu e(t) \\ &\geqslant \gamma_1(t)\big(H_1\big(\alpha_1[u]\big) + (d_{11}I_1 + e_{11}N_1)\big(u(\tau_1)\big)\big) + F_1(u,v)(t) + \mu e(t). \end{aligned}$$

Since

$$\bar{\alpha}_1[u] \leqslant H_1\big(\alpha_1[u]\big) + (d_{11}I_1 + e_{11}N_1)\big(u(\tau_1)\big),$$

we have

$$u(t) \geqslant \gamma_1(t)\bar{\alpha}_1[u] + F_1(u,v)(t) + \mu e(t). \tag{10}$$

Applying $\bar{\alpha}_1$ to both sides of (10) gives

$$\bar{\alpha}_1[u] \geqslant \bar{\alpha}_1[\gamma_1]\bar{\alpha}_1[u] + \bar{\alpha}_1[F_1(u,v)] + \mu\bar{\alpha}_1[e].$$

This can be written in the form

$$\big(1-\bar{\alpha}_1[\gamma_1]\big)\bar{\alpha}_1[u] \geqslant \bar{\alpha}_1\big[F_1(u,v)\big] + \mu\bar{\alpha}_1[e],$$

that is

$$\bar{\alpha}_1[u] \geqslant \frac{\bar{\alpha}_1[F_1(u,v)]}{1-\bar{\alpha}_1[\gamma_1]} + \frac{\mu\bar{\alpha}_1[e]}{1-\bar{\alpha}_1[\gamma_1]}.$$

Thus, (10) becomes

$$\begin{aligned}
u(t) &\geqslant \frac{\gamma_1(t)\bar{\alpha}_1[F_1(u,v)]}{1-\bar{\alpha}_1[\gamma_1]} + \frac{\mu\gamma_1(t)\bar{\alpha}_1[e]}{1-\bar{\alpha}_1[\gamma_1]} + F_1(u,v)(t) + \mu e(t) \\
&= \frac{\gamma_1(t)}{1-\bar{\alpha}_1[\gamma_1]} \int_0^1 \mathcal{K}^1_{\bar{A}_i}(s) g_1(s) f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s + \frac{\mu\gamma_1(t)\bar{\alpha}_1[e]}{1-\bar{\alpha}_1[\gamma_1]} \\
&\quad + \int_0^1 k_1(t,s) g_1(s) f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s + \mu.
\end{aligned}$$

Then we have, for $t \in [a_1,b_1]$,

$$\begin{aligned}
u(t) &\geqslant \frac{c_{\gamma_1}\|\gamma_1\|_\infty}{1-\bar{\alpha}_1[\gamma_1]} \int_{a_1}^{b_1} \mathcal{K}^1_{\bar{A}_1}(s) g_1(s) f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s + \frac{\mu c_{\gamma_i}\|\gamma_1\|_\infty\bar{\alpha}_1[e]}{1-\bar{\alpha}_1[\gamma_1]} \\
&\quad + \int_{a_1}^{b_1} k_1(t,s) g_1(s) f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s + \mu \\
&\geqslant \frac{c_{\gamma_1}\|\gamma_1\|_\infty}{1-\bar{\alpha}_1[\gamma_1]} \int_{a_1}^{b_1} \mathcal{K}^1_{\bar{A}_1}(s) g_1(s) f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s \\
&\quad + \int_{a_1}^{b_1} k_1(t,s) g_1(s) f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s + \mu.
\end{aligned}$$

