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# The solution of an initial-boundary value problem of the filtration theory with nonlocal boundary condition

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**Abstract.** The nonlocal initial and boundary value problem for Lavrentiev–Bitsadze equation is considered. By this problem the nonstacionary one-dimentional motion of a groundwater with horizontal stopping is modeling. The existence and the uniqueness of the classical (in the elliptic part of the domain) and generalized (in the hyperbolic part of the domain) solution of the considered problem is proved.

**Keywords:** motion of groundwater, Boussinesq's equation, equations of mixed type, Lavrentiev–Bitsadze equation, nonlocal problems, Samarskii condition, initial and boundary value problems.

## 1 Introduction

The nonstacionary one-dimentional motion of a groundwater with the weakly varying free surface and with the horizontal water bed is described in the classical filtration theory by the Boussinesq's equation [1]

$$\sigma\frac{\partial h}{\partial t} = k\frac{\partial}{\partial \xi}\left(h\frac{\partial h}{\partial \xi}\right) - \frac{k_0}{M_0}(h - H_0) + w(\xi, t); \tag{1}$$

here by the $\sigma$ and $k$ are denoted the water removal and the filtration coefficient, which are proposed to be constant, $k_0$ is the coefficient of the filtration by the weak permeability of the water bed of constant thickness $M_0$, $H_0$ is the water pressure in the underneath water-bearing layer, $w(\xi, t)$ is the discrepancy between the infiltration and the evaporation, $h = h(\xi, t)$ is the level of the groundwater in a point $\xi$ at moment $t$. We assume that a motion of groundwater is observed on the prognosticated interval $0 < \xi < l$ for $t > 0$.

Many investigations of the groundwater dynamics are based on the mathematical models by a reduction of Boussinessq's equation into linear parabolic equation with non-local condition. The differential problems of such type were proposed and considered, for instance, in [2–4]. Unfortunately, it was shown experimentally that such models describe the motion of the groundwater in porous environment not adequately. Qualitatively new mathematical models of the groundwater dynamics are proposed in [5–7]. It is based on the the elliptic-hyperpolic equation with nonlocal condition and is more suitable for the porous environment.

Our goal in this paper is to reduce the filtration problem described by equation (1) to the classical partial differential equations of the mixed type with nonlocal condition and to obtain the conditions of the solvability of the reduced problem.

If the solution $h$ of equation (1) has the derivatives of a proper order, then it satisfies the equation

$$\sigma\frac{\partial^2 h}{\partial t^2} = k\frac{\partial^2}{\partial \xi^2}\left(h\frac{\partial h}{\partial t}\right) - \frac{k_0}{M_0}\frac{\partial h}{\partial t} + \frac{\partial w(\xi,t)}{\partial t}. \tag{2}$$

We will use below the denotation

$$\delta(t) = \int_0^l h(\xi,t)\,\mathrm{d}\xi.$$

Then $\sigma\delta'(t)$ is the average of groundwater flux on the interval $(0,l)$ at the moment $t$. Let us assume that $\partial h/\partial t$ is proportional with a constant coefficient $\gamma$ to the average of the groundwater flux:

$$\frac{\partial h}{\partial t} = \gamma\sigma\delta'(t). \tag{3}$$

Under this hypothesis, equation (2) takes the form

$$\frac{\partial^2 h}{\partial t^2} = k\gamma\delta'(t)\frac{\partial^2 h}{\partial \xi^2} - \frac{1}{\sigma}\frac{k_0}{M_0}\frac{\partial h}{\partial t} + \frac{1}{\sigma}\frac{\partial w(\xi,t)}{\partial t}. \tag{4}$$

It is evident that a type of this equation depends on the sign of function $\delta'(t)$ and it can be elliptic or hyperbolic as well. It is easily seen that at the line $t = t^*$, for which $\delta'(t^*) = 0$, equation (4) degenerate into parabolic one. It is well known that, for a broad class of the melioration problems of the prognostication of a dynamics of the groundwater, one can take that

$$k\gamma\delta'(t) = c|t - t^*|^m \operatorname{sgn}(t^* - t), \tag{5}$$

where $m \geqslant 0$ and $c > 0$ are some constants, $t^*$ is an extreme time when the groundwater flux attains its maximum on the interval $0 < \xi < l$ and decreases after that till the value, which does not violate the ecology of the aeration area. The law of the groundwater motion described by (5) is reasoned properly in [8].

Due to (5), we get from (4) the equation

$$\frac{\partial^2 h}{\partial t^2} = c|t - t^*|^m \operatorname{sgn}(t^* - t)\frac{\partial^2 h}{\partial \xi^2} - \frac{k_0}{\sigma M_0}\frac{\partial h}{\partial t} + \frac{1}{\sigma}\frac{\partial w(\xi,t)}{\partial t} \tag{6}$$

of mixed type, which is elliptic on semi-plane $\mathbb{R}^2_+ = \{(\xi,t)\colon\ t > t^*\}$, hyperbolic on semi-plane $\mathbb{R}^2_- = \{(\xi,t)\colon\ t < t^*\}$ and parabolic at the line $t = t^*$.

