http://dx.doi.org/10.15388/NA.2014.3.13

The Fučík spectrum for nonlocal BVP with Sturm–Liouville boundary condition*

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Received: 16 February 2014 / **Revised:** 16 March 2014 / **Published online:** 30 June 2014

Abstract. Boundary value problem of the form $x'' = -\mu x^+ + \lambda x^-$, $\alpha x(0) + (1-\alpha)x'(0) = 0$, $\int_0^1 x(s)\,\mathrm{d}s = 0$ is considered, where $\mu, \lambda \in \mathbb{R}$ and $\alpha \in [0,1]$. The explicit formulas for the spectrum of this problem are given and the spectra for some α values are constructed. Special attention is paid to the spectrum behavior at the points close to the coordinate origin.

Keywords: boundary value problem, Sturm–Liouville boundary condition, integral condition, Fučík spectrum.

1 Introduction

This paper studies the Fučík type spectrum for the problem

$$x'' = -\mu x^+ + \lambda x^- \tag{1}$$

with the conditions

$$\alpha x(0) + (1-\alpha)x'(0) = 0, \qquad \int\limits_0^1 x(s)\,\mathrm{d}s = 0. \tag{2}$$

Problems of this type have long history.

The classical Fučík problem is equation (1) with the boundary conditions

$$x(0) = 0, \qquad x(1) = 0. \tag{3}$$

The spectrum of problem (1), (3) was given in [1]. It is a set of such points (μ, λ) that the problem has a nontrivial solution.

In the works of S. Fučík, this spectrum was studied first as an object related to "slightly" nonlinear problems. The interest in these type of problems grew also in connection with the theory of suspension bridges (see e.g. [2]). From the mathematical point of view the Fučík equation became a source of numerous investigations generalizing and refining the results by Fučík.

*This research was supported by Latvian Council of Science through the project 345/2012.

The Fučík equation with Sturm–Liouville conditions has similar structure of spectrum and the description of it can be found in [3] or [4].

In the last time, great attention was paid to problems involving differential equations with asymptotically asymmetric parts. The simplest equation of this type is the Fučík equation. Later nonlinear equations of the type $x'' + f(x^+) - g(x^-) = \varphi(t, x, x')$ were considered in [5], where f and g are positive valued continuous functions, φ is a bounded continuous nonlinearity (often $\varphi = 0$), λ and μ are spectral parameters. Notions of spectral surfaces were introduced in [6] and [7]. Solvability of Dirichlet and Neumann boundary value problems was considered in [8].

Completely different spectrum was obtained for the problem composed of equation (1) with nonlocal conditions

$$x(0) = 0, \qquad \int\limits_0^1 x(s)\,\mathrm{d}s = 0. \tag{4}$$

It is not yet a countable set of hyperbolas but rather a wave spreading along the bisectrix of the first quadrant in the parameter μ, λ-plane. The spectrum of problem (1), (4) was given in [9].

The progress in this direction can be traced by [10] and [11]. Also another type of nonlocal boundary conditions was considered by the author in the work [12].

A recent author's paper [13] deals with the problem composed of equation (1) with conditions

$$x'(0) = 0, \qquad \int\limits_0^1 x(s)\,\mathrm{d}s = 0. \tag{5}$$

Problem (1), (2), which contains conditions (4) and (5), has been considered in this paper.

Below the structure of the spectrum for problem (1), (2) has been described, giving a full set of formulas and discussing some arising problems and differences of this spectrum and previously known ones.

In the case $\mu = \lambda$, problem (1), (2) is Sturm–Liouville problem with one classical third type boundary condition and nonlocal integral condition. The spectrum for such problem was investigated by M. Sapagovas and other lithuanian scientists in [14, 15]. The nonlocal conditions (2) are particular case of more general nonlocal conditions [16, 17], where real spectrum and Green's function were investigated for stationary problem with nonlocal conditions as linear functionals.

This paper is organized as follows. Some basic notations are given in Section 2. In order to describe the properties of the spectrum for problem (1), (2), in Section 3 some information about the spectrum for the problem composed of equation (1) with conditions

$$\alpha x(0) + (1 - \alpha)x'(0) = 0, \qquad x(1) = 0 \tag{6}$$

is provided. In Section 4, the results on the Fučík type problem (1), (2) are presented.

2 Some basic notations

Consider problem (1), (6).

The first result describes decomposition of the spectrum into branches F_i^+ and F_i^- $(i = 0, 1, 2, \dots)$.

Since the spectrum of problem (1), (6) coincides with the classical one in the case $\alpha = 1$ than the notation compatible with the classical spectrum has been chosen. Therefore, branches, which relate to solutions $x(t)$ with $x(0) < 0$, $x'(0) > 0$, have been denoted by F_i^+, but F_i^- is used for branches related to solutions with $x(0) > 0$, $x'(0) < 0$.

The following proposition is valid.

Proposition 1. *The spectrum consists of the set of curves*

$$\begin{aligned} F_i^+ = \bigl\{(\mu,\lambda) \bigm| x'(0) > 0, x(0) < 0, &\ \textit{the nontrivial solution of problem } (1),(6)\ x(t) \\ &\textit{has exactly } i \textit{ zeroes in } (0,1)\bigr\}, \\ F_i^- = \bigl\{(\mu,\lambda) \bigm| x'(0) < 0,\ x(0) > 0, &\ \textit{the nontrivial solution of problem } (1),(6)\ x(t) \\ &\textit{has exactly } i \textit{ zeroes in } (0,1)\bigr\}. \end{aligned}$$

Remark 1. If $\alpha = 1$, then the classical Fučík problem is considered. The notations F_i^+ and F_i^- are used (see e.g. [1]) to distinguish branches corresponding to the solutions $x(t)$ with i zeroes in the interval $(0, 1)$ which have positive derivative at the beginning $x'(0) > 0$ and negative one $x'(0) < 0$.