Taking the minimum over $[a_1,b_1]$ gives

$$\begin{aligned}
\rho &= \min_{t\in[a_1,b_1]} u(t) \geqslant \rho f_{1,(\rho,\rho/c)} \frac{c_{\gamma_1}\|\gamma_1\|_\infty}{1-\bar{\alpha}_1[\gamma_1]} \int_{a_1}^{b_1} \mathcal{K}^1_{\bar{A}_1}(s) g_1(s)\,\mathrm{d}s + \rho f_{1,(\rho,\rho/c)} \frac{1}{M_1} + \mu \\
&= \rho f_{1,(\rho,\rho/c)} \left(\frac{c_{\gamma_1}\|\gamma_1\|_\infty}{1-\bar{\alpha}_1[\gamma_1]} \int_{a_1}^{b_1} \mathcal{K}^1_{\bar{A}_1}(s) g_1(s)\,\mathrm{d}s + \frac{1}{M_1} \right) + \mu.
\end{aligned}$$

Using the hypothesis (9) we obtain $\rho > \rho + \mu$, a contradiction. □

Lemma 6. *Assume that*

$(\mathrm{I}_\rho^0)^\star$ *there exist* $\rho > 0$ *such that for some* $i = 1, 2$

$$f^*_{i,(0,\rho/c)}\left(\frac{c_{\gamma_i}\|\gamma_i\|_\infty}{1-\bar{\alpha}_i[\gamma_i]}\int_{a_i}^{b_i}\mathcal{K}^i_{\bar{A}_i}(s)g_i(s)\,\mathrm{d}s + \frac{1}{M_i}\right) > 1, \tag{11}$$

where

$$f^*_{i,(0,\rho/c)} = \inf\left\{\frac{f_i(t,u,v)}{\rho}\colon (t,u,v) \in [a_i,b_i] \times \left[0,\frac{\rho}{c}\right] \times \left[0,\frac{\rho}{c}\right]\right\}.$$

Then $i_K(T, V_\rho) = 0$.

Proof. Suppose that the condition (11) holds for $i = 1$. Let $e(t) \equiv 1$ for $t \in [0,1]$. Then $(e,e) \in K$. We prove that

$$(u,v) \neq T(u,v) + \mu(e,e) \quad \text{for} \quad (u,v) \in \partial V_\rho \quad \text{and} \quad \mu \geqslant 0.$$

In fact, if this does not happen, there exist $(u,v) \in \partial V_\rho$ and $\mu \geqslant 0$ such that $(u,v) = T(u,v) + \mu(e,e)$. So, for all $t \in [a_1,b_1]$, $\min u(t) \leqslant \rho$ and for $t \in [a_2,b_2]$, $\min v(t) \leqslant \rho$. We obtain, for $t \in [a_1,b_1]$, with the same proof of Lemma 5,

$$\begin{aligned} u(t) \geqslant{}& \frac{\gamma_1(t)}{1-\bar{\alpha}_1[\gamma_1]}\int_{a_1}^{b_1}\mathcal{K}^1_{\bar{A}_1}(s)g_1(s)f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s \\ &+ \int_{a_1}^{b_1} k_1(t,s)g_1(s)f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s + \mu. \end{aligned}$$

Then we have

$$\min_{t\in[a_1,b_1]} u(t) \geqslant \rho f^*_{1,(0,\rho/c)}\frac{c_{\gamma_1}\|\gamma_1\|_\infty}{1-\bar{\alpha}_1[\gamma_1]}\int_{a_1}^{b_1}\mathcal{K}^1_{\bar{A}_1}(s)g_1(s)\,\mathrm{d}s + \rho f^*_{1,(0,\rho/c)}\frac{1}{M_1} + \mu.$$

Using the hypothesis (11) we obtain $\min_{t\in[a_1,b_1]} u(t) > \rho + \mu \geqslant \rho$, a contradiction. □

4 Existence and multiplicity of the solutions

By combining the above results on the index of the sets V_ρ and K_ρ we obtain the following theorem, in which we deal with the existence of at least one, two or three solutions. It is possible to state results for four or more positive solutions by expanding the lists in conditions (S_5), (S_6), see, for example, paper [61] for this type of results.

We omit the proof of Theorem 2 which follows from the properties of fixed point index.