Let us introduce the new variables

$$x = \frac{\xi}{\sqrt{c}}, \qquad y = t - t^*$$

and the function

$$u(x, y) = h\left(\frac{\xi}{\sqrt{c}}, y + t^*\right).$$

Then we obtain for function $u$ the equation

$$|y|^m \operatorname{sgn} y \, u_{xx} + u_{yy} + \beta u_y = f(x, y), \tag{7}$$

where

$$\beta = \frac{k_0}{\sigma M_0}, \qquad f(x, y) = \frac{1}{\sigma} \frac{\partial}{\partial y} w\big(x\sqrt{c}, y + t^*\big).$$

Moreover, taking in account both presumptions (3) and (4), we obtain the relation

$$\frac{\partial}{\partial y} \int\limits_0^r u(x, y) \, \mathrm{d}x = A|y|^m \operatorname{sgn} y, \tag{8}$$

where $r = l/\sqrt{c}$, $A = -\sqrt{c}/(k\gamma)$.

Thus, equation (7) jointly with nonlocal condition (8) represents the linear mathematical model of a groundwater dynamics with weakly varying free surface and with weak permeability of the horizontal water bed. There is important to observe that this nonlocal conditions comments on the finite rate of the spread of any small disturbance of groundwater motion in the porous medium.

Notice, one can assume in the case of a horizontal water bed that it is impervious, i.e., $k_0 = 0$. Then $\beta = 0$, evidently. If $M_0 \gg 1$, i.e., if the thickness is very high, then we can take $\beta = 0$ again. In these both cases, we get from (7) the equation

$$|y|^m u_{xx} \operatorname{sgn} y + u_{yy} = f(x, y).$$

Further, if $m = 0$ in relation (5), then this equation takes the shape

$$u_{xx} \operatorname{sgn} y + u_{yy} = f(x, y),$$

evidently. Obtained equation is well known as Lavrentiev–Bitsadze equation [4]. If an external disturbance on the flow absent, i.e., the infiltration in unit time is zero, then $w(x, y) = 0$ and $f(x, y) = 0$, consequently. In such case, we get the homogeneous case of Lavrentiev–Bitsadze equation

$$u_{xx} \operatorname{sgn} y + u_{yy} = 0. \tag{9}$$

We shall consider below the model described by equation (9) and non-local condition

$$\frac{\partial}{\partial y} \int\limits_0^r u(x, y) \, \mathrm{d}x = A \operatorname{sgn} y \tag{10}$$

obtained from (8) with $m = 0$, which quite adequate corresponds to the filtration problem in porous medium.

## 2 Statement of the nonlocal problem to Lavrentiev–Bitsadze equation

We will analyze equation (9) in the rectangle

$$\Omega_n = \{(x, y)\colon\ 0 < x < a,\ -na < y < b\},$$

where both $a$ and $b$ are positive numbers, $n$ is an natural number.

Introduce the subdomains

$$\Omega^+ = \{(x, y)\colon\ 0 < x < a,\ 0 < y < b\},$$
$$\Omega_n^- = \{(x, y)\colon\ 0 < x < a,\ -na < y < 0\}$$

of domain $\Omega_n$. Notice that equation (9) is elliptic in $\Omega_n^+$ and hyperbolic in $\Omega_n^-$. Let $I$ be the interval $\{(x, y)\colon\ 0 < x < a,\ y = 0\}$. Then $\Omega_n = \Omega^+ \cup I \cup \Omega_n^-$. Besides, let us introduce the characteristic triangle

$$\Delta = \{(x, y)\colon\ 0 < x < a,\ -x < y < x - a\}$$

and the subdomain $\Omega = \Omega^+ \cup I \cup \Delta$. Notice to the purpose that both isosceles sides of the triangle $\Delta$ emerging from the vertex $(a/2, -a/2)$ and lie on the corresponding characteristics $l_1$: $x + y = 0$ and $l_2$: $x - y = a$ of equation (9).

Besides, let us introduce the square subdomains

$$\tilde{\Omega}_k = \{(x, y)\colon\ 0 < x < a,\ -ka < y < (1 - k)a\} \subset \Omega_n^-, \quad k = 1, \dots, n.$$

If $n = 1$, then $\Omega_1^- = \tilde{\Omega}_1$, obviously. It is easily seen that $\Omega_n^- = \bigcup_{k=1}^{n-1} (\tilde{\Omega}_k \cup I_k) \cup \tilde{\Omega}_n$, where $I_k = \{(x, y)\colon\ 0 < x < a,\ y = -ka\}$.

**Definition 1.** We say that a function $u$ is regular solution of equation (9) in some domain $D \subset \Omega_n$ if it satisfies this equation everywhere in $D$ and $u \in C(\overline{D})$.

**Definition 2.** We say, similarly as in [4], that a function $u$ is generalized solution of equation (9) in some domain $D \subset \Omega_n^-$ if it can be expressed by the continuous in $\overline{D}$ wave-functions of the shape

$$u(x, y) = f(x - y) + g(x + y).$$

The latter definition is needful to be introduced because the solution of the problem, which shall be discussed below can be not twice differentiable in a hyperbolic part of domain $\Omega_n$. Such situation emerges due to the insufficient smoothness of the initial data in the hyperbolic type problems for equation (9). Then there is purposeful to define the generalized solution, e.g. by the way given above.

Let us consider the following problem.

**Problem S.** Find the solution $u \in C^2(\Omega^+ \cup \Delta) \cap C^1(\Omega) \cap C(\overline{\Omega}_n)$ of equation (9), which is regular in the both domains $\Omega^+$ and $\Delta$, represents the generalized solution in $\Omega_n \setminus \Omega$ and satisfies the following conditions:

$$\int_0^a u_y(x,y)\,\mathrm{d}x = \mu(y), \quad 0 < y < b, \tag{11}$$

$$u(x,-na) = \varphi_n(x), \quad u(x,b) = \varphi(x), \quad 0 \leqslant x \leqslant a, \tag{12}$$

$$u(a,y) = \psi_a(y), \quad -na \leqslant y \leqslant b, \tag{13}$$

$$u(0,y) = \psi_0(y), \quad -na \leqslant y \leqslant 0, \tag{14}$$

where $\mu$, $\varphi_n$, $\varphi$, $\psi_a$, $\psi_0$ are given continuous functions.

