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Existence of the solution to a nonlocal-in-time evolutional problem

Volodymyr Makarov, Dmytro Sytnyk, Vitalii Vasylyk

Institute of Mathematics, National Academy of Sciences of Ukraine
Tereschenkivs'ka str. 3, Kiev-4, 01601 Ukraine
makarov@imath.kiev.ua; sytnik@imath.kiev.ua

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Abstract. This work is devoted to the study of a nonlocal-in-time evolutional problem for the first order differential equation in Banach space. Our primary approach, although stems from the convenient technique based on the reduction of a nonlocal problem to its classical initial value analogue, uses more advanced analysis. That is a validation of the correctness in definition of the general solution representation via the Dunford–Cauchy formula. Such approach allows us to reduce the given existence problem to the problem of locating zeros of a certain entire function. It results in the necessary and sufficient conditions for the existence of a generalized (mild) solution to the given nonlocal problem. Aside of that we also present new sufficient conditions which in the majority of cases generalize existing results.

Keywords: nonlocal-in-time evolutional problem, unbounded operator coefficient, mild solution, zeros of polynomial.

1 Introduction

The Cauchy problem for differential equation with a nonlocal in time condition

$$\begin{aligned} &u'_t + Au = f(t), \quad t \in [0,T], \\ &u(0) + g(t_1, t_2, \dots, t_k, u) = u_0, \quad 0 \leqslant t_0 < t_1 < \dots < t_k \leqslant T, \end{aligned} \tag{1}$$

with A being a densely defined strongly positive operator (the detailed definition will be given below) and $g : X \to D(A) \subseteq X$, is one of the important topics in the theory of differential equation theory and its application. Interest in such problems stems mainly from the better effect of the nonlocal initial condition than the usual one in treating physical problems. Actually, the nonlocal initial condition from (1) models many interesting nature phenomena [1, 2], where the normal initial condition $u(0) = u_0$ may not fit in. In addition, some models of the control theory [3] and economical management problems may be represented in form (1) as well.

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Even in a much simpler form

$$u'_t + Au = f(t), \quad t \in [0, T], \tag{2a}$$

$$u(0) + \sum_{k=1}^{n} \alpha_k u(t_k) = u_0, \quad 0 < t_1 < t_2 < \cdots < t_n \leqslant T, \tag{2b}$$

the model reflects an important case when one is more likely to have more information about the solution at the times $t = t_1, t_2, \ldots, t_n$ rather than as in the classical case just at $t = 0$. Such situation is common for physical systems where an observer was not able to witness (or measure) characteristics of the system at the initial time. It is also intrinsic for modelling certain physical measurements performed repeatedly by the devices having relaxation time comparable to the delay between the measurements. A simple example of this kind is a recovery of movement captured by a multi-exposure camera (streak-camera).

The particular cases of (2) covers many well-known physical phenomena such as: problems with periodic conditions $u(0) - u(t_1) = 0$, problems with Bitsadze–Samarskii conditions $u(0) + \alpha_1 u(t_1) = \alpha_2 u(t_2)$, regularized backward problems etc. One can not underestimate the role of nonlocal problems having the form (2) in the theory of ill-posed problems, where they appear as natural counterparts of the improperly posed problem in the course of quasi-regularization technique [4].

Historically the first, to the authors best knowledge, work devoted to the problems with nonlocal-in-time conditions was the work of Dezin [5]. In this work, author studied the restrictions of abstract differential operators $u'(t) + A$ in a Hilbert space imposed by the non-local condition (2) (see also [6] and the references therein). Similar theoretical technique in conjunction with a numerical scheme was used to solve nonlocal-in-space boundary value problem for elliptic partial-differential operators in the work of Bitsadze and Samarskii [7]. Next addition to the theory was made by Gordeziani with co-authors [8, 9]. They rather generalized the previous results and proposed iterative problem-solving methods. Eigenvalue problems for elliptic operators with nonlocal conditions were considered in [10], see also [11] for review of recent works in that direction. In [12–14], authors stated the sufficient conditions for the existence and uniqueness of a solution to (1) in a Banach space. Initial analysis of (2) used in the present work was performed in [15]. Starting from the conditions similar to those proposed in [13], authors of [15] develop efficient parallel numerical methods for (2). They use a rather general methodology developed for classical Cauchy problem for equation (2a). Majority of later theoretical works are mainly devoted to the generalization of results, received in [12], to more wider classes of equations.

Aside of that, some efforts have been made in the direction of sharpening the sufficient existence and uniqueness conditions for various specific classes of a nonlocal condition from (1) and an operator coefficient A [16].

Most of previous research concerning the nonlocal problem (2) were done under a strict constraint on quantities α_i, from the nonlocal condition [12–14]

$$\sum_{i=1}^{n} |\alpha_i| \leqslant 1,$$

allowing the transition from (2) to the classical Cauchy problem. This inequality obviously gives only a sufficient condition. Indeed, if one had a solution of ordinary Cauchy problem for equation from (1) he could simply choose such a nonlocal condition that mentioned constrain fails but the solution exist and unique.

In [17], authors, using spectral characteristics of A, obtained new slightly weaker constraint

$$\sum_{i=1}^{n} |\alpha_i| \mathrm{e}^{-\rho t_i} \leqslant 1, \tag{3}$$

here ρ has the same meaning as in (4).