Theorem 2. *System* (4) *has at least one positive solution in K if either of the following conditions hold*:

(S1) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1/c < \rho_2$ such that $(\mathrm{I}^0_{\rho_1})$ [or $(\mathrm{I}^0_{\rho_1})^\star$], $(\mathrm{I}^1_{\rho_2})$ hold.*

(S2) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < \rho_2$ such that $(\mathrm{I}^1_{\rho_1})$, $(\mathrm{I}^0_{\rho_2})$ hold.*

System (4) *has at least two positive solutions in K if one of the following conditions hold*:

(S3) *There exist $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ with $\rho_1/c < \rho_2 < \rho_3$ such that $(\mathrm{I}^0_{\rho_1})$ [or $(\mathrm{I}^0_{\rho_1})^\star$], $(\mathrm{I}^1_{\rho_2})$ and $(\mathrm{I}^0_{\rho_3})$ hold.*

(S4) *There exist $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ with $\rho_1 < \rho_2$ and $\rho_2/c < \rho_3$ such that $(\mathrm{I}^1_{\rho_1})$, $(\mathrm{I}^0_{\rho_2})$ and $(\mathrm{I}^1_{\rho_3})$ hold.*

System (4) *has at least three positive solutions in K if one of the following conditions hold*:

(S5) *There exist $\rho_1, \rho_2, \rho_3, \rho_4 \in (0,\infty)$ with $\rho_1/c < \rho_2 < \rho_3$ and $\rho_3/c < \rho_4$ such that $(\mathrm{I}^0_{\rho_1})$ [or $(\mathrm{I}^0_{\rho_1})^\star$], $(\mathrm{I}^1_{\rho_2})$, $(\mathrm{I}^0_{\rho_3})$ and $(\mathrm{I}^1_{\rho_4})$ hold.*

(S6) *There exist $\rho_1, \rho_2, \rho_3, \rho_4 \in (0,\infty)$ with $\rho_1 < \rho_2$ and $\rho_2/c < \rho_3 < \rho_4$ such that $(\mathrm{I}^1_{\rho_1})$, $(\mathrm{I}^0_{\rho_2})$, $(\mathrm{I}^1_{\rho_3})$ and $(\mathrm{I}^0_{\rho_4})$ hold.*

We illustrate the conditions that occur in the above Theorem in the following example, where multi-point type BCs are considered.

Example 1. Consider the system

$$
\begin{aligned}
&u'' + \frac{1}{8}\big(u^3 + t^3v^3\big) + 2 = 0, \quad v'' + \frac{1}{8}\big(\sqrt{tu} + 13v^2\big) = 0, \quad t \in (0,1),\\
&\Delta u|_{t=1/5} = I_1\left(u\left(\frac{1}{5}\right)\right), \qquad \Delta u'|_{t=1/5} = N_1\left(u\left(\frac{1}{5}\right)\right),\\
&\Delta v|_{t=2/5} = I_2\left(v\left(\frac{2}{5}\right)\right), \qquad \Delta v'|_{t=2/5} = N_2\left(v\left(\frac{2}{5}\right)\right),\\
&u(0) = H_1\left(u\left(\frac{1}{4}\right)\right), \qquad u(1) = L_1\left(v\left(\frac{3}{4}\right)\right),\\
&v(0) = H_2\left(v\left(\frac{1}{3}\right)\right), \qquad v'(1) = L_2\left(u\left(\frac{2}{3}\right)\right).
\end{aligned}
\tag{12}
$$