If $\alpha = 0$, then the spectrum of problem (1), (5) is considered. The notations F_i^+ and F_i^- are used to distinguish branches corresponding to the solutions $x(t)$ with i zeroes in the interval $(0, 1)$ which are negative at the beginning $x(0) < 0$ and positive ones $x(0) > 0$.

In our notation, we use three lower indices in description of some branches. This complication may be justified by the following reasons. Usually only positive μ and λ are considered. The additional first "–" sign at the lower index in a description of a branch shows that the respective μ values is negative and the additional second "–" sign shows that the respective λ values is negative (the corresponding points of spectrum are located in the third quadrant). For instance, the notation $F^+_{1,-,-}$ refers to solutions of the equation $x'' = -\mu x^+ + \lambda x^-$, which have one zero in the interval $(0, 1)$ and $\mu < 0$, $\lambda < 0$. But the notation $F^+_{0,\mathbb{R},-}$ refers to solutions of the equation $x'' = -\mu x^+ + \lambda x^-$ without zeroes in the interval $(0, 1)$ with $\mu \in \mathbb{R}$ and $\lambda < 0$. Similar notations of the branches were used in the other author's works [10–12] devoted to the Fučík type problems with nonlocal (integral) condition.

3 The spectra with Dirichlet condition

Consider problem (1), (6).

In the next two theorems, the Fučík spectrum for this problem is defined for $\alpha = 1$ and $\alpha = 0$, respectively.

Theorem 1. (*See* [1].) *The Fučík spectrum for problem* (1), (6) *for* $\alpha = 1$ *consists of the branches given by*

$$
\begin{aligned}
F_0^+ &= \{(\mu,\lambda) \mid \mu = \pi^2,\ \lambda \in \mathbb{R}\},\\
F_{2i-1}^+ &= \left\{(\mu,\lambda) \,\middle|\, i\frac{\pi}{\sqrt{\mu}} + i\frac{\pi}{\sqrt{\lambda}} = 1\right\},\\
F_{2i}^+ &= \left\{(\mu,\lambda) \,\middle|\, (i+1)\frac{\pi}{\sqrt{\mu}} + i\frac{\pi}{\sqrt{\lambda}} = 1\right\},\\
F_j^- &= \{(\mu,\lambda) \mid (\lambda,\mu) \in F_j^+\},
\end{aligned}
$$

where $i = 1, 2, \dots$, $j = 0, 1, \dots$.

Theorem 2. (*See* [13].) *The Fučík spectrum for problem* (1), (6) *for* $\alpha = 0$ *consists of the branches given by*

$$
\begin{aligned}
F_0^+ &= \left\{(\mu,\lambda) \,\middle|\, \mu \in \mathbb{R},\ \lambda = \frac{\pi^2}{4}\right\},\\
F_{2i-1}^+ &= \left\{(\mu,\lambda) \,\middle|\, i\frac{\pi}{\sqrt{\mu}} + \frac{2i-1}{2}\frac{\pi}{\sqrt{\lambda}} = 1\right\},\\
F_{2i}^+ &= \left\{(\mu,\lambda) \,\middle|\, i\frac{\pi}{\sqrt{\mu}} + \frac{2i+1}{2}\frac{\pi}{\sqrt{\lambda}} = 1\right\},\\
F_j^- &= \{(\mu,\lambda) \mid (\lambda,\mu) \in F_j^+\},
\end{aligned}
$$

where $i = 1, 2, \dots$, $j = 0, 1, \dots$.

Now consider problem (1), (6) for $\alpha \in (0,1)$.

First, consider the solutions of problem (1), (6) without zeroes in the interval $(0,1)$. These solutions correspond to the spectrum branch $F_0^{\pm}$.

The solutions of the problem (1), (6) without zeroes in the interval $(0,1)$ may be of three types. The first type of them can be analytically described with the sine function, the next one with the linear function and the third type of solutions of problem (1), (6) without zeroes in the interval $(0,1)$ can be analytically described with the hyperbolic sine function. Let us denote the branches of the spectrum of the problem (1), (6), which correspond to the solutions of the first type with $F_{0,\mathbb{R},+}^+$ and $F_{0,+,\mathbb{R}}^-$, similarly $F_{0,\mathbb{R},0}^+$ and $F_{0,0,\mathbb{R}}^-$ are the branches corresponding to the second type of solutions and $F_{0,\mathbb{R},-}^+$ and $F_{0,-,\mathbb{R}}^-$ are the branches corresponding to the third type of solutions.

Lemma 1. *The branches* $F_{0,\mathbb{R},+}^+$ *and* $F_{0,+,\mathbb{R}}^-$ *of the spectrum for problem* (1), (6) *exist only for* $\alpha < 1/2$.

Proof. Consider the solutions of the problem (1), (6) without zeroes in the interval $(0,1)$ for positive λ values. We obtain that the considering problem reduces to the problem $x'' = -\lambda x$, $\alpha x(0) + (1-\alpha)x'(0) = 0$, $x(1) = 0$.

The solution of the problem is the function $x(t) = A\sin(\sqrt{\lambda}(t-\arctan((1-\alpha)/\alpha)))$. In view of the boundary conditions, we obtain the equation

$$\frac{\tan\sqrt{\lambda}}{\sqrt{\lambda}} = \frac{1-\alpha}{\alpha}. \tag{7}$$

Taking into account that the solutions of the problem (1), (6) have no zeroes in the interval $(0,1)$, follows that $\lambda < \pi^2/4$.