Here we will introduce more advanced and yet quite straightforward technique for the treatment of nonlocal problems. To do that, in Section 2 we reduce the given problem (2) to the corresponding Cauchy problem with ordinary initial value condition as in [12, 17]. The subsequent analysis, presented in Section 3, study the correctness of the solution operator defined via the Dunford–Cauchy formula on a feasible integration contour. It is equivalent to the calculation of zeros set for a certain entire function or the polynomial corresponding to it. The resulting necessary and sufficient conditions for the existence and uniqueness of the solution take in to account both the nonlocal condition and the spectral information of A. Section 4 is devoted to the situation when zeros of the mentioned entire function can not be calculated directly. That may be caused by a dependence of the nonlocal condition on some additional parameter like in applications of quasi-regularization technique, or because of a large number of time moments t_k in (2), etc. Such situation as it will be shown, can be circumvented using some available zeros estimates [18, 19] for the mentioned entire function in its reduced to the polynomial form. It results in the sufficient conditions for (2) which however still outperform (3) in many important applications.

2 Reduction of nonlocal problem to classical Cauchy problem

From now on we assume that the coefficient A from (1) is a densely defined strongly positive operator acting in Banach space $X \supseteq D(A) \to X$ [20, 21]. Its spectrum $\Sigma(A)$ lies in a sector

$$\Sigma = \left\{ z = \rho + r\mathrm{e}^{\mathrm{i}\varphi}\colon\ r \in [0, \infty),\ |\varphi| \leqslant \theta < \frac{\pi}{2},\ \rho > 0 \right\}, \tag{4}$$

while the resolvent $R(z, A)$ of A satisfies the following estimate on the boundary of spectrum Γ_Σ and outside of it:

$$\left\| (zI - A)^{-1} \right\| \leqslant \frac{M}{1 + |z|} \tag{5}$$

with $\|\cdot\|$ being an operator norm.

The class of strongly positive operators plays an important role in applications of functional analysis to the theory of partial differential equations, dynamical systems, numerical analysis, etc. Strongly-elliptic partial differential operator defined on a bounded

Lipschitz domain is a strongly positive operator with spectral parameters that can be estimated from the coefficients of elliptic operator [22], similar is true for a general elliptic pseudo-differential operator.

The theory of sectorial operators in its present form was developed in the works of Hille, Dunford, Philipps [23]. According to this theory, every closed strongly-positive operator generates a one parameter semi-group $T(t) = \mathrm{e}^{-tA}$ which acts as a propagator for a solution to (2a). That is any solution $u(t) \in D(A)$ of the differential equation (2a) has the following representation:

$$u(t) = \mathrm{e}^{-At}u(0) + \int_0^t \mathrm{e}^{-A(t-\tau)} f(\tau)\,\mathrm{d}\tau. \tag{6}$$

Recall that the opposite is not true in general [23], since even if $u(t)$ satisfies (6) it does not need to belong to the domain of A. It is also well know that $D(A)$ is dense in X so there always exists a sequence of elements from $D(A)$ converging to $u(t) \in X$ defined by (6). A function $u(t)$ satisfying (6) is called a *generalized* [23, p. 30] or sometimes *mild* [24, p. 117] *solution* to problem (2). Formula (6) becomes more convenient than the original equation (2a) in cases where the classical Cauchy problem for (2a) with a given $u(0)$ is considered. In our case of problem (2), that distinction between (6) and (2a) is not so obvious because $u(0)$ is not given directly.

Here we intend to derive a direct representation of the solution to (2). Let us start from representation (6) and assume the existence of an initial value $u(0)$ such that $u(t)$ defined by (6) satisfies nonlocal condition (2b) (the precise conditions for the existence of such $u(0)$ will be stated below).

By substituting the formula for $u(0)$ from (2b) into representation (6) and evaluating the result at t_i, $i = \overline{1,n}$, we obtain the system of equations

$$u(t_i) = \mathrm{e}^{-At_i}\left[u_0 - \sum_{k=1}^{n} \alpha_k u(t_k)\right] + \int_0^{t_i} \mathrm{e}^{-A(t_i-\tau)} f(\tau)\,\mathrm{d}\tau, \quad i = \overline{1,n}. \tag{7}$$

Next, we multiply each part of ith equation by α_i and then sum up the resulting equalities. It gives us the following:

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i u(t_i) &= \sum_{i=1}^{n} \alpha_i \mathrm{e}^{-At_i} u_0 - \sum_{i=1}^{n} \alpha_i \mathrm{e}^{-At_i} \sum_{k=1}^{n} \alpha_k u(t_k) \\ &\quad + \sum_{i=1}^{n} \alpha_i \int_0^{t_i} \mathrm{e}^{-A(t_i-\tau)} f(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{8}$$

Now let us put $\sum_{i=1}^{n} \alpha_i u(t_i) = w$, then (8) can be represented as follows:

$$w = -\sum_{i=1}^{n} \alpha_i \mathrm{e}^{-At_i} w + \sum_{i=1}^{n} \alpha_i \mathrm{e}^{-At_i} u_0 + \sum_{i=1}^{n} \alpha_i \int_0^{t_i} \mathrm{e}^{-A(t_i-\tau)} f(\tau)\,\mathrm{d}\tau,$$

the last equality can be regarded as an operator equation with respect to w. Using the notation $B(A) = I + \sum_{i=1}^{n} \alpha_i e^{-At_i}$, we rewrite it to get

$$Bw = Bu_0 - u_0 + \sum_{i=1}^{n} \alpha_i \int_0^{t_i} \mathrm{e}^{-A(t_i-\tau)} f(\tau)\,\mathrm{d}\tau. \tag{9}$$

Here unknown w appears only on the left-hand side of (9), hence in order to solve (9) for w, we need to assume the existence and boundedness of the operator valued function $B^{-1}(A)$ inverse to $B(A)$.

