Jacobi rational–Gauss collocation method for Lane–Emden equations of astrophysical significance

Eid H. Doha[a], **Ali H. Bhrawy**[b,c], **Ramy M. Hafez**[d], **Robert A. Van Gorder**[e,1]

[a]Faculty of Science, Cairo University
Giza, Egypt
eiddoha@frcu.eun.eg

[b]Faculty of Science, King Abdulaziz University
Jeddah, Saudi Arabia
alibhrawy@yahoo.co.uk

[c]Faculty of Science, Beni-Suef University
Beni-Suef, Egypt

[d]Institute of Information Technology, Modern Academy
Cairo, Egypt
r_mhafez@yahoo.com

[e]Department of Mathematics, University of Central Florida
Orlando, FL 32816, USA
rav@knights.ucf.edu

Abstract. In this paper, a new spectral collocation method is applied to solve Lane–Emden equations on a semi-infinite domain. The method allows us to overcome difficulty in both the nonlinearity and the singularity inherent in such problems. This Jacobi rational–Gauss method, based on Jacobi rational functions and Gauss quadrature integration, is implemented for the nonlinear Lane–Emden equation. Once we have developed the method, numerical results are provided to demonstrate the method. Physically interesting examples include Lane–Emden equations of both first and second kind. In the examples given, by selecting relatively few Jacobi rational–Gauss collocation points, we are able to get very accurate approximations, and we are thus able to demonstrate the utility of our approach over other analytical or numerical methods. In this way, the numerical examples provided demonstrate the accuracy, efficiency, and versatility of the method.

Keywords: Lane–Emden equation, isothermal gas spheres, collocation method, Jacobi rational–Gauss quadrature, Jacobi rational polynomials.

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1 Introduction

The fundamental goal of this paper is to develop a suitable way to approximate the singular nonlinear Lane–Emden equation on the interval $x \in (0, \infty)$ using the Jacobi rational polynomials. To this end, consider

$$u''(x) + \frac{a}{x}u' = g\big(x, u(x)\big), \quad 0 < x < \infty, \tag{1}$$

subject to $u(0) = b_0$ and $u'(0) = b_1$, where the prime denotes differentiation with respect to x, and $a > 0$, b_0 and b_1 are constants. Lane–Emden equations model many phenomena in mathematical physics and astrophysics. This equation is a generalization of some of the basic equations in the theory of stellar structure, and has been the focus of many studies [1–7]. When $a = 2$ and $g = u^m$, we recover the Lane–Emden equation of the first kind, while when $g = \exp(u)$, we recover the Lane–Emden equation of the second kind.

Many mathematical problems arising in science and engineering are defined over unbounded domains. To make matters more complicated, many such problems are nonlinear. Several spectral methods have been successfully applied in the approximation of problems on unbounded domains. The common methods for dealing with such problems are the Hermite spectral method [8, 9], the Laguerre spectral method [10–12], mapping the original problem in an unbounded domain to a problem in a bounded domain [13, 14] and rational approximations [15–17].

The solution of nonlinear singular initial value problems of Lane–Emden type is numerically challenging because of the singularity at the origin, in addition to the strong nonlinearity. Approximate solutions to the Lane–Emden equation were given by implicit series solution [18] and the homotopy perturbation method [19,20]. In [21], the Boubaker polynomials expansion scheme is applied successfully in order to obtain analytical-numerical solutions for two kind of Lane–Emden problems. The enhanced Lagrangian formulation method and the Boubaker polynomials expansion scheme have been confirmed in [22] to solve the related generalized Lane–Emden equation for polytropic star structure analysis under Bonnor–Ebert gas sphere astrophysical configuration. Moreover, Danish et al. [23] introduced an optimal homotopy analysis method to overcome the presence of singularity of some related boundary value problems arising in engineering and applied sciences.

Polynomial approximations can be quite useful for expressing the solution of a differential equation [24]. One such approach would be the spectral methods. A well-known advantage of a spectral method is that it achieves high accuracy with relatively fewer spatial grid points when compared with other numerical or analytical methods. Recently, Bhrawy et al. [25] proposed the shifted Jacobi collocation spectral method for solving the nonlinear Lane–Emden type equation, while the spatial approximation is based on shifted Jacobi polynomials with their parameters α and β and used the collocation nodes of shifted Jacobi–Gauss points. Adibi and Rismani [26] proposed an approximation algorithm for the solution of (1) using modified Legendre-spectral method. Recently, the sinc-collocation method and Hermite function collocation method is introduced in [27] and [28] for the solution of Lane–Emden type equations. A modified generalized Laguerre

functions Lagrangian method and the rational Legendre pseudospectral approach are also introduced in [29, 30]. More recently, Pandey et al. [31], and Pandey and Kumar [32] developed two numerical methods for solving Lane–Emden type equations using Legendre and Bernstein operational matrices of differentiation, respectively.