Consider the left side of equation (7) as a function. The range of values of this function is $(1,+\infty)$ for $\lambda \in (0,\pi^2/4)$.

Then consider the right side of equation (7). This expression is more than one only for $\alpha < 1/2$. This proves lemma. The prove for $F^-_{0,+,\mathbb{R}}$ is similar. □

Lemma 2. *The branches $F^+_{0,\mathbb{R},0}$ and $F^-_{0,0,\mathbb{R}}$ of the spectrum of problem* (1), (6) *exist only for $\alpha = 1/2$.*

Proof. Consider the solutions of the problem (1), (6) without zeroes in the interval $(0,1)$ for $\lambda = 0$. We obtain that the considered problem reduces to the problem $x'' = 0$, $\alpha x(0) + (1-\alpha)x'(0) = 0$, $x(1) = 0$. The solution of the problem is the function $x(t) = A(t-(1-\alpha)/\alpha)$. In view of condition, we obtain that $\alpha = 1/2$. Similarly for $F^-_{0,0,\mathbb{R}}$. This proves lemma. □

Remark 2. The branches $F^+_{0,\mathbb{R},0}$ and $F^-_{0,0,\mathbb{R}}$ of the spectrum for problem (1), (6) coinside with axes.

Lemma 3. *The branches $F^+_{0,\mathbb{R},-}$ and $F^-_{0,-,\mathbb{R}}$ of the spectrum of problem* (1), (6) *exist only for $\alpha > 1/2$.*

Proof. Consider the solutions of the problem (1), (6) without zeroes in the interval $(0,1)$ for $\lambda < 0$. We obtain that the considering problem reduces to the problem $x'' = \delta x$, $\alpha x(0) + (1-\alpha)x'(0) = 0$, $x(1) = 0$, where $\delta = -\lambda > 0$.

The solution of the problem is the function $x(t) = A\sinh(\sqrt{\delta}(t-\operatorname{arctanh}((1-\alpha)/\alpha)))$. In view of the boundary conditions, it follows

$$\frac{\tanh\sqrt{\delta}}{\sqrt{\delta}} = \frac{1-\alpha}{\alpha}. \tag{8}$$

Consider the left side of equation (8) as a function. The range of values of this function for $\delta > 0$ is $(0,1)$. The expression on the right side of equation (8) is less than one only for $\alpha > 1/2$. This proves lemma for $F^+_{0,\mathbb{R},-}$. The prove for $F^-_{0,-,\mathbb{R}}$ is similar. □

Now consider the solutions of problem (1), (6) with one zero in the interval $(0,1)$. Let us denote this zero with τ. These solutions correspond to the spectrum branch $F_1^{\pm}$.

Using the geometrical type arguments we obtain that the solutions of problem (1), (6) with one zero in the interval $(0,1)$ may be of the three different types.

The first type of them can be analytically described with the sine function in the interval $(0, \tau)$ and with the sine function in the interval $(\tau, 1)$. The notations of corresponding branches are $F^{\pm}_{1,+,+}$.

The second type of the solutions can be analytically described with the linear function in the interval $(0, \tau)$ and the with sine function in the interval $(\tau, 1)$. The notations of such (μ, λ) points are $F^{+}_{1,+,0}$ and $F^{-}_{1,0,+}$.

The third type of them can be analytically described with the hyperbolic sine function in the interval $(0, \tau)$ and with the sine function in the interval $(\tau, 1)$. The branches $F^{+}_{1,+,-}$ and $F^{-}_{1,-,+}$ correspond to the such (μ, λ) points.

The next theorem is valid.

Theorem 3. *The Fučík spectrum for problem* (1), (6) *for* $\alpha \in (0, 1)$ *consists of the branches* (*if these branches exist for corresponding* α *value*) *given by*

$$\begin{aligned}
F^{+}_{0,\mathbb{R},+} &= \left\{(\mu, \lambda) \;\middle|\; \mu \in \mathbb{R},\ \frac{\tan\sqrt{\lambda}}{\sqrt{\lambda}} = \frac{1-\alpha}{\alpha},\ \lambda \in \left(0, \frac{\pi^2}{4}\right)\right\},\\
F^{+}_{0,\mathbb{R},0} &= \bigl\{(\mu, \lambda) \bigm| \mu \in \mathbb{R},\ \lambda = 0\bigr\},\\
F^{+}_{0,\mathbb{R},-} &= \left\{(\mu, \lambda) \;\middle|\; \mu \in \mathbb{R},\ \frac{\tanh\sqrt{-\lambda}}{\sqrt{-\lambda}} = \frac{1-\alpha}{\alpha},\ \lambda < 0\right\},\\
F^{+}_{1} &= F^{+}_{1,+,+} \cup F^{+}_{1,+,0} \cup F^{+}_{1,+,-},\\
F^{+}_{1,+,+} &= \left\{(\mu, \lambda) \;\middle|\; \frac{\pi}{\sqrt{\mu}} + \frac{\arctan(\frac{1-\alpha}{\alpha}\sqrt{\lambda})}{\sqrt{\lambda}} = 1\right\},\\
F^{+}_{1,+,0} &= \left\{(\mu, \lambda) \;\middle|\; \frac{\pi}{\sqrt{\mu}} + \frac{1-\alpha}{\alpha} = 1,\ \lambda = 0\right\},\\
F^{+}_{1,+,-} &= \left\{(\mu, \lambda) \;\middle|\; \frac{\pi}{\sqrt{\mu}} + \frac{\operatorname{arctanh}(\frac{1-\alpha}{\alpha}\sqrt{-\lambda})}{\sqrt{-\lambda}} = 1\right\},\\
F^{+}_{2i} &= \left\{(\mu, \lambda) \;\middle|\; i\frac{\pi}{\sqrt{\mu}} + \frac{\arctan(\frac{1-\alpha}{\alpha}\sqrt{\lambda})}{\sqrt{\lambda}} + i\frac{\pi}{\sqrt{\lambda}} = 1\right\},\\
F^{+}_{2i+1} &= \left\{(\mu, \lambda) \;\middle|\; (i+1)\frac{\pi}{\sqrt{\mu}} + \frac{\arctan(\frac{1-\alpha}{\alpha}\sqrt{\lambda})}{\sqrt{\lambda}} + i\frac{\pi}{\sqrt{\lambda}} = 1\right\},\\
F^{-}_{j} &= \bigl\{(\mu, \lambda) \bigm| (\lambda, \mu) \in F^{+}_{j}\bigr\},
\end{aligned}$$