As it will turn out later (see Theorem 1), such assumptions with respect to $B^{-1}(A)$ is the only thing we need to reduce the question about the existence of solution to (2) to the question about the existence of corresponding solution to a classical Cauchy problem associated with (2a). For the time being, let us assume that $B^{-1}(A)$ is a properly defined bounded operator valued function, in that case,

$$w = u_0 - B^{-1}u_0 + B^{-1} \sum_{i=1}^{n} \alpha_i \int_0^{t_i} \mathrm{e}^{-A(t_i-\tau)} f(\tau)\,\mathrm{d}\tau.$$

Our last step is to substitute $u(0) = u_0 - W$ (6), and finally get a direct representation of solution to (2) free of the unknown values $u(t_k)$:

$$u(t) = \mathrm{e}^{-At}\left[B^{-1}u_0 - B^{-1} \sum_{i=1}^{n} \alpha_i \int_0^{t_i} \mathrm{e}^{-A(t_i-\tau)} f(\tau)\,\mathrm{d}\tau\right] + \int_0^{t} \mathrm{e}^{-A(t-\tau)} f(\tau)\,\mathrm{d}\tau. \tag{10}$$

For further analysis of relationship between the solution to (2) and the initial data α_i, t_i, we will need a following definition from the operator function calculus [20, p. 167], [25].

Definition 1. Let $f(z)$ be a complex valued function analytic in the neighbourhood of the spectrum $\Sigma(A) \subset \mathbb{C}$ and at the infinity. Suppose that there exist an open set $V \supset \Sigma(A)$ with the boundary Γ consisting of a finite number of rectifying Jourdan curves such that $f(z)$ is analytic in $V \cap \Gamma$, then $f(A)$ can be defined as follows:

$$f(A)x = f(\infty)I + \frac{1}{2\pi\mathrm{i}} \int_{\Gamma} f(z)R(z, A)x\,\mathrm{d}z, \tag{11}$$

here we assume that the integral is taken over the positively oriented contour Γ which may pass trough ∞.

The integral in (11) is called a Dunford–Cauchy integral.

Representation (11) once applied to $B^{-1}(A)$ lead us to the formula

$$B^{-1}(A)u = I + \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_A} \frac{1}{1 + \sum_{k=1}^{n} \alpha_k \mathrm{e}^{(-t_k z)}} R(z, A)u\,\mathrm{d}z, \tag{12}$$

from which it is clear that the only possible source of singularities of $B^{-1}(z)$ would be a set of zeros of denominator from (12). Thus the function $B^{-1}(A)u$ is properly defined in the sense of Definition 1 if and only if all the zeros of

$$B(z) = 1 + \sum_{k=1}^{n} \alpha_k \mathrm{e}^{(-t_k z)}, \quad z \in \mathbb{C}_+, \tag{13}$$

belong to a set $\mathbb{C} \setminus \Sigma$. Now we can formalize our previous analysis as a theorem.

Theorem 1. *Let A be a strongly positive linear operator with the spectral parameters (ρ, θ), and $f(t) \in L^1((0;T), X)$ be a given function. Then the generalized solution* (10) *exists if and only if the set of zeros* $\operatorname{Ker}(B(z)) \equiv \{z\colon\ B(z) = 0,\ z \in C\}$ *satisfy the inclusion*

$$\operatorname{Ker}\bigl(B(z)\bigr) \subset \mathbb{C} \setminus \Sigma, \tag{14}$$

where $B(z)$ associated with nonlocal condition (2b).

Proof. 1. Necessary conditions. At first, let us consider the homogeneous case $f(t) \equiv 0$. If the theorem's assumptions regarding A are valid, then the classical Cauchy problem associated with equation (2a) is *well posed* in the sense of [23, p. 29] and therefore has a unique solution for any initial state $u_0 \in D(A)$ [20, Thm. 23.8.1].

Any solution of (2a) can be represented in the form $v(t) = \mathrm{e}^{-At}v(0)$. Assume that $u(t)$ is a solution to nonlocal problem (2) for all $t \geqslant 0$, $u(t) \in D(A)$ and consider a function $w(t) = \mathrm{e}^{-At}u(0)$. Both $w(t)$, $u(t)$ satisfy differential equation (2a), and so does their difference $p(t) = w(t) - u(t)$. The difference $p(t) \equiv 0$ everywhere since it the only solution to (2a) with the zero initial condition. Consequently, solution to (2) posses a representation

$$u(t) = \mathrm{e}^{-At}u(0).$$

Equivalent existence results for inhomogeneous case can be found in [24, Prop. 3.1.16].

Once the validity of above representation (and (6) in the inhomogeneous case) is established, we can use it along with (2b) to get operator equation (9), in a way described above. This equation, as we already discovered, has a nontrivial solution only if the inclusion (14) is valid.