The use of Jacobi polynomials for solving differential equations has gained increasing popularity in recent years (see [33–37]). The main concern of this paper is to develop a spectral Jacobi rational–Gauss collocation (JRC) method to find an approximate solution $u_N(x)$ of singular Lane–Emden type initial value problems on the semi-infinite domain $(0, \infty)$. We first derive an algorithm for the general Lane–Emden model so that we may apply the Jacobi rational–Gauss collocation method to determine solutions. Then we apply the algorithm to some physically reasonable examples, namely, the Lane–Emden equations of first and second kind, in order to demonstrate the method. We show that the proposed method is both accurate and efficient compared with alternative methods.

This paper is organized as follows. In Section 2, we construct collocation algorithm for Lane–Emden equation using the Jacobi rational polynomials. Then, in Section 3, the proposed method is applied to various types of Lane–Emden equations, and the results are compared with existing analytic or exact solutions that were reported in other published works in the literature.

2 Jacobi rational–Gauss collocation method

In this section, we use the Jacobi rational–Gauss collocation method to solve numerically the following model problem:

$$u''(x) = f\big(x, u(x), u'(x)\big), \quad 0 < x < \infty, \tag{2}$$

subject to

$$u(0) = d_0, \qquad u'(0) = d_1, \tag{3}$$

where the values of d_0 and d_1 describe the initial state of $u(x)$ and $f(x, u, u')$ is a nonlinear function of x, u and u' which may be singular at $x = 0$. It is well known that the Lane–Emden equations of first and second kind are special cases of (2)–(3).

It should be noted that for a second-order differential equation with the singularity at $x = 0$ in the interval $[0, \infty)$, one is unable to apply the collocation method with Jacobi rational–Gauss–Radau points because the fixed node $x = 0$ is necessary to use as a collocation node. Therefore, the collocation method with Jacobi rational–Gauss nodes are used to overcome the difficulty of such a singular point at $x = 0$; i.e., we collocate the singular nonlinear ODE only at the $N - 1$ Jacobi Rational–Gauss points that are the $N - 1$ zeros of the Jacobi rational polynomial on $(0, \infty)$. These equations together with two initial conditions generate $N + 1$ nonlinear algebraic equations which can be solved.

Let us first introduce some basic notation that will be used. To begin with, some mathematical preliminaries are laid out in the Appendix. We use the results presented there to construct our algorithm. To begin with, we set

$$S_N(0, \infty) = \text{span}\big\{ R_0^{(\alpha, \beta)}(x), R_1^{(\alpha, \beta)}(x), \ldots, R_N^{(\alpha, \beta)}(x) \big\}, \tag{4}$$

while we define the discrete inner product and norm as

$$(u, v)_{\chi_R^{(\alpha,\beta)}, N} = \sum_{j=0}^{N} u\big(x_{R,N,j}^{(\alpha,\beta)}\big) v\big(x_{R,N,j}^{(\alpha,\beta)}\big) \varpi_{R,N,j}^{(\alpha,\beta)},$$

$$\|u\|_{\chi_R^{(\alpha,\beta)}, N} = \sqrt{(u, u)_{\chi_R^{(\alpha,\beta)}, N}}. \tag{5}$$

Here $x_{R,N,j}^{(\alpha,\beta)}$ and $\varpi_{R,N,j}^{(\alpha,\beta)}$ are the nodes and the corresponding weights of the Jacobi rational–Gauss quadrature formula on the interval $(0, \infty)$, respectively. Obviously,

$$(u, v)_{\chi_R^{(\alpha,\beta)}, N} = (u, v)_{\chi_R^{(\alpha,\beta)}} \quad \forall u, v \in S_{2N-1}. \tag{6}$$

Thus, for any $u \in S_N(0, \infty)$, the norms $\|u\|_{\chi_R^{(\alpha,\beta)}, N}$ and $\|u\|_{\chi_R^{(\alpha,\beta)}}$ coincide.

Associating with this quadrature rule, we denote by $I_N^{R_T^{(\alpha,\beta)}}$ the Jacobi rational–Gauss interpolation

$$I_N^{R_T^{(\alpha,\beta)}} u\big(x_{R,N,j}^{(\alpha,\beta)}\big) = u\big(x_{R,N,j}^{(\alpha,\beta)}\big), \quad 0 \leqslant k \leqslant N.$$

The Jacobi rational–Gauss collocation method for solving (2) and (3) is to seek $u_N(x) \in S_N(0, \infty)$ such that

$$u''\big(x_{R,N,k}^{(\alpha,\beta)}\big) = f\big(x_{R,N,k}^{(\alpha,\beta)}, u\big(x_{R,N,k}^{(\alpha,\beta)}\big), u'\big(x_{R,N,k}^{(\alpha,\beta)}\big)\big), \quad k = 0, 1, \ldots, N-2,$$

$$u_N^{(i)}(0) = d_i, \quad i = 0, 1. \tag{7}$$

Now, we derive the algorithm for solving the singular second-order differential equation (2) and (3). Let

$$u_N(x) = \sum_{j=0}^{N} a_j R_j^{(\alpha,\beta)}(x), \quad \mathbf{a} = (a_0, a_1, \ldots, a_N)^{\mathrm{T}}. \tag{8}$$