where $i = 1, 2, \ldots$, $j = 0, 1, \ldots$.

Proof. The proof for the branches which correspond to the solutions without zeroes follows from Lemmas 1–3.

Consider the solutions of problem (1), (6) with one zero at $t = \tau$ in the interval $(0, 1)$ and $x'(0) > 0$, $x(0) < 0$.

In the interval $(\tau, 1)$, we obtain the problem $x'' = -\mu x$, $x(\tau) = 0$, $x(1) = 0$. It is clear that $1 - \tau = \pi/\sqrt{\mu}$.

If $\lambda > 0$, then problem (1), (6) reduces to the problem

$$x'' = -\lambda x, \qquad \alpha x(0) + (1-\alpha)x'(0) = 0, \qquad x(\tau) = 0$$

in the interval $(0, \tau)$. The solution of it is $x(t) = A\sin(\sqrt{\lambda}(t-\tau))$. In view of boundary conditions, we obtain that $\tau = \arctan(((1-\alpha)/\alpha)\sqrt{\lambda})/\sqrt{\lambda}$. It follows the explicit formula for $F^+_{1,+,+}$.

If $\lambda = 0$, then problem (1), (6) reduces to the problem

$$x'' = 0, \qquad \alpha x(0) + (1-\alpha)x'(0) = 0, \qquad x(\tau) = 0$$

in the interval $(0, \tau)$. The solution of it is $x(t) = A(t-\tau)$. In view of boundary conditions, we obtain that $\tau = (1-\alpha)/\alpha$. It follows the explicit formula for $F^+_{1,+,0}$.

If $\lambda < 0$, then problem (1), (6) reduces to the problem

$$x'' = \delta x, \qquad \alpha x(0) + (1-\alpha)x'(0) = 0, \qquad x(\tau) = 0$$

(where $\delta = -\lambda > 0$) in the interval $(0, \tau)$. The solution of it is the function $x(t) = A\sinh(\sqrt{\delta}(t-\tau))$. Therefore, $\tau = \operatorname{arctanh}(((1-\alpha)/\alpha)\sqrt{\delta})/\sqrt{\delta}$. It follows the explicit formula for $F^+_{1,+,-}$.

The proof for other branches is similar. For example, consider the solutions of problem (1), (6) with $2i$ zeroes in the interval $(0, 1)$ and $x'(0) > 0$, $x(0) < 0$. Let us denote the first zero with τ_1. It is clear that $1 - \tau_1 = i\pi/\sqrt{\mu} + i\pi/\sqrt{\lambda}$ and $\tau_1 = \arctan(((1-\alpha)/\alpha)\sqrt{\lambda})/\sqrt{\lambda}$. In view of it, the explicit formula for F^+_{2i} is obtained. □

Remark 3. The $F^+_{1,+,0}$ ($F^-_{1,0,+}$) is a point on the μ axis (λ axis).

4 The spectra with integral condition

Now consider problem (1), (2).

The meaning of the notation of the spectrum for problem (1), (2) is the same as before.

Lemma 4. *The branches $F^\pm_0$ of the spectrum for problem* (1), (2) *do not exist.*

Proof. It is clear that the solution of problem (1), (2) must have at least one zero in the interval $(0, 1)$ in order to meet the second condition in (2). □

Remark 4. If $\alpha = 1$ then we obtain problem (1), (4) which was described in the author's work [10]. The spectrum of problem (1), (5) (for $\alpha = 0$ in problem (1), (2)) can be found in the work [13]. Some branches of the spectra for these two problems are depicted in Fig. 1.

4.1 About the branch related to the solutions with one zero

Consider the solutions of problem (1), (2) with one zero in the interval $(0, 1)$. Let us denote this zero with τ. These solutions correspond to the branches F^+_1 and F^-_1.

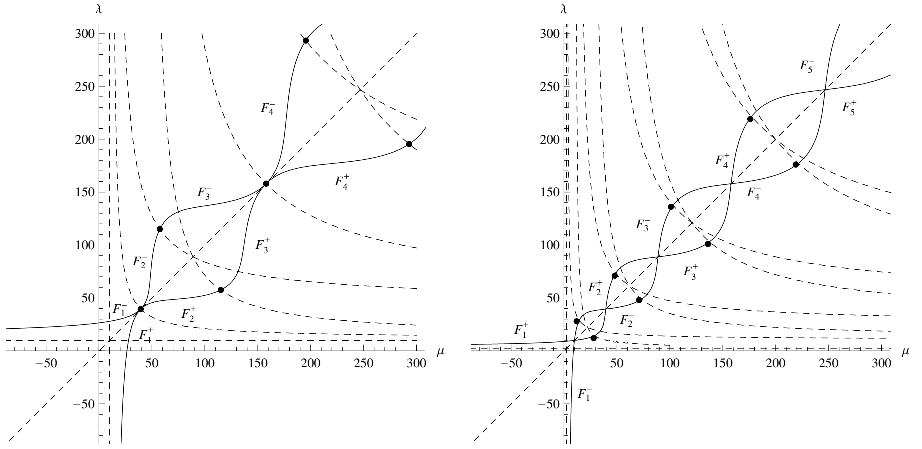

Fig. 1. The spectrum for problem (1), (4) and for problem (1), (5).