2. Sufficient conditions. To prove the sufficiency let us focus our attention on function $u(t)$ defined by (10). Operator valued function $\mathrm{e}^{-A(t-s)}$ from the integrands appearing in (10) is differentiable so the integrals are convergent for any $f \in X$, and therefore $u(t)$ is properly defined and bounded once $B^{-1}(A)$ is properly defined. Next, from the Dunford–Cauchy integral representation (12) we observe that $B^{-1}(A)x$ is properly defined and bounded for all $x \in X$ as long as the propositions of the theorem regarding the spectrum of A are fulfilled and (14) is valid. It remains to show that $u(t)$ defined by (10) is indeed a generalized solution to (2). This can be easily done by substitution of the representation of $u(t)$ in to (2b). □

Example 1. To demonstrate the application of Theorem 1 let us consider the following nonlocal problem:

$$\begin{aligned} &u'_t + Au = f(t), \quad t \in [0, T], \\ &u(0) + \alpha_1 u(t_1) = u_0, \quad 0 < t_1 \leqslant T. \end{aligned} \tag{15}$$

For such nonlocal condition $B(z)$, will have the form

$$B(z) = 1 + \alpha_1 \mathrm{e}^{-zt_1}.$$

Its representation permits us to write the set of zeros $\operatorname{Ker} B(z)$ in a closed form

$$\begin{aligned} \operatorname{Ker}\bigl(B(z)\bigr) &= -\frac{1}{t_1} \ln\left(-\frac{1}{\alpha_1}\right) \\ &= -\frac{1}{t_1}\left[\ln\left|\frac{1}{\alpha_1}\right| + \mathrm{i}\left(\operatorname{Arg}\left(-\frac{1}{\alpha_1}\right) + 2\pi m\right)\right], \quad m \in \mathbb{Z}, \end{aligned} \tag{16}$$

here $\operatorname{Arg}(\cdot)$ stands for the principal value of argument. Assuming that the operator A has the spectral parameters (ρ, θ),

$$z = x + \mathrm{i}y \in \mathbb{C} \setminus \Sigma \quad \Longleftrightarrow \quad |y| > (x - \rho)\tan\theta.$$

One can observe from the last inequality that if the principal value of logarithm satisfies condition (16), the same is true for the entire set of the logarithm values, so one can safely put $m = 0$ in (16).

Condition (14) for problem (15) therefore is equivalent to the following inequality:

$$\left|\operatorname{Arg}\left(-\frac{1}{\alpha_1}\right)\right| > \bigl(\ln|\alpha_1| - t_1\rho\bigr)\tan\theta. \tag{17}$$

Note that unlike (3) or earlier estimates by Byszewski, this inequality takes into account both spectral parameters of A. Closed form representation for $\operatorname{Ker} B(z)$ and the proposition of Theorem 1 guarantee that (17) forms a necessary and sufficient conditions for the existence of generalized solution to (15).

Example 1 is instructive in a sense that, for such nonlocal problem, one can easily compare conditions (3) and (17) graphically. This comparison is given on Fig. 1, where we depict three sets of admissible values of $\alpha_1 \in \mathbb{C}$ for $t_1 = 1$. Observe that, for the operator A with spectral parameters $(1, \pi/4)$ ($\rho = 1$, $\theta = \pi/4$), the set of admissible α_1 obtained from (17) (interior of the region coloured in dark grey contains in itself as a subset the admissible set obtained by (3) (coloured in black). This set remains the same for the whole family of sectorial operator coefficients with some fixed ρ and, for all $\theta \in [0, \pi/2]$, since (3) are independent of θ. While in reality the admissible set grows larger when we make θ smaller. Check, for example, the corresponding set for the case $\theta = \pi/6$ obtained using (17) which is coloured in light grey on Fig. 1. In the limiting case of $\theta = 0$ when A is self-adjoint, this set becomes equal to $\mathbb{C} \setminus (-\infty; -\mathrm{e}^{t_1\rho})$.

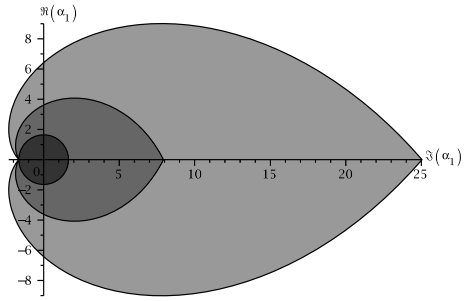


Fig. 1. Admissible values of parameter $\alpha_1 \in \mathbb{C}$ from nonlocal condition of problem (15) obtained by estimate (3) with $\theta = \pi/4$ (black), estimate (17) with $\theta = \pi/4$ (dark grey) and estimate (17) with $\theta = \pi/6$ (light grey). Spectral parameter $\rho = 1$ everywhere.

As partial case of (15), one can consider the problem with $A = -\mathrm{d}^2/\mathrm{d}x^2 + \pi^2 - 1$, $D(A) = \{v(x)\colon\ v \in H^2,\ v(0) = v(1) = 0\}$, $\rho = 1$ and (2b) in the form $u(0) + 2\mathrm{e}u(1) = 3\sin(\pi x)$. Even thought the solution to such problem exists and has the form $u(x,t) = \mathrm{e}^{-t}\sin(\pi x)$ condition (3) fails. Meanwhile the approach described above shows that solution to the mentioned problem exists for all $\alpha_1 \in \mathbb{C} \setminus (-\infty; -\mathrm{e})$.