We first approximate $u(x)$, $u'(x)$ and $u''(x)$, as Eq. (8). By substituting these approximations in Eq. (2), we get

$$\sum_{j=0}^{N} a_j D^2 R_j^{(\alpha,\beta)}(x) = f\left(x, \sum_{j=0}^{N} a_j R_j^{(\alpha,\beta)}(x), \sum_{j=0}^{N} a_j D R_j^{(\alpha,\beta)}(x)\right). \tag{9}$$

Therefore, we deduce from (A.11) and (A.12) that

$$\sum_{j=0}^{N} a_j \big[(j + \alpha + \beta + 1)_2 (x+1)^{-4} R_{j-2}^{(\alpha+2,\beta+2)}(x)$$

$$- 2(j + \alpha + \beta + 1)(x+1)^{-3} R_{j-1}^{(\alpha+1,\beta+1)}(x)\big]$$

$$= f\left(x, \sum_{j=0}^{N} a_j R_j^{(\alpha,\beta)}(x), \sum_{j=0}^{N} a_j (j + \alpha + \beta + 1)(x+1)^{-2} R_{j-1}^{(\alpha+1,\beta+1)}(x)\right). \tag{10}$$

Substitution of (8) into (3) yields

$$\sum_{j=0}^{N} a_j D^i R_j^{(\alpha,\beta)}(0) = d_i, \quad i = 0, 1. \tag{11}$$

To find the solution $u_N(x)$, we first collocate Eq. (10) at the $N - 1$ Jacobi rational roots, yields

$$\sum_{j=0}^{N} a_j \big[(j + \alpha + \beta + 1)_2 \big(x_{R,N,k}^{(\alpha,\beta)} + 1 \big)^{-4} R_{j-2}^{(\alpha+2,\beta+2)} \big(x_{R,N,k}^{(\alpha,\beta)} \big)$$

$$- 2(j + \alpha + \beta + 1) \big(x_{R,N,k}^{(\alpha,\beta)} + 1 \big)^{-3} R_{j-1}^{(\alpha+1,\beta+1)} \big(x_{R,N,k}^{(\alpha,\beta)} \big) \big]$$

$$= f \bigg(x_{R,N,k}^{(\alpha,\beta)}, \sum_{j=0}^{N} a_j R_j^{(\alpha,\beta)} \big(x_{R,N,k}^{(\alpha,\beta)} \big),$$

$$\sum_{j=0}^{N} a_j (j + \alpha + \beta + 1) \big(x_{R,N,k}^{(\alpha,\beta)} + 1 \big)^{-2} R_{j-1}^{(\alpha+1,\beta+1)} \big(x_{R,N,k}^{(\alpha,\beta)} \big) \bigg). \tag{12}$$

Next, Eq. (11), after using (A.9) and (A.10), can be written as

$$\sum_{j=0}^{N} (-1)^j \frac{\Gamma(j + \beta + 1)}{\Gamma(\beta + 1)j!} a_j = d_0, \tag{13}$$

$$\sum_{j=1}^{N} (-1)^{j-1} \frac{(j + \alpha + \beta + 1)\Gamma(j + \beta + 1)}{(j - 1)!\,\Gamma(\beta + 2)} a_j = d_1. \tag{14}$$

Finally, from (12), (13) and (14), we get $N + 1$ nonlinear algebraic equations which can be solved for the unknown coefficients a_j by using any standard iteration technique, like Newton's iteration method. Consequently, $u_N(x)$ given in Eq. (8) can be evaluated.

3 Numerical results

We report in this section some numerical results obtained with the algorithms presented in the previous section. Comparisons of the results obtained by the present method with those obtained by other methods reveal that the present method is very accurate and efficient. We consider the two examples, both of which are physically relevant.

3.1 Lane–Emden equation of the first kind

The nonlinear problem we shall consider is the Lane–Emden equation of the first kind, of index m. The equation is given by

$$u''(x) + \frac{2}{x} u'(x) + u^m(x) = 0, \quad x > 0, \tag{15}$$

subject to the conditions

$$u(0) = 1, \qquad u'(0) = 0, \tag{16}$$

where u is a function of x and for physical interest the polytropic index m lies between 0 and 5 [5, 38, 39]. Exact solutions corresponding to taking $m = 0, 1$ and 5 are given by

$$u(x) = 1 - \frac{1}{3!}x^2, \quad u(x) = \frac{\sin x}{x} \quad \text{and} \quad u(x) = \left(1 + \frac{x^2}{3}\right)^{-1/2},$$

respectively.