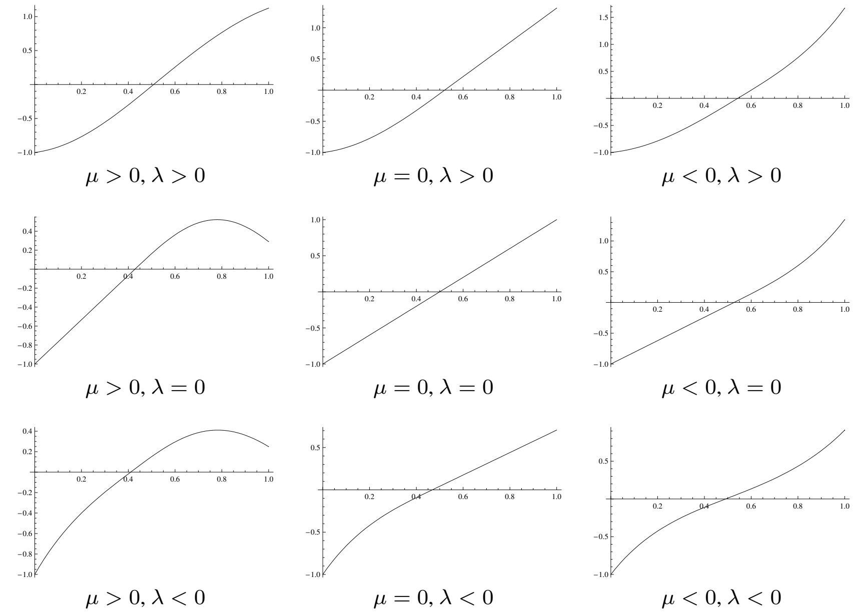

Fig. 2. The different types of solutions for problem (1), (2) with one zero in the interval $(0, 1)$.

There are nine different types of solutions for problem (1), (2) with one zero in the interval $(0, 1)$. It may be explained by the following reasons. Sine function, linear function or hyperbolic sine function in the interval $(0, \tau)$ may be continued by sine function, linear function or hyperbolic sine function in the interval $(\tau, 1)$.

All types of solutions for problem (1), (2) with one zero in the interval $(0, 1)$ are shown in Fig. 2.

Theorem 4. *The branch $F_1^{\pm}$ of the spectrum for problem* (1), (2) *for $\alpha \in (0,1)$ given by* (*if the respective part of branch exists for corresponding value of α*)

$$F^+_{1,+,+} = \biggl\{(\mu,\lambda) \Bigm| \frac{1}{\lambda} - \frac{1}{\mu} - \frac{1}{\lambda}\frac{1}{\sqrt{1+(\frac{1-\alpha}{\alpha})^2\lambda}} + \frac{1}{\mu}\cos\biggl(\sqrt{\mu} - \sqrt{\frac{\mu}{\lambda}}\arctan\biggl(\frac{1-\alpha}{\alpha}\sqrt{\lambda}\biggr)\biggr) = 0,$$
$$\frac{\arctan(\frac{1-\alpha}{\alpha}\sqrt{\lambda})}{\sqrt{\lambda}} < 1,\ \ \frac{\arctan(\frac{1-\alpha}{\alpha}\sqrt{\lambda})}{\sqrt{\lambda}} + \frac{\pi}{\sqrt{\mu}} \geqslant 1\biggr\},$$

$$F^+_{1,0,+} = \biggl\{(\mu,\lambda) \Bigm| \frac{1}{\lambda} - \frac{1}{\lambda}\frac{1}{\sqrt{1+(\frac{1-\alpha}{\alpha})^2\lambda}} - \frac{(1-\arctan(\frac{1-\alpha}{\alpha}\sqrt{\lambda}))^2}{2} = 0,$$
$$\frac{\arctan(\frac{1-\alpha}{\alpha}\sqrt{\lambda})}{\sqrt{\lambda}} < 1,\ \ \mu = 0\biggr\},$$

$$F^+_{1,-,+} = \biggl\{(\mu,\lambda) \Bigm| \frac{1}{\lambda} - \frac{1}{\mu} - \frac{1}{\lambda}\frac{1}{\sqrt{1+(\frac{1-\alpha}{\alpha})^2\lambda}} + \frac{1}{\mu}\cosh\biggl(\sqrt{-\mu} - \sqrt{-\frac{\mu}{\lambda}}\arctan\biggl(\frac{1-\alpha}{\alpha}\sqrt{\lambda}\biggr)\biggr) = 0,$$
$$\frac{\arctan(\frac{1-\alpha}{\alpha}\sqrt{\lambda})}{\sqrt{\lambda}} < 1,\ \ \mu < 0\biggr\},$$

$$F^+_{1,+,0} = \biggl\{(\mu,\lambda) \Bigm| \frac{1}{\mu} - \frac{1}{\mu}\cos\biggl(\sqrt{\mu}\biggl(1 - \frac{1-\alpha}{\alpha}\biggr)\biggr) - \frac{1}{2}\biggl(\frac{1-\alpha}{\alpha}\biggr)^2 = 0,$$
$$\frac{1-\alpha}{\alpha} < 1,\ \ \frac{1-\alpha}{\alpha} + \frac{\pi}{\sqrt{\mu}} \geqslant 1\biggr\},$$