To convince the reader that situation complicates when the nonlocal condition consists of more than one value of unknown at the given times, we study a problem with two-point nonlocal condition.

Example 2. Let us consider the problem

$$\begin{aligned} &u'_t + Au = f(t), \quad t \in [0,T], \\ &u(0) + \alpha_1 u(t_1) + \alpha_2 u(t_2) = u_0, \quad 0 < t_1 < t_2 \leqslant T. \end{aligned} \tag{18}$$

Such nonlocal condition yields the following $B(z)$:

$$B(z) = 1 + \alpha_1 \mathrm{e}^{-zt_1} + \alpha_2 \mathrm{e}^{-zt_2},$$

whence it is clear that a closed form representation of $\operatorname{Ker} B(z)$ is not available in general. The function $B(z)$ remains entire for any fixed $\alpha_1, \alpha_2, t_1, t_2$. So its roots can be accurately approximated numerically using various methods [26] (Newtons method and its modifications, gradient methods, numerical methods based on the argument principle and numerical quadratures, etc.) which implementations are available as a part of many modern mathematical programs (Octave, Maxima, Matlab, Maple).

By fixing the values

$$\alpha_1 = -0.13, \quad \alpha_2 = 3, \quad t_1 = \frac{1}{2}, \quad t_2 = 1, \tag{19}$$

we get

$$B(z) = 1 - 0.13\mathrm{e}^{-z/2} + 3\mathrm{e}^{-z}.$$

If we additionally assume that operator A has the spectral parameters $(0, \theta)$, it becomes obvious that condition (3) is inapplicable in such situation $(0.13 + 3 > 1)$. The approximate calculation of zeros of $B(z)$ carried by Maple package or, to be more exact, the function `Analytic` (being the implementation of modified Newton's method) gives us

$$\operatorname{Ker}\bigl(B(z)\bigr) = -2.09255541146 + 4\pi \mathrm{i},$$

here all given digits are significant. Combining this information, Theorem 1 and the fact that the spectrum of A lies in a right half-plane of $\mathbb{C}$, we conclude that the generalized solution to problem (18) exists for any $\theta \in [0, \pi/2]$.

All in all, the performed numerical analysis will always allow us to clarify the existence of a solution to (2) as long as the nonlocal parameters from (2b) are fixed. For many application of (1) with $n > 2$, this is not enough as one still would like to have some a priory information about the admissible parameters set rather than simply check the existence of solution for some fixed values of nonlocal parameters. This often happens in applications to control theory, where one must guarantee the solution's existence for a certain submanifold in the space of parameter values. In the remaining part of the work, we propose the technique how to estimate $\operatorname{Ker} B(z)$ by means of some well-known bounds on roots of polynomial.

3 Zeros of $B(z)$ and equivalent problem for polynomial

At first, we assume that all t_k from nonlocal condition (2b) are rational numbers. This assumption in itself is quite adequate in practice because the computer representation of t_k rely on a fixed size mantissa [27]. Every t_k admits the representation

$$t_k = \frac{\lambda_k}{\mu_k}, \quad \lambda_k \in \mathbb{Z}, \quad \mu_k \in \mathbb{N},$$

Next, we set $c_k = Q\lambda_k/\mu_k$ with $Q = \mathrm{LCM}(\mu_1, \mu_2, \dots, \mu_n)$. The function $B(z)$ is periodic with period $2\pi Q\mathrm{i}$. Thus, using the arguments from Example 15, the set $\mathbb{C} \setminus \Sigma$ can be safely reduced to $D_Q \setminus \Omega_Q$,

$$\Omega_Q = \Sigma(A) \cap D_Q,$$

here D_Q is a strip around the real axis with the width $2\pi Q$ (see Fig. 2a).

Now we would like to make use of Theorem 1. For that one needs to check whether $\operatorname{Ker} B(z) \subset D_Q \setminus \Omega_Q$. This problem is just as difficult as the corresponding problem for $\mathbb{C} \setminus \Sigma$.

A mapping

$$\varphi(z) = \exp\left(-\frac{z}{Q}\right) \tag{20}$$

transforms (13) into the following form:

$$P(z) = 1 + \sum_{k=1}^{n} \alpha_k z^{c_k}. \tag{21}$$

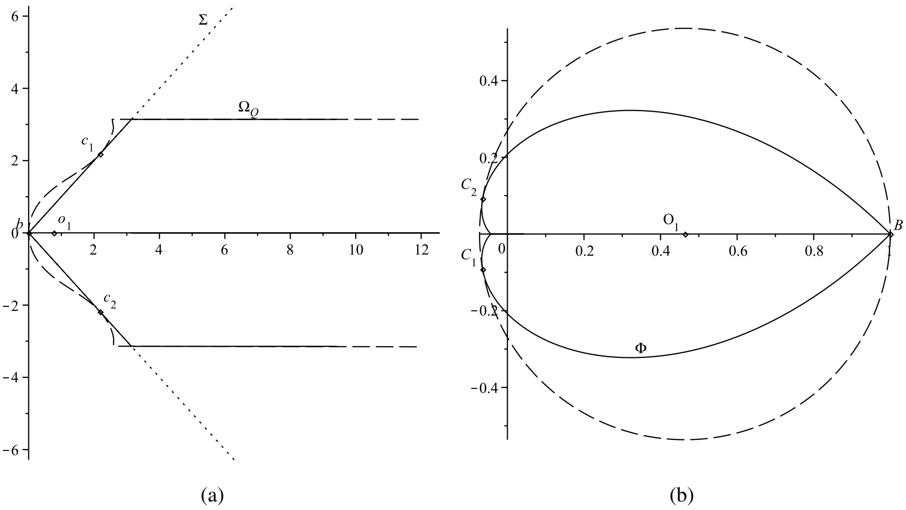


Fig. 2. The regions $\varSigma$, D_Q and their images for A with spectral parameters $\theta = \pi/4$, $\rho = 0$ ($Q = 1$): (a) intersection of spectrum $\varSigma$ and the set D_Q; (b) the region $\varPhi$ and its encompassing circle. Points $c_{1/2}, b, o_1$ are the preimages of $C_{1/2}, B, O_1$.

It is well known [26] that (13) is one-to-one conformal mapping of $\varOmega_Q$ onto $\varPhi$ (see Fig. 2b). By using it we achieved two goals. First of all, the selected mapping transforms the entire function $B(z)$ into polynomial $P(z)$ with real coefficients (provided that all α_i are real). Second, this mapping reduces the zeros finding problem for the exterior of $\varSigma$ to the same problem for the exterior of a bounded set $\varPhi \subset \mathbb{C}$. Speaking more precisely, the conditions guarantying that all roots of $P(z)$ lie outside $\varPhi$ or equivalently $\operatorname{Ker} B(z) \subset D_Q \setminus \varOmega_Q$ would be necessary and sufficient to prove that existence and uniquenesses of solution (10). The majority of results related to such conditions for polynomials are devoted to the situation when a circle is considered in place of $\varPhi$ (to review existent results in that field, see [18, 19] as well as [26, 28]).

That is why we firstly encircle $\varPhi$ and then use readily available zero-free conditions for that circle. Such approach will make the resulting conditions only sufficient for all $\theta \in [0, \pi/2)$ except for the limiting case $\theta = \pi/2$ when $\varPhi$ is a circle by construction.

For any given operator A with spectral parameters (ρ, θ), the boundary of $\varPhi$ can be parametrized as follows:

$$\partial\varPhi = \left\{ \exp\left(-\frac{Z(x)}{Q}\right) \colon x \in [0, +\infty] \right\},$$

where $Z(x)$ is a parametrization of $\partial\varOmega_Q$:

$$Z(x) = \rho + x + \mathrm{i} \begin{cases} x \tan\theta, & x\tan\theta < Q\pi, \\ Q\pi, & x \tan\theta \geqslant Q\pi. \end{cases}$$

A closer look at the expression for $\partial\varPhi$ unveils that a vertical linear diameter of $\varPhi$ is proportional to the magnitude of spectral angle, and the horizontal diameter of $\varPhi$ is reversely proportional to ρ. This observation suggests us to describe the encompassing circle as a circumcircle of a triangle with the vertices

$$B = \max_{z\in\partial\Phi} \Re(z) + 0\mathrm{i} = \exp\left(-\frac{\rho}{Q}\right),$$

and $C_{1/2} \in \partial\Phi$ which are symmetric with respect to the real axis. The coordinates of C_1 are chosen to maximize the distance $|O_1 - B|$ under the constrain $|O_1 - B|^2 = |O_1 - C_i|^2$, here O_1 is a circumcentre of $\triangle_{BC_1C_2}$. Using the definition of $Z(x)$, (20) and some basic facts from calculus, we reduce the mentioned maximization problem to the following equation:

$$\exp\left(-\frac{2x}{Q}\right)\left[\cos\left(\frac{x\tan\theta}{Q}\right) - \tan\theta\sin\left(\frac{x\tan\theta}{Q}\right)\right] \\ + \cos\left(\frac{x\tan\theta}{Q}\right) + \tan\theta\sin\left(\frac{x\tan\theta}{Q}\right) = 2\exp\left(-\frac{x}{Q}\right). \tag{22}$$

It has a positive solution for $Q \in \mathbb{N}$ and for all $\theta \in [0, \pi/2]$. Assume that x_d is a solution of (22), then

$$C_{1/2} = \varphi(\rho + x_d \pm \mathrm{i}x_d\tan\theta), \qquad O_1 = \frac{\varphi(2\rho) - \Re(C_1)^2 - \Im(C_1)^2}{2(\varphi(\rho) - \Re(C_1))},$$

while the radius of circumcircle $r = \varphi(\rho) - O_1$ (the picture of Φ, its encompassing circle along with their inverse images are shown in Fig. 2).