Here condition (14) means in a physical sense that the law of the groundwater dynamics is observed at the point $x = 0$ on the all time interval $[-na, 0]$, i.e., it is known till the critical time $y = 0$. Thus, there appears in Problem S the local conditions (12)–(13) and nonlocal in Samarskii sense condition (11) of type (10). This problem can be included to the class of problems for so-called loaded differential equations [9].

## 3 Uniqueness of the solution of Problem S

Let us introduce the square subdomains

$$\tilde{\Omega}_k^- = \{(x,y)\colon\ 0 < x < a,\ -ka < y < (1-k)a\} \subset \Omega_n^-, \quad k = 1, \dots, n.$$

If $n = 1$, then $\Omega_1^- = \tilde{\Omega}_1^-$, obviously. It is easily seen that $\Omega_n^- = \bigcup_{k=1}^{n-1} (\tilde{\Omega}_k^- \cup I_k) \cup \tilde{\Omega}_n^-$, where $I_k = \{(x,y)\colon\ 0 < x < a,\ y = -ka\}$. We shall prove the following lemma.

**Lemma 1.** *Let $u(x,y)$ be a solution of Problem S. Then there holds the relation*

$$\begin{aligned} u(x,0) = \sum_{k=1}^{n} \{&\alpha_k[\psi_a(x-ka) + \psi_0(-x-(k-1)a)] \\ &- \alpha_{k+1}[\psi_a(-x-(k-1)a) + \psi_0(x-ka)]\} \\ &+ \Phi_n(x), \quad 0 \leqslant x \leqslant a, \end{aligned} \tag{15}$$

*where*

$$\alpha_k = \begin{cases} 1, & k = 1 \ (\mathrm{mod}\ 2), \\ 0, & k = 0 \ (\mathrm{mod}\ 2), \end{cases} \qquad \Phi_n(x) = \begin{cases} -\varphi_n(a-x), & n = 1 \ (\mathrm{mod}\ 2), \\ \varphi_n(x), & n = 0 \ (\mathrm{mod}\ 2). \end{cases}$$

*Proof.* We shall prove this lemma by the mathematical induction using the well-known theorem on an intermediate value of the solutions of the one-dimentional wave equation.

Let us check the validity of equality (15) for $n = 1$. Evidently, in this case, any collection of the four points $(x,0)$, $(a, x-a)$, $(a-x,-a)$ and $(0,-x)$, $x \in (0,a)$, represents the angular points of characteristic rectangular[1] belonging to closed rectangular $\tilde{\Omega}_1^- \cup \partial\tilde{\Omega}_1^-$. According to the theorem on the intermediate value of the solutions of wave equation (9) ($y < 0$), the relation

$$u(x,0) + u(a-x,-a) = u(a,x-a) + u(0,-x), \quad 0 \leqslant x \leqslant a,$$

holds. Then we get from conditions (12)–(14) that

$$u(x,0) = \psi_a(x-a) + \psi_0(-x) - \varphi_1(a-x), \quad 0 \leqslant x \leqslant a,$$

i.e., formula (15) is true.

Assume that $n > 1$ and assume that formula (15) is true in the case of domain $\Omega_n^-$ with $n = m-1$, $m > 2$, when $m$ is an even:

$$\begin{aligned} u(x,0) = \sum_{k=1}^{m-1} \bigl\{ &\alpha_k \bigl[\psi_a(x-ka) + \psi_0\bigl(-x-(k-1)a\bigr)\bigr] \\ &- \alpha_{k+1}\bigl[\psi_a\bigl(-x-(k-1)a\bigr) + \psi_0(x-ka)\bigr]\bigr\} \\ &- \varphi_{m-1}(x), \quad 0 \leqslant x \leqslant a. \end{aligned} \tag{16}$$

We shall show that that formula (15) is true for $n = m$. Let us consider the characteristic rectangular with the corners $(x,-(m-1)a)$, $(a,x-ma)$, $(a-x,-am)$ and $(0,-x-(m-1)a)$, $x \in (0,a)$. Then the equality

$$\begin{aligned} &u\bigl(x,-(m-1)a\bigr) + u(a-x,-am) \\ &\quad = u(a,x-ma) + u\bigl(0,-x-(m-1)a\bigr), \quad 0 \leqslant x \leqslant a, \end{aligned}$$

holds because of the analogous factor as in the case $n = 1$. Taking into account the conditions (12)–(14) again, we obtain from this equality that

$$\varphi_{m-1}(a-x) = \psi_a\bigl(-x-(m-1)r\bigr) + \psi_0(x-na) - \varphi_m(x), \quad 0 \leqslant x \leqslant a.$$

Putting obtained expression of $\varphi_{m-1}$ into (16), we get relation (15)

$$\begin{aligned} u(x,0) = \sum_{k=1}^{m} \bigl\{ &\alpha_k \bigl[\psi_a(x-ka) + \psi_0\bigl(-x-(k-1)a\bigr)\bigr] \\ &- \alpha_{k+1}\bigl[\psi_a\bigl(-x-(k-1)a\bigr) + \psi_0(x-ka)\bigr]\bigr\} \\ &+ \varphi_m(x), \quad 0 \leqslant x \leqslant a, \end{aligned}$$

if $m = 0 \pmod 2$.