$$F^+_{1,0,0} = \{(\mu,\lambda) \mid \mu = 0,\ \lambda = 0\},$$

$$F^+_{1,-,0} = \biggl\{(\mu,\lambda) \Bigm| \frac{1}{\mu} - \frac{1}{\mu}\cosh\biggl(\sqrt{-\mu}\biggl(1 - \frac{1-\alpha}{\alpha}\biggr)\biggr) - \frac{1}{2}\biggl(\frac{1-\alpha}{\alpha}\biggr)^2 = 0,$$
$$\frac{1-\alpha}{\alpha} < 1,\ \ \mu < 0\biggr\},$$

$$F^+_{1,+,-} = \biggl\{(\mu,\lambda) \Bigm| \frac{1}{\lambda} - \frac{1}{\mu} - \frac{1}{\lambda}\frac{1}{\sqrt{1+(\frac{1-\alpha}{\alpha})^2\lambda}} + \frac{1}{\mu}\cos\biggl(\sqrt{\mu} - \sqrt{-\frac{\mu}{\lambda}}\,\mathrm{arctanh}\biggl(\frac{1-\alpha}{\alpha}\sqrt{-\lambda}\biggr)\biggr) = 0,$$
$$\frac{\mathrm{arctanh}(\frac{1-\alpha}{\alpha}\sqrt{-\lambda})}{\sqrt{-\lambda}} < 1,\ \ \frac{\mathrm{arctanh}(\frac{1-\alpha}{\alpha}\sqrt{-\lambda})}{\sqrt{-\lambda}} + \frac{\pi}{\sqrt{\mu}} \geqslant 1\biggr\},$$

$$F^+_{1,0,-} = \Big\{(\mu,\lambda) \;\Big|\; \frac{1}{\lambda} - \frac{1}{\lambda}\frac{1}{\sqrt{1+(\frac{1-\alpha}{\alpha})^2\lambda}} - \frac{(1-\operatorname{arctanh}(\frac{1-\alpha}{\alpha}\sqrt{-\lambda}))^2}{2} = 0,$$
$$\frac{\operatorname{arctanh}(\frac{1-\alpha}{\alpha}\sqrt{-\lambda})}{\sqrt{-\lambda}} < 1,\;\; \mu = 0\Big\},$$
$$F^+_{1,-,-} = \Big\{(\mu,\lambda) \;\Big|\; \frac{1}{\lambda} - \frac{1}{\mu} - \frac{1}{\lambda}\frac{1}{\sqrt{1+(\frac{1-\alpha}{\alpha})^2\lambda}}$$
$$+ \frac{1}{\mu}\cosh\bigg(\sqrt{-\mu} - \sqrt{\frac{\mu}{\lambda}}\operatorname{arctanh}\bigg(\frac{1-\alpha}{\alpha}\sqrt{-\lambda}\bigg)\bigg) = 0,$$
$$\frac{\operatorname{arctanh}(\frac{1-\alpha}{\alpha}\sqrt{-\lambda})}{\sqrt{-\lambda}} < 1,\;\; \mu < 0\Big\},$$
$$F^-_1 = \big\{(\mu,\lambda) \;\big|\; (\lambda,\mu) \in F^+_1\big\}.$$

Proof. The ideas of the proof of all branches are similar. That is why we prove only some of them.

Consider the solutions of problem (1), (2) with one zero in the interval $(0,1)$ in the case of $\lambda > 0$.

We obtain that the problem in the interval $(0,\tau)$ reduces to the linear eigenvalue problem $x'' = -\lambda x$, $\alpha x(0) + (1-\alpha)x'(0) = 0$, $x(\tau) = 0$. The corresponding solution is $x(t) = A\sin(\sqrt{\lambda}(t-\tau))$. Therefore, $\tau = \arctan(((1-\alpha)/\alpha)\sqrt{\lambda})/\sqrt{\lambda}$ and

$$\int_0^\tau x(s)\,\mathrm{d}s = \frac{A}{\sqrt{\lambda}}\bigg(\frac{1}{\sqrt{1+(\frac{1-\alpha}{\alpha})^2\lambda}} - 1\bigg). \tag{9}$$

Now consider the solution of problem (1), (2) in the interval $(\tau,1)$ with $\mu > 0$. We obtain the problem $x'' = -\mu x$, $x(\tau) = 0$, $x'(\tau) = A\sqrt{\lambda}$ for the positive value of μ. The solution of the last problem is the function $x(t) = B\sin(\sqrt{\mu}(t-\tau))$ or $x(t) = A\sqrt{\lambda/\mu}\sin(\sqrt{\mu}(t-\tau))$.

It follows that

$$\int_\tau^1 x(s)\,\mathrm{d}s = -A\frac{\sqrt{\lambda}}{\mu}\big(\cos\big(\sqrt{\mu}(1-\tau)\big) - 1\big). \tag{10}$$

From the (9) and (10) we obtain the expression for $F^+_{1,+,+}$. In view of the structure of solution which corresponds to $F^+_{1,+,+}$, we obtain that

$$\frac{\arctan(\frac{1-\alpha}{\alpha}\sqrt{\lambda})}{\sqrt{\lambda}} < 1 \leqslant \frac{\arctan(\frac{1-\alpha}{\alpha}\sqrt{\lambda})}{\sqrt{\lambda}} + \frac{\pi}{\sqrt{\mu}}.$$

Now consider the solution of problem (1), (2) in the interval $(\tau,1)$ with $\mu = 0$. We obtain the problem $x'' = 0$, $x(\tau) = 0$, $x'(\tau) = A\sqrt{\lambda}$. The solution of the last problem is

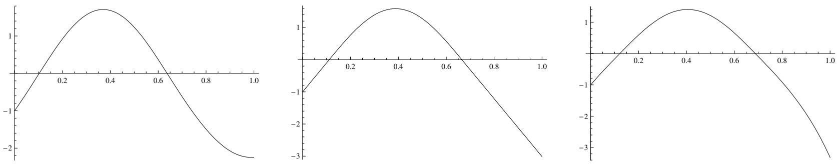

Fig. 3. The different types of solutions for problem (1), (2) with two zeroes in the interval $(0, 1)$.

the function $x(t) = A\sqrt{\lambda}(t - \tau)$. In view of it,

$$\int_\tau^1 x(s)\,\mathrm{d}s = A\sqrt{\lambda}\frac{(1-\tau)^2}{2}. \tag{11}$$

From the (9) and (11) we obtain the expression for $F^+_{1,0,+}$.