The Lane–Emden equation of index m is a basic equation in the theory of stellar structure [5]. The thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics is modelled by this equation. Unfortunately, analytical solutions to (15)–(16) in closed form are possible only for values of the polytropic index $m = 0, 1$ and 5. For other values of m, only numerical solutions or approximate analytical solutions are available in the literature. It is worthy noting here that Eq. (15) is linear for $m = 0$ and 1, and nonlinear otherwise. So, the $m = 5$ case is the only case in which there exists an exact solution to the nonlinear problem.

For the sake of comparison with others methods, we consider the following two cases:

(i) In the case of $m = 4$, we introduce Table 1, where the maximum absolute errors using the present JRC method, those obtained by the Hermite functions collocation method (HFC, see [28]), and the values obtained by Horedt [40] are compared.

(ii) In the case of $m = 5$, Tables 2, 3 show the maximum absolute errors using JRC method at $N = 20$ and $N = 20, 36$ respectively with two choices of α and β. Table 2, shows the maximum absolute errors of our method are compared with homotopy-perturbation method (HPM [19]).

Moreover, the curves of the approximate and exact solutions for $m = 0, 1, 5$ for the special value of Jacobi parameters $\alpha = \beta = -0.5$ and $x \in [0, 8]$, $N = 20$ are shown in Fig. 1a. From this figure, we see the agreement between the exact and approximate solutions. In the case of $\alpha = \beta = 1$ and $x \in [0, 320]$, $N = 40$ the approximate solution by the presented method is shown in Fig. 1b, to make it easier to compare with

Table 1. Absolute errors using the JRC method (present method) and the HFC method [28] compared with references values of Horedt [40] for (15)–(16).

x	α	β	JRC method	HFC [28]	x	α	β	JRC method	HFC [28]
0.0	0.5	0.5	0	0	5.0	0.5	0.5	$1.46 \cdot 10^{-8}$	$8.59 \cdot 10^{-5}$
	1	1	0			1	1	$1.07 \cdot 10^{-8}$	
0.1	0.5	0.5	$4.02 \cdot 10^{-8}$	$2.51 \cdot 10^{-4}$	10.0	0.5	0.5	$4.11 \cdot 10^{-7}$	$6.22 \cdot 10^{-5}$
	1	1	$4.06 \cdot 10^{-8}$			1	1	$1.33 \cdot 10^{-6}$	
0.2	0.5	0.5	$1.35 \cdot 10^{-8}$	$2.48 \cdot 10^{-4}$	14.0	0.5	0.5	$1.77 \cdot 10^{-6}$	$2.47 \cdot 10^{-5}$
	1	1	$1.36 \cdot 10^{-8}$			1	1	$4.04 \cdot 10^{-6}$	
0.5	0.5	0.5	$3.09 \cdot 10^{-9}$	$2.05 \cdot 10^{-4}$	14.9	0.5	0.5	$1.33 \cdot 10^{-6}$	$4.59 \cdot 10^{-7}$
	1	1	$1.82 \cdot 10^{-8}$			1	1	$1.77 \cdot 10^{-6}$	
1.0	0.5	0.5	$1.15 \cdot 10^{-8}$	$1.93 \cdot 10^{-4}$					
	1	1	$1.29 \cdot 10^{-8}$						

Table 2. Absolute errors using the JRC method (present method) with $N = 20$ and the HPM [19] compared with reference values from the exact solution when in the case $m = 5$ for (15)–(16).

x	α	β	JRC method	HPM [19]	x	α	β	JRC method	HPM [19]
0.0	0.5	0.5	0	0	0.6	0.5	0.5	$2.759 \cdot 10^{-10}$	$5.118 \cdot 10^{-5}$
	0	0	0			0	0	$4.704 \cdot 10^{-10}$	
0.2	0.5	0.5	$3.763 \cdot 10^{-11}$	$8.540 \cdot 10^{-9}$	0.8	0.5	0.5	$5.977 \cdot 10^{-10}$	$4.754 \cdot 10^{-4}$
	0	0	$1.269 \cdot 10^{-11}$			0	0	$9.625 \cdot 10^{-10}$	
0.4	0.5	0.5	$4.084 \cdot 10^{-11}$	$2.111 \cdot 10^{-6}$	1.0	0.5	0.5	$5.795 \cdot 10^{-10}$	$2.599 \cdot 10^{-3}$
	0	0	$9.122 \cdot 10^{-11}$			0	0	$8.573 \cdot 10^{-10}$	

Table 3. Absolute errors using the JRC method (present method) with $N = 20, 36$ in the case $m = 5$ for (15)–(16).