4 Sufficient conditions for existence of solution

Let us review what we have done so far. Starting from Theorem 1, we reduce the problem of clarifying whether $\operatorname{Ker} B(z) \subset \mathbb{C} \setminus \Sigma$ to the corresponding problem for zeros of the polynomial $P(z)$ lying in the exterior of the circle $|z - O_1| \leqslant r$. Conditions guarantying such layout of roots [18, 19] are obtained, as a rule, from equivalent conditions for the interior of the circle. That is why most of the related results are formulated for the circle with centre at origin. Some of them, in addition, operate with the unit circle only. To accommodate this observation, we introduce two alternative forms of (21): with the given circle transformed to the unit circle centered at the origin

$$P_1(z') = P(O_1 + rz') = \sum_{k=0}^{c_n} \alpha'_k z'^k, \tag{23}$$

and with the given circle transformed to the circle centered at the origin

$$P_2(z'') = P(O_1 + z'') = \sum_{k=0}^{c_n} \alpha''_k z''^k. \tag{24}$$

At this point, we need to make use of several results estimating the radius of zero-free circle in terms of the polynomial coefficients. First of these results is a so-called Schur–Cohn test [26]. It establishes the necessary and sufficient conditions for the roots of polynomial to lie in the region $|z| > 1$.

Definition 2. Given $P^\star(z) = z^n\overline{P(1/\overline{z})}$ we define the Schur transform T of polynomial $P(z)$ by

$$TP(z) := \overline{\alpha_0}P(z) - \alpha_n P^\star(z) = \sum_{k=0}^{n-1}(\overline{\alpha_0}\alpha_k - \alpha_n\overline{\alpha_{n-k}})z^k.$$

Theorem 2. (*See* [26].) *Let $P(z)$ is a polynomial of degree $n > 0$. All zeros of $P(z)$ lie in the exterior of the circle $|z| \leqslant 1$ if and only if, for all $k = 1, 2, \dots, n$,*

$$\Gamma_k > 0, \tag{25}$$

where $\Gamma_k := T^kP(0)$ and $T^kP = T(T^{k-1}P)$.

Corollary 1. *If the coefficients α_k satisfy*

$$\alpha_0 \geqslant \alpha_1 \geqslant \dots \geqslant \alpha_{n-1} \geqslant \alpha_n > 0,$$

then all zeros of $P(z)$ lie outside the circle $|z| \leqslant 1$.

The application Schur–Cohn test to $P_1(z)$ produces a system of c_nth innequalities for the coefficients of nonlocal condition (2b) which is sufficient for the solution's existence. Next four Lemmas are more convenient than Theorem 2 from the computational standpoint since the number of the produced inequalities are independent on the polynomial degree.

Lemma 1. *All zeros of $P(z)$ lie in th region*

$$|z| \geqslant \frac{|\alpha_0|}{|\alpha_0| + M},$$

where $M = \max_{1\leqslant k\leqslant n}|\alpha_k|$.

Lemma 2. *All zeros of $P(z)$ satisfy the inequality*

$$|z| \geqslant \frac{|\alpha_0|}{[|\alpha_0| + M^q]^{1/q}}, \quad M = \left(\sum_{k=1}^n |\alpha_k|^p\right)^{1/p}, \quad p, q \in \mathbb{R}_+, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Next estimate is due to Fujiwara [29]. It is an optimal homogeneous estimate in the space of polynomials [30].

Lemma 3. *All zeros of $P(z)$ belong to the region*

$$|z| \geqslant \frac{1}{2}\min_{\alpha_i\neq 0}\left\{\left|\frac{\alpha_0}{\alpha_1}\right|, \left|\frac{\alpha_0}{\alpha_2}\right|^{1/2}, \dots, \left|\frac{\alpha_0}{\alpha_{n-1}}\right|^{1/(n-1)}, \left|\frac{2\alpha_0}{\alpha_n}\right|^{1/n}\right\}.$$

The last of the estimates given here was proved by H. Linden. This estimate in its original form gives bounds on the real and imaginary part of zeros separately. It has been adapted to fit within the framework studied here.

Lemma 4. *All zeros of $P(z)$ belong to the region $|z| \geqslant \max\{V_1^{-1}, V_2^{-1}\}$, where*

$$V_1 = \cos\frac{\pi}{n+1} + \frac{|\alpha_n|}{2|\alpha_0|}\left(\left|\frac{\alpha_1}{\alpha_n}\right| + \sqrt{1 + \sum_{k=1}^{n-1}\left|\frac{\alpha_k}{\alpha_n}\right|^2}\right),$$

$$V_2 = \frac{1}{2}\left(\left|\frac{\alpha_1}{\alpha_0}\right| + \cos\frac{\pi}{n}\right) + \frac{1}{2}\left[\left(\left|\frac{\alpha_1}{\alpha_0}\right| - \cos\frac{\pi}{n}\right)^2 + \left(1 + \left|\frac{\alpha_n}{\alpha_0}\right|\sqrt{1 + \sum_{k=2}^{n-1}\left|\frac{\alpha_k}{\alpha_n}\right|^2}\right)^2\right]^{1/2}.$$

By combining the estimates given by Theorem 2 or Lemmas 1–4 with Theorem 1 we obtain new sufficient conditions for the existence and uniquenesses of the solution to (2).

Theorem 3. *Assume that A, $f(t)$ and u_0 satisfy the conditions of Theorem* 1. *The generalized solution* (10) *of nonlocal problem* (2) *exists if either of the following is true*:

(i) $\exists z > 1$*, which along with the coefficients of $P(\varphi(\rho)z)$ from* (21) *satisfies the conditions of Theorem* 2 *or at least one of Lemmas* 1–4.

(ii) $\exists z > 1$*, which along with the coefficients of $P_1(z)$ from* (23) *satisfies the conditions of Theorem* 2 *or at least one of Lemmas* 1–4.