Now consider the solution of problem (1), (2) in the interval $(\tau, 1)$ with $\mu < 0$. We obtain the problem $x'' = \delta x$, $x(\tau) = 0$, $x'(\tau) = A\sqrt{\lambda}$, where $\delta = -\lambda$. The solution of the last problem is the function $x(t) = A\sqrt{\lambda/\delta}\sinh(\sqrt{\delta}(t-\tau))$. In view of it, $\int_\tau^1 x(s)\,\mathrm{d}s = A(\sqrt{\lambda}/\delta)(\cosh(\sqrt{\delta}(1-\tau)) - 1)$ or

$$\int_\tau^1 x(s)\,\mathrm{d}s = -A\frac{\sqrt{\lambda}}{\mu}\bigl(\cosh\bigl(\sqrt{-\mu}(1-\tau)\bigr) - 1\bigr). \tag{12}$$

From the (9) and (12) we obtain the expression for $F^+_{1,-,+}$.

The proof for other branches is similar. □

Remark 5. The $F^+_{1,0,+}$, $F^+_{1,+,0}$, $F^+_{1,0,0}$, $F^+_{1,-,0}$ and $F^+_{1,0,-}$ are the points at the axis.

4.2 About the branch related to the solutions with two zeroes

Consider the solutions of problem (1), (2) with two zeroes in the interval $(0, 1)$.

The solutions of problem (1), (2) with two zeroes τ_1 and τ_2 in the interval $(0, 1)$ may be of the three types. Sine function, linear function or hyperbolic sine function in the interval $(0, \tau_1)$ must be continued by sine function in the interval (τ_1, τ_2) and then with sine function, linear function or hyperbolic sine function in the interval $(\tau_2, 1)$.

All types of solutions of problem (1), (2) with two zeroes in the interval $(0, 1)$ are shown in Fig. 3.

Theorem 5. *The branches $F^\pm_2$ of the spectrum for problem* (1), (2) *for $\alpha \in (0, 1)$ given by* (*if the respective part of branch exists for corresponding value of α*)

$$F^+_2 = F^+_{2,+,+} \cup F^+_{2,+,0} \cup F^+_{2,+,-},$$

$$
\begin{aligned}
F^+_{2,+,+} = \Bigg\{(\mu,\lambda) \;\Big|\; &\frac{2}{\lambda} - \frac{2}{\mu} - \frac{1}{\lambda}\frac{1}{\sqrt{1+(\frac{1-\alpha}{\alpha})^2\lambda}} \\
&- \frac{1}{\lambda}\cos\biggl(\sqrt{\lambda}\biggl(1 - \frac{\pi}{\sqrt{\mu}} - \frac{\arctan(\frac{1-\alpha}{\alpha}\sqrt{\lambda})}{\sqrt{\lambda}}\biggr)\biggr) = 0, \\
&\frac{\arctan(\frac{1-\alpha}{\alpha}\sqrt{\lambda})}{\sqrt{\lambda}} + \frac{\pi}{\sqrt{\mu}} < 1,\;\; \frac{\arctan(\frac{1-\alpha}{\alpha}\sqrt{\lambda})}{\sqrt{\lambda}} + \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\lambda}} \geqslant 1\Bigg\}, \\
F^+_{2,+,0} = \Bigg\{(\mu,\lambda) \;\Big|\; &\left(\frac{1-\alpha}{\alpha}\right)^2 - \frac{4}{\mu} + \left(1 - \frac{1-\alpha}{\alpha} - \frac{\pi}{\sqrt{\mu}}\right)^2 = 0, \\
&\frac{1-\alpha}{\alpha} + \frac{\pi}{\sqrt{\mu}} < 1,\;\; \lambda = 0\Bigg\}, \\
F^+_{2,+,-} = \Bigg\{(\mu,\lambda) \;\Big|\; &\frac{2}{\lambda} - \frac{2}{\mu} - \frac{1}{\lambda}\frac{1}{\sqrt{1+(\frac{1-\alpha}{\alpha})^2\lambda}} \\
&- \frac{1}{\lambda}\cosh\biggl(\sqrt{-\lambda}\biggl(1 - \frac{\pi}{\sqrt{\mu}} - \frac{\operatorname{arctanh}(\frac{1-\alpha}{\alpha}\sqrt{-\lambda})}{\sqrt{-\lambda}}\biggr)\biggr) = 0, \\
&\frac{\operatorname{arctanh}(\frac{1-\alpha}{\alpha}\sqrt{-\lambda})}{\sqrt{-\lambda}} + \frac{\pi}{\sqrt{\mu}} < 1,\;\; \lambda < 0\Bigg\}, \\
F^-_2 = \bigl\{(\mu,\lambda) \;\big|\; &(\lambda,\mu) \in F^+_2\bigr\}.
\end{aligned}
$$

Proof. The proof of this theorem is similar to the proof of Theorem 4. □

4.3 About other branches of spectrum

The explicit formulas for the other branches of spectrum for problem (1), (2) are provided in the next theorem.