In the case when $n$ is odd, the proof is analogous. □

[1] Since the sides of this rectangular lie on he characteristics, it is named as characteristic one. Notice, if $x = 0$ or $x = a$, then last-mentioned rectangular degenerates into corresponding segments lying on the characterictics.

Let a function $u$ be harmonic in a plane domain $D \subset Oxy$. Then it satisfies in $D$ the obvious equality

$$\frac{\partial}{\partial x}\left(u_x^2 - u_y^2\right) + 2\frac{\partial}{\partial y}(u_x u_y) = 0. \tag{17}$$

If the boundary $\partial D$ of domain $D$ is piecewise smooth, then using the Green's formula, we get from (17) that

$$\begin{aligned} 0 &= \iint\limits_D \left[\frac{\partial}{\partial x}\left(u_x^2 - u_y^2\right) + 2\frac{\partial}{\partial y}(u_x u_y)\right] \mathrm{d}x\,\mathrm{d}y \\ &= \int\limits_{\partial D} \left(u_x^2 - u_y^2\right) \mathrm{d}y - 2u_x u_y\,\mathrm{d}x. \end{aligned} \tag{18}$$

Certainly, this relation is valid only in the case when there exists the curvilinear integral on right-hand side.

Let us propose one important property of the solution $u(x, y)$ of Problem S. Assuming that $u_{xx}(x, y) \in L(0, a)$ for all $y \in (0, b)$ and according to condition (11), we get that

$$\int\limits_0^a u_{xx}(x, y)\,\mathrm{d}x = -\int\limits_0^a u_{yy}(x, y)\,\mathrm{d}x = -\mu'(y), \quad 0 < y < b.$$

On the other,

$$\int\limits_0^a u_{xx}(x, y)\,\mathrm{d}x = u_x(a, y) - u_x(0, y), \quad 0 < y < b.$$

Thus, if $u(x, y)$ is differentiable with respect to $x$ at the points $(0, y)$ and $(a, y)$ for $0 < y < b$, then relation

$$u_x(a, y) - u_x(0, y) = -\mu'(y), \quad 0 < y < b, \tag{19}$$

holds.

**Theorem 1.** *Let $u(x, y)$ be the solution of the homogeneous problem corresponding to Problem* S*, which is continuously differentiable everywhere in the domain $\overline{\Omega}^+$ except, may be, its angular points, and is such that the following conditions are fulfilled*: (i) $u_y(0, y) \in L_2(0, b)$, (ii) $u_y(x, 0)$ *and* $u_y(x, b) \in L(0, a)$. *Then $u(x, y) \equiv 0$ in $\Omega_n$.*

*Proof.* Firstly, we shall prove that $u(x, y) \equiv 0$ in $\Omega^+$. It follows from the homogeneity of local conditions (12)–(13) that

$$u(x, b) = 0 \quad \forall x \in [0, a], \qquad u(a, y) = 0 \quad \forall y \in [0, b], \tag{20}$$

immediately, and

$$u(x, 0) = 0 \quad \forall x \in [0, a] \tag{21}$$

due to Lemma 1. Therefore, the following equalities

$$u_x(x,0) = u_x(x,b) = 0 \quad \forall x \in (0,a), \qquad u_y(a,y) = 0 \quad \forall y \in (0,b) \tag{22}$$

are valid.

Further, the homogeneity of nonlocal condition (11) and condition (ii) yield the equality

$$u_x(0,y) = u_x(a,y) \quad \forall y \in (0,b) \tag{23}$$

because of relation (19). Applying relation (18) with respect to domain $\Omega^+$ and taking in account equalities (22) and (23), we obtain due to condition (i) that

$$\begin{aligned}
0 &= \int\limits_{\partial\Omega^+} \left(u_x^2 - u_y^2\right) \mathrm{d}y - 2u_x u_y \,\mathrm{d}x \\
&= \int\limits_0^b \left[u_x^2(0,y) - u_y^2(0,y) - u_x^2(a,y) + u_y^2(a,y)\right] \mathrm{d}y \\
&\quad + 2\int\limits_0^a \left[u_x(x,b)u_y(x,b) - u_x(x,0)u_y(x,0)\right] \mathrm{d}x \\
&= -\int\limits_0^b u_y^2(0,y)\,\mathrm{d}y \quad \forall y \in (0,y).
\end{aligned}$$

Hence, $u_y(0,y) = 0$, i.e., $u(0,y) = \text{const}$ on the interval $(0,b)$. Since $u(0,0) = 0$, $u \in C(\overline{\Omega^+})$, we get that $u(0,y) = 0$ for all $y \in [0,b]$ due to the continuity of solution $u(x,y)$ in $\overline{\Omega^+}$. That jointly with (20), (21) yield the condition $u|_{\partial\Omega^+} = 0$. Then $u(x,y) \equiv 0$ in $\Omega^+$ according to the maximum principle for harmonic functions.

There is important to observe that $u_y(x,0) = 0$ on the interval $I$. Indeed, if $u_y(x_0,0) \neq 0$, $x_0 \in (0,a)$, then it follows from the Zarembo–Giraud principle [4, 10] that $u(x,0) \neq 0$ in some $\varepsilon$-neighborhood $\{(x,y)\colon (x-x_0)^2 + y^2 < \varepsilon^2\} \cap \Omega^+$ of the point $(x_0,0)$, but that is in a contradiction with the condition $u(x,y) \equiv 0$ in $\Omega^+$.

Thus, it remains to show that $u(x,y) \equiv 0$ in the domain $\Omega_n^-$.