x	α	β	JRC method		x	α	β	JRC method	
			$N = 20$	$N = 36$				$N = 20$	$N = 36$
0.0	0.5	−0.5	0	0	40.0	0.5	−0.5	$2.594 \cdot 10^{-7}$	$5.068 \cdot 10^{-14}$
	0	0	0	0		0	0	$6.625 \cdot 10^{-8}$	$2.292 \cdot 10^{-13}$
5.0	0.5	−0.5	$1.072 \cdot 10^{-9}$	$3.830 \cdot 10^{-14}$	80.0	0.5	−0.5	$1.136 \cdot 10^{-6}$	$1.739 \cdot 10^{-13}$
	0	0	$1.013 \cdot 10^{-8}$	$2.237 \cdot 10^{-14}$		0	0	$3.257 \cdot 10^{-6}$	$1.038 \cdot 10^{-12}$
10.0	0.5	−0.5	$6.531 \cdot 10^{-8}$	$6.841 \cdot 10^{-14}$	160.0	0.5	−0.5	$3.898 \cdot 10^{-6}$	$1.357 \cdot 10^{-13}$
	0	0	$5.040 \cdot 10^{-8}$	$7.790 \cdot 10^{-14}$		0	0	$8.410 \cdot 10^{-6}$	$3.762 \cdot 10^{-12}$
20.0	0.5	−0.5	$9.730 \cdot 10^{-8}$	$1.424 \cdot 10^{-13}$	320.0	0.5	−0.5	$6.333 \cdot 10^{-6}$	$7.613 \cdot 10^{-12}$
	0	0	$1.847 \cdot 10^{-8}$	$1.645 \cdot 10^{-13}$		0	0	$1.265 \cdot 10^{-5}$	$2.578 \cdot 10^{-11}$

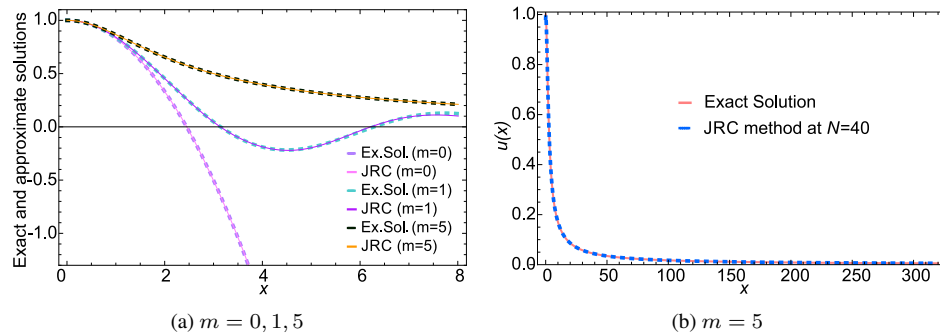(a) $m = 0, 1, 5$ (b) $m = 5$

Fig. 1. Comparison of the approximate and exact solutions to (15)-(16) for the Lane–Emden equation of the first kind.

the analytic solution. Therefore, this example indicates that the obtained numerical results are accurate and that the spectral Jacobi rational–Gauss collocation method is compared favorably with the analytical solution.

3.2 Lane–Emden equation of the second kind

Consider the second-order nonlinear ordinary differential equation (see [41])

$$u''(x) + \frac{2}{x}u'(x) - \mathrm{e}^{-u} = 0,$$

subject to the conditions $u(0) = 0$, $u'(0) = 0$.

Table 4. Approximate solutions for the Lane–Emden equation of the second kind with $N = 22$.

| x	$\alpha = \beta = -0.5$	$\alpha = \beta = 0$	$\alpha = \beta = 0.5$	x	$\alpha = \beta = -0.5$	$\alpha = \beta = 0$	$\alpha = \beta = 0.5$
0.0	0.000000	0.000000	0.000000	2.5	0.806373	0.806357	0.806349
0.5	0.041156	0.041155	0.041154	3.0	1.063340	1.063330	1.063330
1.0	0.158825	0.158827	0.158827	3.5	1.320710	1.320740	1.320760
1.5	0.338034	0.338025	0.338022	4.0	1.52722	1.527223	1.527224
2.0	0.559801	0.559814	0.559819				
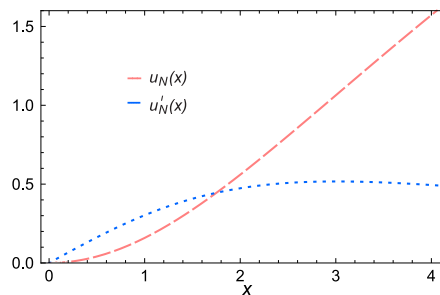
Fig. 2. Graph of the approximation $u_N(x)$ (dotted line) and $u'_N(x)$ (dashed line) for $\alpha = \beta = -0.5$ at $N = 22$ for the Lane–Emden equation of the second kind.

(a) $N = 20$　　　　　　　　　　　　　　　　　(b) $N = 40$
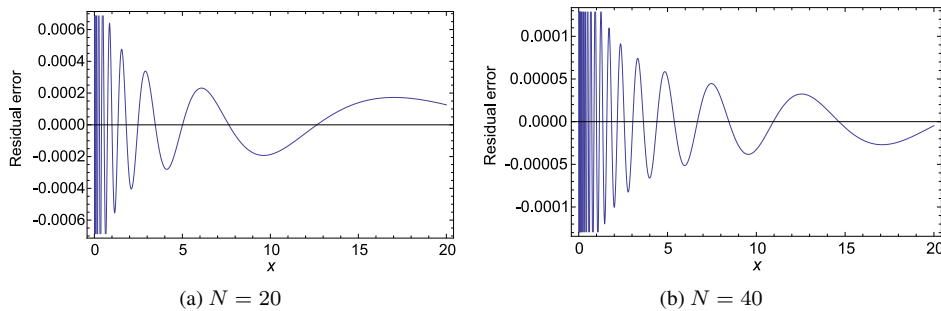
Fig. 3. Graph of residual error functions for $\alpha = \beta = 0$.