This differential system can be rewritten in the integral form

$$
\begin{aligned}
u(t) = {}&(1-t)H_1\left(u\left(\frac{1}{4}\right)\right) + tL_1\left(v\left(\frac{3}{4}\right)\right) + G_1(u)(t)\\
&+ \int_0^1 k_1(t,s)g_1(s)f_1\big(s,u(s),v(s)\big)\,\mathrm{d}s,
\end{aligned}
$$

$$v(t) = H_2\left(v\left(\frac{1}{3}\right)\right) + tL_2\left(u\left(\frac{2}{3}\right)\right) + G_2(v)(t) + \int_0^1 k_2(t,s)g_2(s)f_2\big(s,u(s),v(s)\big)\,\mathrm{d}s,$$

where the Green's functions

$$k_1(t,s) = \begin{cases} s(1-t), & s \leqslant t, \\ t(1-s), & s > t, \end{cases} \quad \text{and} \quad k_2(t,s) = \begin{cases} s, & s \leqslant t, \\ t, & s > t, \end{cases}$$

are non-negative continuous functions on $[0,1]\times[0,1]$. Here $\gamma_1(t) = 1-t$, $\gamma_2(t) = 1$, $\delta_1(t) = t$, $\delta_2(t) = t$, $c_{\gamma_1} = 1-b_1$, $c_{\gamma_2} = 1$, $c_{\delta_1} = a_1$ and $c_{\delta_2} = a_2$. The intervals $[a_1,b_1]$ may be chosen arbitrarily in $(1/5,1)$ and $[a_2,b_2]$ can be chosen arbitrarily in $(2/5,1]$. It is easy to check that

$$k_1(t,s) \leqslant s(1-s) := \Phi_1(s), \qquad \min_{t\in[a_1,b_1]} k_1(t,s) \geqslant c_{\Phi_1} s(1-s),$$

where $c_{\Phi_1} = \min\{1-b_1, a_1\}$. Furthermore we have that

$$k_2(t,s) \leqslant s := \Phi_2(s), \qquad \min_{t\in[a_2,b_2]} k_2(t,s) \geqslant c_{\Phi_2}\Phi_2(s),$$

where $c_{\Phi_2} = a_2$. The choice $[a_1,b_1] = [1/4,3/4]$ and $[a_2,b_2] = [1/2,1]$ gives

$$c = \frac{1}{4}, \quad m_1 = 8, \quad M_1 = 16, \quad m_2 = 2, \quad M_2 = 4.$$

In our example, the nonlinearities used to illustrate the constants that occur in our theory are taken in a similar way as in [6,7,51,57]. We consider

$$H_1(w) = \begin{cases} \frac{5}{6}w, & 0 \leqslant w \leqslant 1, \\ \frac{1}{3}w + \frac{1}{2}, & w \geqslant 1, \end{cases} \qquad L_1(w) = \frac{1}{30}\big(1+\sin(w)\big),$$

$$H_2(w) = \begin{cases} \frac{1}{19}w, & 0 \leqslant w \leqslant 2, \\ \frac{1}{25}w + \frac{12}{475}, & w \geqslant 2, \end{cases} \qquad L_2(w) = \frac{1}{38}\big(1+\cos(w)\big).$$

The functions H_i and L_i satisfy the conditions

$$h_{i1}w \leqslant H_i(w) \leqslant h_{i2}w, \qquad L_i(w) \leqslant l_{i2}w$$

with

$$h_{11} = \frac{1}{3}, \quad h_{12} = \frac{5}{6}, \quad h_{21} = \frac{1}{25}, \quad h_{22} = \frac{1}{19}, \quad l_{12} = \frac{1}{15}, \quad l_{22} = \frac{1}{75}.$$

The functions

$$I_1(w)=\begin{cases}\frac{1}{100}w, & 0\leqslant w\leqslant 1,\\ \frac{13}{1400}w+\frac{1}{1400}, & w\geqslant 1,\end{cases}\qquad N_1(w)=\begin{cases}-\frac{3}{100}w, & 0\leqslant w\leqslant 1,\\ -\frac{39}{1400}w-\frac{3}{1400}, & w\geqslant 1,\end{cases}$$

$$I_2(w)=\begin{cases}\frac{1}{300}w, & 0\leqslant w\leqslant 1,\\ \frac{1}{400}w+\frac{1}{1200}, & w\geqslant 1,\end{cases}\qquad N_2(w)=\begin{cases}-\frac{1}{30}w, & 0\leqslant w\leqslant 1,\\ -\frac{1}{40}w-\frac{1}{120}, & w\geqslant 1,\end{cases}$$