Let us introduce in addition to the triangle $\Delta$ the following nonintersecting isosceles triangles:

$$\begin{aligned}
\Delta_1 &= \left\{(x,y)\colon 0 < x < \frac{a}{2},\ x - a < y < -x\right\}, \\
\Delta_2 &= \left\{(x,y)\colon \frac{a}{2} < x < a,\ -x < y < x - a\right\}, \\
\Delta_3 &= \left\{(x,y)\colon y + a < x < -y,\ -a < y < -\frac{a}{2}\right\}.
\end{aligned}$$

Notice to a purpose that namely the characteristics $x + y = 0$ and $x - y = a$ divides the square $\tilde{\Omega}_1$ into these four characteristic triangles.

Firstly, let us consider the solution $u(x,y)$ in $\Delta$. Since $u(x,0) = 0$ on the closure $\overline{I}$ of interval $I$ and $u_y(x,0) = 0$ on $I$, and condition (ii) holds, we get due to principle of continuos extension of the solution of Cauchy problem from $\overline{I}$ into $\overline{\Delta}$ the identity $u(x,y) \equiv 0$ in $\overline{\Delta}$.

Let us get into the triangle $\Delta_1$. Notice that we have the following situation: $u(0,y)=0$ for all $y \in [-a,0]$ according to the assumption of the theorem, and, as we obtain thereto-fore, $u(x,y) = 0$ on the segment $0 \leqslant x \leqslant a/2$ of the characteristics $x+y=0$. Thus, it follows from the uniqueness of the solution of Darboux problem [11] that $u(x,y) \equiv 0$ in the $\overline{\Delta}_1$. Further, we get by the analogous reasoning that $u(x,y) \equiv 0$ in the closed triangle $\overline{\Delta}_2$ due to the homogeneous data on the its basis and on the its lateral side lying on the characteristics $l_2$.

So, we get that $u(x,y) = 0$ on the lateral sides of the characteristic triangle $\Delta_3$. Therefore, $u(x,y) \equiv 0$ in $\overline{\Delta}_3$ because of the uniqueness of the solution of Goursat's problem [11, 12] to the wave equation.

One can prove by the coherent continuation of this process that $u(x,y) \equiv 0$ in all $\tilde{\Omega}_k^-$ $(k = 2, \dots, n)$. The theorem is proved. □

## 4 Existence of the solution of Problem S

Primarily, we shall consider the following auxiliary problem.

**Problem S$^+$.** Find the harmonic function $u(x,y) \in C^2(\Omega^+) \cap C(\overline{\Omega}^+)$ satisfying the boundary value conditions

$$u(x,0) = \tau(x), \quad u(x,b) = \varphi(x), \quad 0 \leqslant x \leqslant a, \tag{24}$$

$$u(a,y) = \psi_a(y), \quad 0 \leqslant y \leqslant b, \tag{25}$$

and the non-local condition

$$u_x(a,y) - u_x(0,y) = \nu(y), \quad 0 < y < b. \tag{26}$$

Here $\tau$, $\varphi$, $\psi_a$ and $\nu$ are given continuous functions.

**Theorem 2.** *Let $\tau$ and $\varphi \in C^1[0,a]$, $\psi_a \in C^1[0,b]$ and $\nu \in C[0,b] \cup C^1(0,b)$, and let the following compatibility conditions are fulfilled*:

$$\tau(a) = \psi_a(0), \qquad \varphi(a) = \psi_a(b), \tag{27}$$

$$\tau'(a) - \tau'(0) = \nu(0), \qquad \varphi'(a) - \varphi'(0) = \nu(b). \tag{28}$$

*Then there exists an unique solution of Problem* S$^+$ *from the class of functions, which are differentiable everywhere in the domain $\Omega^+$ except, may be, its angular points.*

*Proof.* Let $u(x,y)$ be the solution of Problem S$^+$. Introduce the following functions:

$$u^+(x,y) = \frac{1}{2}\big[u(x,y) + u(a-x,y)\big], \tag{29}$$

$$u^-(x,y) = \frac{1}{2}\big[u(x,y) - u(a-x,y)\big]. \tag{30}$$

Obviously, both of them are harmonic in $\Omega^+$ and the equalities

$$u^+(x,y) = u^+(a-x,y), \qquad u^-(x,y) = -u^-(a-x,y)$$

hold for all $x \in [0,a]$ and for all $y \in [0,b]$. Taking in account the conditions (24)–(26), we obtain after straightforward calculation that

$$u^+(x,0) = \tau_1(x), \quad u^+(x,b) = \varphi_1(x), \quad 0 \leqslant x \leqslant a, \tag{31}$$

$$u_x^+(0,y) = -\frac{1}{2}\nu(y), \quad u_x^+(a,y) = \frac{1}{2}\nu(y), \quad 0 < y < b; \tag{32}$$

$$u^-(x,0) = \tau_2(x), \quad u^-(x,b) = \varphi_2(x), \quad 0 \leqslant x \leqslant a, \tag{33}$$

$$u^-(0,y) = u^+(0,y) - \psi_a(y), \quad u^-(a,y) = \psi_a(y) - u^+(0,y), \quad 0 \leqslant y \leqslant b, \tag{34}$$

where

$$2\tau_1(x) = \tau(x) + \tau(a-x), \qquad 2\tau_2(x) = \tau(x) - \tau(a-x),$$
$$2\varphi_1(x) = \varphi(x) + \varphi(a-x), \qquad 2\varphi_2(x) = \varphi(x) - \varphi(a-x).$$

Notice, the functions $u^+$ and $u^-$ are related only by conditions (34). That yields the possibility to reduce the Problem $\mathrm{S}^+$ into two problems, which can be solved one-by-one. So, firstly we shall look for harmonic in $\Omega^+$ function $u^+$ satisfying the local initial and boundary value conditions (31), (32), and thereafter we shall solve problem (33), (34) with respect to harmonic in $\Omega^+$ function $u^-$, assuming that $u^+$ and $u^- \in C(\overline{\Omega}^+)$.