(iii) $\exists z > r$*, which along with the coefficients of $P_2(z)$ from* (24) *satisfies the conditions of at least one of Lemmas* 1–4.

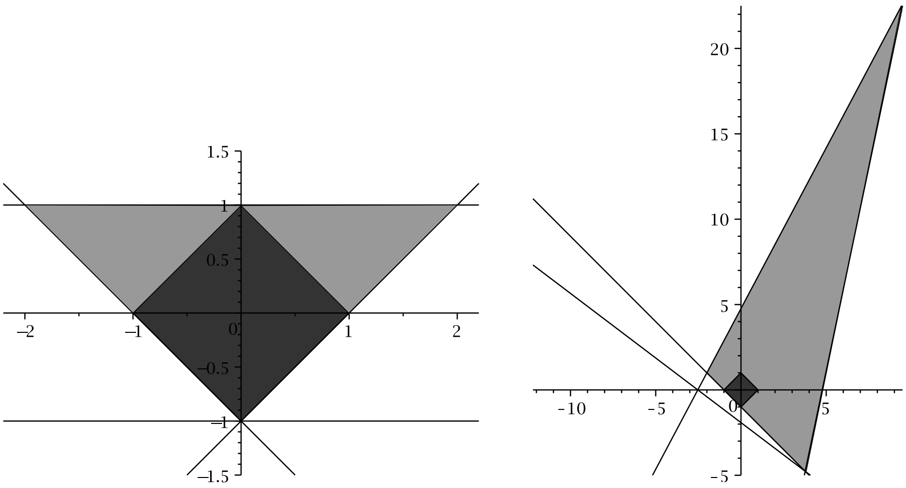
Proof. The results formulated in Theorem 2 and Lemmas 1–4 guaranty that as soon as at least one of three theorem's propositions is valid all zeros of $B(z)$ will lie outside Φ. Hence, according to Theorem 1, the solution of (2) exists and is unique. □

Propositions (i)–(iii) of Theorem 3 are ordered in such a way that the first proposition deals with a circle obtained by putting $\theta = \pi/2$. It is therefore valid for A with any $\theta \in (0; \pi/2]$. The other two propositions use the parameters of encompassing circle defined above. These propositions will lead to more general sufficient conditions for $\theta < \pi/2$. To illustrate these facts, we apply different proposition stated in Theorem 3 to some concrete examples of nonlocal conditions.

Example 3. Let us again consider the nonlocal problem (2) with operator coefficient A $(\theta = \theta_0, \rho = 0)$ and the Bitsadze–Samarskii-type nonlocal condition $u(0) + \alpha_1 u(t_1) = \alpha_2 u(t_2)$. As we have mentioned in Example 2, the set $\mathrm{Ker}(B(z))$ can not be found in a closed form.

Estimate (3) yields:

$$|\alpha_1| + |\alpha_2| < 1.$$



(a) (b)

Fig. 3. The admissible sets of $\alpha_{1,2} \in \mathbb{R}$ from (2b) with $t_1 = 1$, $t_2 = 2$ using condition (3) (black) and: (a) proposition (i) of Theorem 3 with $\rho = 0$, $\theta_0 \in (0, \pi/2]$ (grey); (b) proposition (ii) of Theorem 3 and estimate (25) with $\rho = 0$, $\theta = \pi/3$ (grey).

Meanwhile the application of proposition (i) from Theorem 3 together with Schur–Cohn algorithm with $\theta_0 = \pi/2$ lead us to system of inequalities

$$|\alpha_2| < 1, \qquad \bigl|1 - \alpha_2^2\bigr| > \bigl|\alpha_1(1 - \alpha_2)\bigr|. \tag{26}$$

Here we assumed that $t_1 = 1$, $t_2 = 2$ and $Q = 1$. Solution of (26) are graphically compared to with set of pairs (α_1, α_2) satisfying (3) in Fig. 3a. Our approach apparently gives more general conditions than (3) or its particular case from [13] even though we made sufficient conditions (26) independent on θ_0.

Propostion (ii) of Theorem 3 ought to be more advantageous for operator coefficients with smaller θ. Let us fix $\theta_0 = \pi/3$. We get

$$O_1 = 0.3950734246, \quad r = 0.6049265754$$

for the parameters of encompassing circle and

$$P_1(z') \approx 0.37\alpha_2 {z'}^2 + (0.6\alpha_1 + 0.48\alpha_2)z' + 1 + 0.4\alpha_1 + 0.16\alpha_2,$$
$$P_2(z'') \approx \alpha_2 {z''}^2 + (\alpha_1 + 0.79\alpha_2)z'' + 1 + 0.4\alpha_1 + 0.16\alpha_2$$

for polynomials (23), (24), correspondingly.

Application of the mentioned proposition along with (25) (setting $t_1 = 1$, $t_2 = 2$ as before) gives us the set of admissible (α_1, α_2) depicted in Fig. 3b. One can see that this set contains both admissible parameters sets obtained from proposition (i) of the same theorem and condition (3).

More comparisons of different combinations of propositions from Theorem 3 and Lemmas 1–4 against previously available existence conditions can be found in [31]. In this work, it is also described how to generalize the results of Section 4 to the case when some of t_i are irrational. All coding for the generation of figures, circle parameters calculation and numerical checks of the presented necessary conditions are to be found at `www.imath.kiev.ua/~sytnik/research/works/nonlocal-2014/`.

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