satisfy the conditions for $w\in[0,\infty)$

$$p_{i11}w\leqslant (d_{i1}I_i+e_{i1}N_i)(w)\leqslant p_{i12}w,\qquad 0\leqslant (d_{i2}I_i+e_{i2}N_i)(w)\leqslant p_{i22}w$$

with

$$d_{11}=1,\quad d_{21}=1,\quad e_{11}=-\frac{1}{5},\quad e_{21}=-\frac{2}{5},$$
$$d_{12}=-1,\quad d_{22}=0,\quad e_{12}=-\frac{4}{5},\quad e_{22}=-1,$$

$$p_{111}=\frac{1}{70},\quad p_{112}=\frac{1}{50},\quad p_{122}=\frac{1}{40},\quad p_{211}=\frac{1}{80},\quad p_{212}=\frac{1}{60},\quad p_{222}=\frac{1}{30}.$$

We have that

$$\tilde{\alpha}_1[\gamma_1]=\frac{641}{1000},\qquad \tilde{\alpha}_2[\gamma_2]=\frac{79}{1140},\qquad \tilde{\alpha}_1[\delta_1]=\frac{634}{3000},\qquad \tilde{\alpha}_2[\delta_2]=\frac{23}{950},$$
$$\bar{\alpha}_1[\gamma_1]=\frac{183}{700},\qquad \bar{\alpha}_2[\gamma_2]=\frac{21}{400},\qquad \beta_1[1]=\beta_2[1]=1,$$

$$\int_0^1 \mathcal{K}^1_{\tilde{A}_1}(s)\,\mathrm{d}s=\frac{3189}{40000},\qquad \int_0^1 \mathcal{K}^2_{\tilde{A}_2}(s)\,\mathrm{d}s=\frac{853}{42750},$$
$$\int_{1/4}^{3/4} \mathcal{K}^1_{\bar{A}_1}(s)\,\mathrm{d}s=\frac{181}{8400},\qquad \int_{1/2}^{1} \mathcal{K}^2_{\bar{A}_2}(s)\,\mathrm{d}s=\frac{11}{1200}.$$

The existence of multiple solutions of system (12) follows from Theorem 2. Then, for $\rho_1=1/8$, $\rho_2=1$ and $\rho_3=11$, we have (the constants that follow have been rounded to 2 decimal places unless exact)

$$\inf\left\{f_1(t,u,v)\colon (t,u,v)\in\left[\frac14,\frac34\right]\times\left[0,\frac12\right]\times\left[0,\frac12\right]\right\}=f_1\left(\frac14,0,0\right)>14.33\rho_1,$$
$$\sup\{f_1(t,u,v)\colon (t,u,v)\in[0,1]\times[0,1]\times[0,1]\}=f_1(1,1,1)<2.46\rho_2,$$
$$\sup\{f_2(t,u,v)\colon (t,u,v)\in[0,1]\times[0,1]\times[0,1]\}=f_2(1,1,1)<1.82\rho_2,$$
$$\inf\left\{f_1(t,u,v)\colon (t,u,v)\in\left[\frac14,\frac34\right]\times[11,44]\times[0,44]\right\}=f_1\left(\frac14,11,0\right)>14.33\rho_3,$$
$$\inf\left\{f_2(t,u,v)\colon (t,u,v)\in\left[\frac12,1\right]\times[0,44]\times[11,44]\right\}=f_2\left(\frac12,0,11\right)>3.86\rho_3,$$

that is the conditions $(\mathrm{I}^0_{\rho_1})^\star$, $(\mathrm{I}^1_{\rho_2})$ and $(\mathrm{I}^0_{\rho_3})$ are satisfied; therefore system (12) has at least two positive solutions in K.

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