Table 4 lists the results obtained by the Jacobi rational collocation method in terms of approximate solutions at $N = 22$ with $\alpha = \beta = -0.5$ (which reduces to the first kind Chebyshev rational collocation method), and $\alpha = \beta = 0$ (which reduces to the Legendre rational collocation method) and $\alpha = \beta = 0.5$ (which reduces to the second kind Chebyshev rational collocation method). The resulting graphs of the approximate solution and its first derivative for $\alpha = \beta = -0.5$ at $N = 22$ are shown in Fig. 2. Moreover, Figs. 3a and 3b show the residual error functions in the interval $[0, 20]$ for $\alpha = \beta = 0$ at $N = 20$ and $N = 40$, respectively. As expected, the number of nodes is larger for this example than it was for the first, owing to the fact that the exponential nonlinearity $\exp(-u)$ is harder to work with than polynomial nonlinearity of the form u^m.

4 Conclusions

We have applied a Jacobi rational–Gauss collocation method to solve Lane–Emden equations on a semi-infinite domain. We then provided numerical results to demonstrate the utility of the method. Physically interesting examples include Lane–Emden equations of both first and second kind. In the examples given, by selecting relatively few Jacobi rational–Gauss collocation points, we are able to get very accurate approximations, and we are thus able to demonstrate the utility of our approach over other analytical or numerical methods such as other collocation methods or perturbation methods. The solutions also agree strongly with exact solutions from the literature, in cases where such exact solutions exist. As many problems arising in theoretical physics and astrophysics are singular and nonlinear, it stands to reason that the present method can be used to solve a number of related problems efficiently and accurately. Indeed, with the freedom to select the parameters α and β, the method can be calibrated for a wide variety of problems.

Appendix: Jacobi rational interpolation

In this appendix, we detail the mathematical properties of Jacobi polynomials and Jacobi rational functions that are used to construct the JRC method.

The Jacobi polynomials $P_k^{(\alpha,\beta)}(y)$, $k = 0, 1, 2, \ldots$, are the eigenfunctions of the Sturm–Liouville problem

$$\partial_y\big((1-y)^{\alpha+1}(1+y)^{\beta+1}\partial_y v(y)\big) + \lambda(1-y)^\alpha(1+y)^\beta v(y) = 0,$$
$$y \in I = [-1, 1]. \tag{A.1}$$

Their corresponding eigenvalues are $\lambda_k^{(\alpha,\beta)} = k(k + \alpha + \beta + 1)$, $k = 0, 1, 2, \ldots$.

Let $\Gamma(x)$ be the Gamma function, then it is to be noted that

$$P_k^{(\alpha,\beta)}(-y) = (-1)^k P_k^{(\beta,\alpha)}(y), \qquad P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k+\alpha+1)}{k!\,\Gamma(\alpha+1)},$$
$$P_k^{(\alpha,\beta)}(-1) = \frac{(-1)^k \Gamma(k+\beta+1)}{k!\,\Gamma(\beta+1)}. \tag{A.2}$$

The Jacobi polynomials fulfill the recurrence relations (see [42])

$$P_k^{(\alpha,\beta-1)}(y) - P_k^{(\alpha-1,\beta)}(y) = P_{k-1}^{(\alpha,\beta)}(y), \tag{A.3}$$

$$(k+\alpha+\beta)P_k^{(\alpha,\beta)}(y) = (k+\beta)P_k^{(\alpha,\beta-1)}(y) + (k+\alpha)P_k^{(\alpha-1,\beta)}(y), \tag{A.4}$$

$$\partial_y P_k^{(\alpha,\beta)}(y) = \frac{1}{2}(k+\alpha+\beta+1)P_{k-1}^{(\alpha+1,\beta+1)}(y), \tag{A.5}$$

and

$$\partial_y^2 P_k^{(\alpha,\beta)}(y) = \frac{1}{4}(k+\alpha+\beta+1)(k+\alpha+\beta+2)P_{k-2}^{(\alpha+2,\beta+2)}(y). \tag{A.6}$$