In the first step, let us consider for harmonic in $\Omega^+$ function $v \in C(\overline{\Omega}^+)$ the following Dirichlet type problem:

$$v(x,0) = \tau_1'(x), \quad v(x,b) = \varphi_1'(x), \quad 0 \leqslant x \leqslant a,$$
$$v(0,y) = -\frac{1}{2}\nu(y), \quad v(a,y) = \frac{1}{2}\nu(y), \quad 0 \leqslant y \leqslant b.$$

Observe that, according to compatibility conditions (28), this problem is well-possed in Hadamard's sense, i.e., it has the solution $v \in C^2(\Omega^+) \cap C(\overline{\Omega}^+)$, which is stable with respect to the norm $\|v\|_{C(\overline{\Omega}^+)} = \max_{\overline{\Omega}^+}|v|$. Moreover, because of the maximum principle, the estimate

$$\|v\|_{C(\overline{\Omega}^+)} \leqslant M = \max\left\{\|\tau_1'\|_{C[0,a]}, \|\varphi_1'\|_{C[0,a]}, \frac{1}{2}\|\nu\|_{C[0,b]}\right\}$$

hold. That yields the uniqueness of the solution $v$, obviously.

We shall show that solution $u^+$ of problem (31), (32) can be obtained from the differential equation

$$v = u_x^+. \tag{35}$$

It is easily seen that the general solution of equation (35) is of the shape

$$u^+(x,y) = \int_0^x v(\xi,y)\,\mathrm{d}x + \omega(y).$$

Here we assume that $\omega \in C^2(0,b) \cap C^1[0,b]$.

Putting the obtained expression of $u^+$ into Laplace's equation $u^+_{xx} + u^+_{yy} = 0$ and taking in account the relation $v_x = u^+_{xx}$, we get for the function $\omega$ the equation

$$\omega''(y) = -v_x(0, y),$$

the general solution of which can be presented by the formula

$$\omega(y) = c_1 y + c_2 - \int\limits_0^y v_x(0, \eta)\,\mathrm{d}\eta.$$

Hence,

$$u^+(x, y) = \int\limits_0^x v(\xi, y)\,\mathrm{d}\xi - \int\limits_0^y v_x(0, \eta)\,\mathrm{d}\eta + c_1 y + c_2.$$

Then taking in account conditions (31), we get by straightforward calculation the following expression of the solution of problem (31), (32):

$$\begin{aligned} u^+(x, y) = {} & \int\limits_0^x v(\xi, y)\,\mathrm{d}\xi - \int\limits_0^y v_x(0, \eta)\,\mathrm{d}\eta + \frac{y}{b}\int\limits_0^b v_x(0, \eta)\,\mathrm{d}\eta \\ & + \frac{y}{b}\big[\varphi_1(0) - \tau_1(0)\big]y + \tau_1(0). \end{aligned} \tag{36}$$

The uniqueness of the solution $u^+(x, y)$ follows from the following standard reasoning. If $w \in C^2(\Omega^+) \cap C(\overline{\Omega}^+)$ is any nonconstant function harmonic in $\Omega^+$ and satisfying the homogenous conditions of the type (31), (32)

$$\begin{aligned} w(x, 0) = w(x, b) = 0, &\quad 0 \leqslant x \leqslant a, \\ w_x(0, y) = w_x(a, y) = 0, &\quad 0 < y < b, \end{aligned}$$

then this function can't attain any nontrivial extremum in the rectangle $\Omega^+$ and on the both lower and upper bases of it. By the Zaremba–Giraud principle it can't attain any positive maximum or an negative minimum on the lateral sides $x = 0$ and $x = a$ $(0 < y < b)$ of $\Omega^+$. Further, an nontrivial extremum can be not attained at the corners of this rectangle because it is equal to zero at them. Thus, $w(x, y) \equiv 0$ in $\Omega^+$.

Next, we are ready to undertake the Dirichlet problem (33), (34) for the harmonic function $u^-$, whereas the function $u^+(0, y)$ is known already, i.e., it can be determined by (36). The existence of the solution $u^- \in C^2(\Omega^+) \cap C(\overline{\Omega}^+)$ of this problem follows by virtue of the continuity and of the compatibility of the boundary value conditions (33), (34). The uniqueness of this solution follows in accordance with the maximum principle for the harmonic functions.

Finally, we get the solution $u \in C^2(\Omega^+) \cap C^1(\Omega^+_0) \cap C(\overline{\Omega}^+)$ of Problem $S^+$ from (29) and (30) by the formula

$$u(x, y) = u^+(x, y) + u^-(x, y), \tag{37}$$

where by $\Omega^+_0$ is denoted the domain $\overline{\Omega}^+$ without its all angular points.

The inclusion $u \in C^1(\Omega_0^+)$ follows from the smoothness of boundary functions in (31)–(34).

So, the proof of the theorem is complete. $\square$

Theorem 2 yields the following.