Theorem 6. *The branches $F^\pm_i$ of the spectrum for problem* (1), (2) *for $\alpha \in (0,1)$ given by*

$$
\begin{aligned}
F^+_{2i+1} = \Bigg\{(\mu,\lambda) \;\Big|\; &\frac{2i+1}{\lambda} - \frac{2i+1}{\mu} - \frac{1}{\lambda}\frac{1}{\sqrt{1+(\frac{1-\alpha}{\alpha})^2\lambda}} + \frac{1}{\mu}\cos\bigl(\sqrt{\mu}(1-\tau_{2i+1})\bigr) = 0, \\
&\tau_{2i+1} < 1,\;\; \tau_{2i+1} + \frac{\pi}{\sqrt{\mu}} \geqslant 1\Bigg\}, \\
F^+_{2i+2} = \Bigg\{(\mu,\lambda) \;\Big|\; &\frac{2i+2}{\lambda} - \frac{2i+2}{\mu} - \frac{1}{\lambda}\frac{1}{\sqrt{1+(\frac{1-\alpha}{\alpha})^2\lambda}} - \frac{1}{\lambda}\cos\bigl(\sqrt{\lambda}(1-\tau_{2i+2})\bigr) = 0, \\
&\tau_{2i+2} < 1,\;\; \tau_{2i+2} + \frac{\pi}{\sqrt{\lambda}} \geqslant 1\Bigg\}, \\
F^-_j = \bigl\{(\mu,\lambda) \;\big|\; &(\lambda,\mu) \in F^+_j\bigr\},
\end{aligned}
$$

where $\tau_{2i+1} = \arctan(((1-\alpha)/\alpha)\sqrt{\lambda})/\sqrt{\lambda} + i\pi/\sqrt{\mu} + i\pi/\sqrt{\lambda}$, $\tau_{2i+2} = \arctan(((1-\alpha)/\alpha)\sqrt{\lambda})/\sqrt{\lambda} + (i+1)\pi/\sqrt{\mu} + i\pi/\sqrt{\lambda}$, $i = 1, 2, \dots$, $j = 3, 4, \dots$.

Proof. The idea of the proof for Theorem 6 is similar to the proof of Theorem 4. We consider the eigenvalue problems in the intervals between two consecutive zeroes of the solution and use the conditions of the solutions for these problems. $\square$

Some branches of the spectrum for problem (1), (2) are depicted in Fig. 4 for some different α values.

Remark 6. The branches of the spectrum for problem (1), (2) are bounded by the respective branches of the spectrum for problem (1), (6). The branches of the spectrum for problem (1), (6) are shown by dashed curves in Fig. 4.

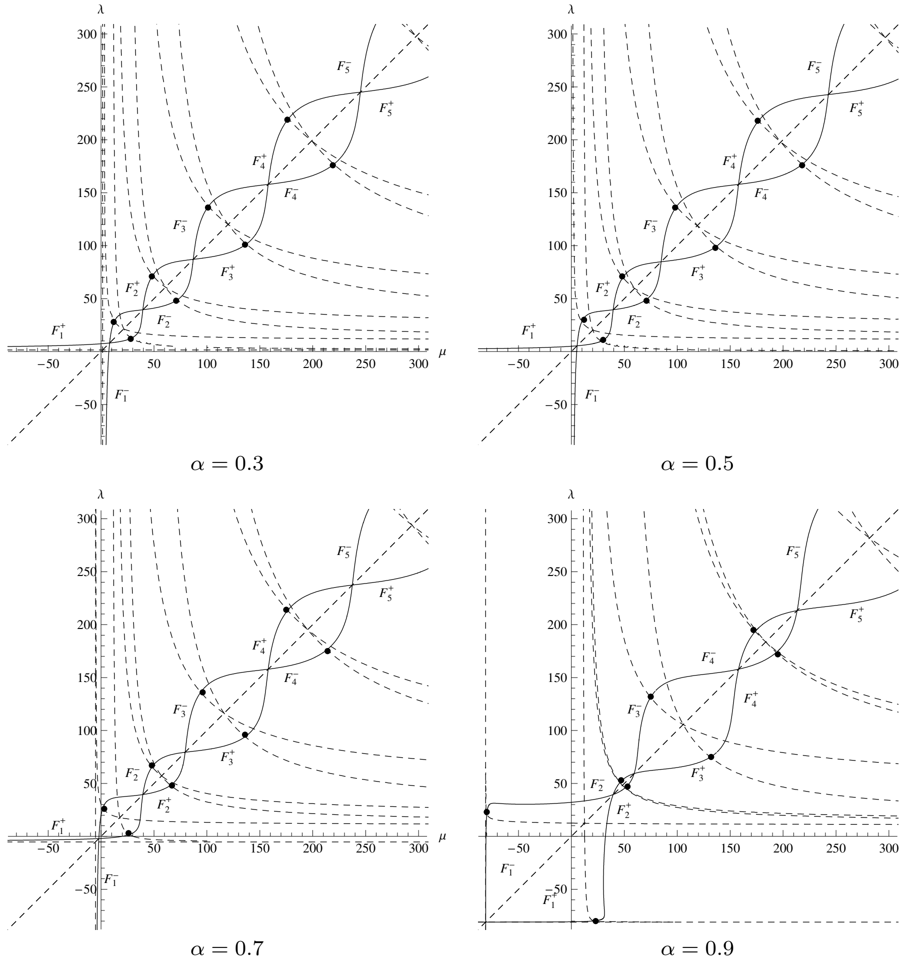

Fig. 4. The spectrum for problem (1), (2) for different α values.

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