Let $w^{(\alpha,\beta)}(y) = (1-y)^\alpha (1+y)^\beta$. Then for $\alpha, \beta > -1$, the set of Jacobi polynomials is a complete $L^2_{w^{(\alpha,\beta)}}(I)$-orthogonal system, i.e.,

$$\int\limits_I P_k^{(\alpha,\beta)}(y) P_l^{(\alpha,\beta)}(y) w^{(\alpha,\beta)}(y) \, \mathrm{d}y = h_k^{(\alpha,\beta)} \delta_{k,l}, \tag{A.7}$$

where $\delta_{k,l}$ is the Kronecker function and

$$h_k^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1} \Gamma(k+\alpha+1) \Gamma(k+\beta+1)}{(2k+\alpha+\beta+1) \Gamma(k+1) \Gamma(k+\alpha+\beta+1)}. \tag{A.8}$$

We denote the norm and semi-norm of the weighted Sobolev space $H^r_{w^{(\alpha,\beta)}}(I)$ by $\|v\|_{r,w^{(\alpha,\beta)},I}$ and $|v|_{r,w^{(\alpha,\beta)},I}$, respectively. In particular, $L^2_{w^{(\alpha,\beta)}}(I) = H^0_{w^{(\alpha,\beta)}}(I)$ and $\|v\|_{w^{(\alpha,\beta)},I} = \|v\|_{0,w^{(\alpha,\beta)},I}$.

The Jacobi rational functions, denoted by $R_k^{(\alpha,\beta)}(x)$, are defined as follows:

$$R_k^{(\alpha,\beta)}(x) = P_k^{(\alpha,\beta)}\left(\frac{x-1}{x+1}\right), \quad k = 0, 1, 2, \ldots.$$

According to (A.1), $R_k^{(\alpha,\beta)}(x)$ are the eigenfunctions of the singular Sturm–Liouville problem

$$\partial_x\left(x^{\beta+1}(x+1)^{-\alpha-\beta}\partial_x v(x)\right) + \lambda x^\beta (x+1)^{-\alpha-\beta-2} v(x) = 0, \quad x \in \Lambda = (0, \infty).$$

Their corresponding eigenvalues are $\lambda_k^{(\alpha,\beta)} = k(k+\alpha+\beta+1)$, $k = 0, 1, 2, \ldots$. Moreover, the recurrence relations (A.2)–(A.6) imply that

$$R_k^{(\alpha,\beta)}(x) = (-1)^k R_k^{(\beta,\alpha)}\left(\frac{1}{x}\right), \quad R_k^{(\alpha,\beta)}(\infty) = \frac{\Gamma(k+\alpha+1)}{k!\Gamma(\alpha+1)},$$
$$R_k^{(\alpha,\beta)}(0) = (-1)^k \frac{\Gamma(k+\beta+1)}{k!\Gamma(\beta+1)}, \tag{A.9}$$

$$DR_k^{(\alpha,\beta)}(0) = \frac{(-1)^{k-1}\Gamma(k+\beta+1)(k+\alpha+\beta+1)}{(k-1)!\Gamma(\beta+2)}, \tag{A.10}$$

$$(k+\alpha+1)R_k^{(\alpha,\beta)}(x) - (k+1)R_{k+1}^{(\alpha,\beta)}(x)$$
$$= (2k+\alpha+\beta+2)(x+1)^{-1}R_k^{(\alpha+1,\beta)}(x),$$

$$R_k^{(\alpha,\beta-1)}(x) - R_k^{(\alpha-1,\beta)}(x) = R_{k-1}^{(\alpha,\beta)}(x),$$

$$(k+\alpha+\beta)R_k^{(\alpha,\beta)}(x) = (k+\beta)R_k^{(\alpha,\beta-1)}(x) + (k+\alpha)R_k^{(\alpha-1,\beta)}(x),$$

$$\partial_x R_k^{(\alpha,\beta)}(x) = (k+\alpha+\beta+1)(x+1)^{-2}R_{k-1}^{(\alpha+1,\beta+1)}(x), \quad k \geqslant 1, \tag{A.11}$$

and

$$\partial_x^2 R_k^{(\alpha,\beta)}(x) = (k+\alpha+\beta+1)(k+\alpha+\beta+2)(x+1)^{-4}R_{k-2}^{(\alpha+2,\beta+2)}(x)$$
$$- 2(k+\alpha+\beta+1)(x+1)^{-3}R_{k-1}^{(\alpha+1,\beta+1)}(x), \quad k \geqslant 2. \qquad \text{(A.12)}$$

Let $\chi_R^{(\alpha,\beta)}(x) = x^\beta(x+1)^{-\alpha-\beta-2}$, $\alpha,\beta > -1$. Thanks to (A.7) and (A.8), the Jacobi rational functions form a complete $L_{\chi_R^{(\alpha,\beta)}}^2(\Lambda)$-orthogonal system, i.e.,

$$\int_\Lambda R_k^{(\alpha,\beta)}(x) R_l^{(\alpha,\beta)}(x) \chi_R^{(\alpha,\beta)}(x)\,\mathrm{d}x = \gamma_k^{(\alpha,\beta)} \delta_{k,l},$$

where

$$\gamma_k^{(\alpha,\beta)} = \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}.$$