**Corollary.** *Determine the functions $\tau$ and $\nu$ in* (24)*) and* (26) *as follows*:

$$\tau(x) \textit{ is right-hand side of } (15), \qquad \nu(y) = -\mu'(y).$$

*Assume that the conditions of Theorem* 2 *are fulfilled. Then, according to this theorem, under conditions* (i)–(iii) *of Theorem* 1*, there exists the unique function $u^+ \in C^2(\Omega^+) \cap C^1(\Omega_0^+) \cap C(\overline{\Omega}^+)$ harmonic in $\Omega^+$ and satisfying the nonlocal condition*

$$\int_0^a u_y^+(x,y)\,\mathrm{d}x = \mu(y), \quad 0 < y < b,$$

*and the following local conditions*:

$$u^+(x,0) = \tau(x), \quad u^+(x,b) = \varphi(x), \quad 0 \leqslant x \leqslant a.$$

*Let us take note that $u^+(x,0) \in C^1[0,b]$ and $u_y(0,x) \in C^1(0,b)$, but we have any guarantee that $u_y(x,0)$ is continuous at the points $x = a$ and $x = b$.*

Let the functions $\tau$ and $\nu$ be the same as in Corollary. Now we shall prove the final theorem.

**Theorem 3.** *Let $\mu \in C^1[0,b] \cup C^2(0,b)$ and*

$$\varphi, \varphi_n \in C^1[0,a], \qquad \psi_0 \in C^1[-na,0], \qquad \psi_a \in C^1[-na,b], \tag{38}$$

*and let the following compatibility conditions* (27)*,* (28) *be fulfilled. Then there exists the unique solution $u$ of Problem* S *in the class of the functions $u \in C^1(\Omega_0^+)$, which satisfy conditions* (i)–(iii) *of Theorem* 1.

*Proof.* Note that $\tau \in C^1[0,a]$ under conditions (38). Let $u^+$ be the harmonic in $\Omega^+$ function from Corollary. Notice that $u_y^+(x,0) \in C^1(0,a) \cap C[0,a]$ and $u_y^+(x,0) \in L(0,a)$ due to assumption (ii), and remember that $\Delta$ is characteristic triangle defined above. Then the wave-function

$$u^-(x,y) = \frac{\tau(x-y) + \tau(x+y)}{2} + \frac{1}{2}\int_{x-y}^{x+y} u_y^+(s,0)\,\mathrm{d}s \tag{39}$$

is the regular in $\Delta$ solution of equation (9) from the class $C^2(\Delta_1) \cap C(\overline{\Delta}_1)$. It satisfies the initial conditions

$$u^-(x,0) = \tau(x), \quad 0 \leqslant x \leqslant a,$$
$$\partial u_y^-(x,0) = u_y^+(x,0), \quad 0 < x < a,$$

obviously. Hence, the function

$$u(x,y) = \begin{cases} u^+(x,y), & (x,y) \in \Omega^+, \\ u^-(x,y), & (x,y) \in \Delta, \end{cases}$$

is the regular in $\Omega^+ \cup \Delta$ solution of the class $C^2(\Omega^+ \cup \Delta) \cap C^1(\Omega) \cap C(\overline{\Omega})$ of equation (9) satisfying nonlocal condition (11) and the local boundary value conditions

$$\begin{aligned} u(x,b) &= \varphi(x), \quad 0 \leqslant x \leqslant a, \\ u(a,y) &= \psi_a(y), \quad 0 \leqslant y \leqslant b. \end{aligned}$$

Let us consider the traces

$$\gamma_1(x) := u_1(x,x) = \frac{\tau(2x)+\tau(0)}{2} + \frac{1}{2}\int_0^{2x} u_y^+(s,0)\,\mathrm{d}s, \quad 0 \leqslant x \leqslant \frac{a}{2},$$

and

$$\gamma_2(x) := u_1(x,x-a) = \frac{\tau(2x-a)+\tau(a)}{2} + \frac{1}{2}\int_{2x-a}^{a} u_y^+(s,0)\,\mathrm{d}s, \quad \frac{a}{2} \leqslant x \leqslant a.$$

of function (39) on the corresponding segments $0 \leqslant x \leqslant a/2$ and $a/2 \leqslant x \leqslant a$ of the characteristics $l_1$ and $l_2$. Both functions $\gamma_1$ and $\gamma_2$ are continuous on the mentioned intervals, but both of them can be not differentiable because of the possible discontinuity of the function $u_y^+(x,0)$ at the points $x=0$ and $x=a$. Then the wave-functions

$$u_1^-(x,y) = \gamma_1\left(\frac{x-y}{2}\right) - \gamma_1\left(\frac{-x-y}{2}\right) + \psi_0(x+y), \quad 0 \leqslant x \leqslant \frac{a}{2},$$

and

$$u_2^-(x,y) = \gamma_2\left(\frac{x+y+a}{2}\right) - \gamma_2\left(\frac{y-x+3a}{2}\right) + \psi_a(y-x+a), \quad \frac{a}{2} \leqslant x \leqslant a,$$

represent the generalized solutions in $\Delta_1$ and $\Delta_2$ of the corresponding Darboux problems

$$\begin{aligned} u_1^-(0,y) &= \psi_0(y), \quad -a \leqslant y \leqslant 0, \\ u_1^-(x,-x) &= \gamma_1(x), \quad 0 \leqslant x \leqslant \frac{a}{2}, \end{aligned}$$

and

$$\begin{aligned} u_2^-(a,y) &= \psi_a(y), \quad -a \leqslant y \leqslant 0, \\ u_2^-(x,x-a) &= \gamma_2(x), \quad \frac{a}{2} \leqslant x \leqslant a, \end{aligned}$$

to equation (9).

Notice, the functions $u_1^-(x,y)$ and $u_2^-(x,y)$ are continuous in $\overline{\Delta}_2$ and $\overline{\Delta}_3$, consequently. Therefore, one can define the functions

$$\begin{aligned}\vartheta_1(x) &:= u_2^-(x, x-a), \quad 0 \leqslant x \leqslant \frac{a}{2},\\ \vartheta_2(x) &:= u_2^-(x, -x), \qquad \frac{a}{2} \leqslant x \leqslant a,\end{aligned}$$

which are continuous on the isosceles sides of the triangle $\Delta_3$ lying on the characteristics $l_1$ and $l_2$. That enable to get the generalized solution

$$u_3^-(x,y) = \vartheta_1\left(\frac{x-y}{2}\right) + \vartheta_2\left(\frac{x+y+a}{2}\right) - \vartheta_1\left(\frac{a}{2}\right) \tag{40}$$

of the Goursat's problem

$$\begin{aligned}u_3^-(x, x-a) &= \vartheta_1(x), \quad 0 \leqslant x \leqslant \frac{a}{2},\\ u_3^-(x, -x) &= \vartheta_2(x), \quad \frac{a}{2} \leqslant x \leqslant a.\end{aligned}$$

So, we obtain the generalized solution

$$\tilde{u}_1^-(x,y) = \begin{cases} u_1^-(x,y), & (x,y) \in \Delta_1,\\ u_2^-(x,y), & (x,y) \in \Delta_2,\\ u_3^-(x,y), & (x,y) \in \Delta_3,\end{cases}$$

to equation (9) in the domain $\tilde{\Omega}_1^- \setminus \Delta$, which is continuous in the closure of this domain and such that

$$\tilde{u}_1^-(0,y) = \psi_0(y), \quad \tilde{u}_1^-(a,y) = \psi_a(y), \quad -a \leqslant y \leqslant 0.$$

If $n = 1$, then taking in account the denotation $\tau(x) = \psi_a(x-a) + \psi_0(-x) - \varphi_1(a-x)$, we obtain from (40) that

$$\begin{aligned}\tilde{u}_1^-(x,-a) &= \vartheta_1\left(\frac{x+a}{2}\right) + \vartheta_2\left(\frac{x}{2}\right) - \vartheta_1\left(\frac{a}{2}\right)\\ &= \psi_a(-x) + \psi_0(x-a) - \tau(a-x) = \varphi_1(x), \quad 0 \leqslant x \leqslant a.\end{aligned} \tag{41}$$

Thus, in this case, the prove of the theorem is complete.

If $n > 1$, then we get by formula (40) the continuous extension of generalized solution into closure $\overline{\Delta}^{(1)}$ of the the upper characteristic triangle

$$\Delta^{(1)} = \bigl\{(x,y)\colon 0 < x < a,\ -x-a < y < x-2a\bigr\} \subset \tilde{\Omega}_1^-.$$

Thereafter we compose coherently in the rest domains $\tilde{\Omega}_k^-$, $k = 2, \dots, n$, the corresponding generalized solutions $\tilde{u}_k^-(x,y)$ satisfying the boundary value conditions

$$\tilde{u}_k^-(0,y) = \psi_0(y), \quad \tilde{u}_k^-(a,y) = \psi_a(y), \quad -(k-1)a \leqslant y \leqslant -ka,$$

by the continuous extension in the same way as above. The validity of the boundary value condition $\tilde{u}_n^-(x,-a) = \varphi_n(x)$, $0 \leqslant x \leqslant a$, is implicated by the composition of the generalized solutions $\tilde{u}_k^-(x,y)$, $k = 1,\dots,n-1$, given before.

The theorem is proved. $\square$

## References


1. P.Ya. Polubarinova-Kochina, *Theory of the Groundwater Motion*, Nauka, Moscow, 1977 (in Russian).
2. N.I. Ionkin, Solution of the boundary value problem of the theory of heat conductivity with nonclassical boundary condition, *Differ. Uravn.*, **13**(2):294–304, 1977 (in Russian).
3. N.I. Ionkin, E.I. Moiseev, On the problem for heat conductivity equation with the point-to-point type boundary conditions, *Differ. Uravn.*, **15**(7):294–304, 1979 (in Russian).
4. A.M. Nakhushev, *Equations of the Mathematical Biology*, Vysshaya Shkola, Moscow, 1995 (in Russian).
5. L.I. Serbina, On a boundary value problem for Lavrentiev–Bitsadze equation with the nonlocal Samarskii condition, *Doklady Adygskoi (Cherkesskoi) Mezhdunarodnoi Akademii Nauk*, **3**(2):6–9, 1998 (in Russian).
6. L.I. Serbina, On an essentially new boundary value problem for Lavrentiev–Bitsadze equation, *Proceedings of the Institute of Mathematics of the Belorussian National Academy of Sciences*, **10**:142–146, 2001 (in Russian).
7. L.I. Serbina, *Nonlocal Mathematical Models of a Transfer in the Aquifer Systems*, Nauka, Moscow, 2007 (in Russian).
8. L.I. Serbina, A problem for the linearized Boussinesq's equation with a nonlocal Samarskii condition, *Differ. Equ.*, **38**(8):1187–1194, 2002.
9. A.M. Nakhushev, *Loaded Equations and Their Applications*, Nauka, Moscow, 2012 (in Russian).
10. A.V. Bitsadze, *Boundary Value Problems for the Elliptic Equations of Second Order*, Nauka, Moscow, 1966 (in Russian).
11. R. Courant, *Partial Differential Equations*, Interscience Publishers, New York, London, 1962.
12. M. Kszyźanski, *Równania różniczkowe zhąnstkowe rzędu drugiego, Cz. II*, Panstwowe wydavnictwo naukowe, Warszawa, 1962.