For any $\upsilon \in L_{\chi_R^{(\alpha,\beta)}}^2(\Lambda)$,

$$\upsilon(x) = \sum_{j=0}^\infty a_j^{(\alpha,\beta)} R_j^{(\alpha,\beta)}(x), \quad a_j^{(\alpha,\beta)} = \left(\gamma_k^{(\alpha,\beta)}\right)^{-1} \int_\Lambda \upsilon(x) R_j^{(\alpha,\beta)}(x) \chi_R^{(\alpha,\beta)}(x)\,\mathrm{d}x.$$

We turn to the Jacobi–Gauss interpolation. We denote by $x_{N,j}^{(\alpha,\beta)}$, $0 \leqslant j \leqslant N$, the nodes of the standard Jacobi–Gauss interpolation on the interval $(-1,1)$. Their corresponding Christoffel numbers are $\varpi_{N,j}^{(\alpha,\beta)}$, $0 \leqslant j \leqslant N$. The nodes of the Jacobi rational–Gauss interpolation on the interval $(0,\infty)$ are the zeros of $R_{N+1}^{(\alpha,\beta)}(x)$, which we denote by $x_{R,N,j}^{(\alpha,\beta)}$, $0 \leqslant j \leqslant N$. Clearly, $x_{R,N,j}^{(\alpha,\beta)} = (1+x_{N,j}^{(\alpha,\beta)})/(1-x_{N,j}^{(\alpha,\beta)})$, and their corresponding Christoffel numbers are $\varpi_{R,N,j}^{(\alpha,\beta)} = 1/(2^{\alpha+\beta+1})\varpi_{N,j}^{(\alpha,\beta)}$, $0 \leqslant j \leqslant N$. Let $S_N(0,\infty)$ be the set of polynomials of degree at most N. Thanks to the property of the standard Jacobi–Gauss quadrature, it follows that for any $\phi \in S_{2N+1}(0,\infty)$,

$$\int_0^\infty x^\beta(x+1)^{-\alpha-\beta-2}\phi(x)\,\mathrm{d}x$$
$$= \frac{1}{2^{\alpha+\beta+1}} \int_{-1}^1 (1-x)^\alpha(1+x)^\beta \phi\left(\frac{1+x}{1-x}\right)\,\mathrm{d}x$$
$$= \frac{1}{2^{\alpha+\beta+1}} \sum_{j=0}^N \varpi_{N,j}^{(\alpha,\beta)} \phi\left(\frac{1+x_{N,j}^{(\alpha,\beta)}}{1-x_{N,j}^{(\alpha,\beta)}}\right) = \sum_{j=0}^N \varpi_{R,N,j}^{(\alpha,\beta)} \phi(x_{R,N,j}^{(\alpha,\beta)}),$$

where

$$\varpi_{R,N,j}^{(\alpha,\beta)} = \frac{(2N+\alpha+\beta+2)\Gamma(N+\alpha+1)\Gamma(N+\beta+1)}{2P_N^{(\alpha,\beta)}(x_{N,j}^{(\alpha,\beta)})\partial_x P_{N+1}^{(\alpha,\beta)}(x_{N,j}^{(\alpha,\beta)})},$$

consider the orthogonal projection $\mathrm{P}_{N,\alpha,\beta} : L_{\chi_R^{(\alpha,\beta)}}(\Lambda) \to \mathcal{R}_N$. It is defined by

$$(\mathrm{P}_{N,\alpha,\beta}\upsilon - \upsilon, \phi)_{\chi_R^{(\alpha,\beta)}} = 0 \quad \forall\phi \in \mathcal{R}_N.$$

In order to present the approximation results precisely, we introduce the space $H^r_{\chi_R^{(\alpha,\beta)},\Lambda}(\Lambda)$, $r \in \mathbb{N}$, with the following semi-norm and norm:

$$|v|_{r,\chi_R^{(\alpha,\beta)},\Lambda} = \left(\sum_{k=r}^{\infty} \left(\lambda_k^{(\alpha,\beta)}\right)^r |a_k|^2 \gamma_k^{(\alpha,\beta)} \right)^{1/2},$$

$$\|v\|_{r,\chi_R^{(\alpha,\beta)},\Lambda} = \left(\sum_{l=0}^{r} |v|^2_{l,\chi_R^{(\alpha,\beta)},\Lambda} \right)^{1/2}.$$

For any $r > 0$, we define the space $H^r_{\chi_R^{(\alpha,\beta)},\Lambda}(\Lambda)$ and its norm by space interpolation as in [43].

Theorem. *For any $v \in H^r_{\chi_R^{(\alpha,\beta)},\Lambda}(\Lambda)$, $r \in \mathbb{N}$, and $0 \leqslant \mu \leqslant r$,*

$$\|P_{N,\alpha,\beta} v - v\|_{\mu,\chi_R^{(\alpha,\beta)},\Lambda} \leqslant C N^{\mu-r} |v|_{r,\chi_R^{(\alpha,\beta)},\Lambda}.$$

A complete proof of the theorem and discussion on convergence are given in [44].

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