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Solitary wave solutions for a generalized KdV–mKdV equation with distributed delays*

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Abstract. This paper deals with a generalized KdV–mKdV equation with time delay. By employing the geometrical singular perturbation theory and the linear chain trick, we establish the existence result of solitary wave solutions when the average delay is sufficiently small, for a special convolution kernel.

Keywords: KdV–mKdV equation, solitary wave solutions, geometric singular perturbation, homoclinic orbit, time delay.

1 Introduction

In the past three decades, traveling wave solutions to the Korteweg–de Vries equation have been studied extensively and a large number of theoretical issues concerning the KdV equation have received considerable attention. These wave solutions when they exist can enable us to well understand the mechanism of the complicated physical phenomena and dynamical processes modeled by these nonlinear evolution equations. One can easily find abundant reports about it, such as [1–7]. And many powerful methods to construct exact solutions of KdV have been established and developed. Among these methods we mainly cite, for example, the bifurcation method of dynamical systems [8], (G'/G)-expansion method [9–11] and the sub-ODE method [12], Lie group theoretical methods [13] and so on. In this paper, we will use the geometrical singular perturbation theory and the linear chain trick to investigate solitary wave solutions of the generalized KdV–mKdV equation.

Let us consider a generalized Korteweg–de Vries–modified Korteweg–de Vries (KdV–mKdV) equation

$$U_t + \left(\alpha + \beta U^p + \gamma U^{2p}\right)U_x + U_{xxx} = 0, \tag{1}$$

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where $p > 0$, α, β and $\gamma \neq 0$ are real constants. It is necessary to point out that, when the parameters are taken as different values, some celebrated equations can be derived from Eq. (1), such as when $p = 1$, $\alpha = 0$, $\beta = \pm 6$ and $\gamma = 0$, Eq. (1) becomes the KdV equation [14, 15]

$$U_t \pm 6UU_x + U_{xxx} = 0.$$

When $p = 1$, $\alpha = 0$, $\beta = 6$ and $\gamma = \pm 6$, Eq. (1) becomes the combined KdV and mKdV equation [16]

$$U_t + 6UU_x \pm 6U^2 U_x + U_{xxx} = 0.$$

When $\alpha = 0, \beta \neq 0$ and $\gamma = 0$, Eq. (1) becomes the higher-order KdV equation [17]

$$U_t + \beta U^p U_x + U_{xxx} = 0.$$

When $p \in Z^+$, $\alpha \neq 0$, $\beta \neq 0$ and $\gamma = 0$, Eq. (1) becomes the generalized KdV–mKdV equation

$$U_t + \alpha U_x + \beta U^p U_x + U_{xxx} = 0. \tag{2}$$

The generalized KdV–mKdV equation with time delay has more actual significance. It is natural to ask weather the generalized KdV–mKdV equation with the delay possesses traveling wave solution. Zhao and Xu [18, 19] discussed generalized KdV equations with time delay and established existence results of solitary waves solutions.

Our purpose is to apply the geometric singular perturbation theorem to establish the existence of solitary wave solutions of Eq. (2) with time delay, such as

$$U_t + \alpha U_x + \beta (f * U) U^{p-1} U_x + U_{xxx} + \tau U_{xx} = 0, \tag{3}$$

where τ is time delay, U_{xxx} represents the dispersion effect and U_{xx} means the backward diffusion, when τ is small, which also is the perturbation, the convolution $f * U$ is denoted by

$$(f * U)(x, t) = \int_{-\infty}^{t} f(t - s) U(x, s)\, \mathrm{d}s,$$

where the kernel $f : [0, +\infty) \to [0, +\infty)$ satisfies the following normalization assumption:

$$f(t) \geqslant 0 \quad \text{for all } t \geqslant 0 \quad \text{and} \quad \int_0^{+\infty} f(t)\, \mathrm{d}t = 1, \quad t f(t) \in L^1\big((0, +\infty), R\big).$$

The average delay for the distributed delay kernel $f(t)$ is defined as

$$\tau = \int_0^{+\infty} t f(t)\, \mathrm{d}t.$$

Here we consider the average delay τ is small. Note that equations of various types can be derived from Eq. (3) by taking different delay kernels. For instance, when we take the kernel to be $f(t) = \delta(t)$, where δ denotes Dirac's delta function, Eq. (3) becomes the corresponding undelayed perturbed mKdV equation

$$U_t + \alpha U_x + \beta U^p U_x + U_{xxx} + \tau U_{xx} = 0. \tag{4}$$

While taking $f(t) = \delta(t-\tau)$, Eq. (3) becomes the following equation with discrete delay:

$$U_t + \alpha U_x + \beta U(x, t-\tau)U^{p-1}U_x + U_{xxx} + \tau U_{xx} = 0.$$

We usually use the gamma distribution delay kernel (see [20])

$$f(t) = \frac{\alpha^n t^{n-1} e^{-\alpha t}}{(n-1)!}, \quad n = 1, 2, \ldots,$$

where $\alpha > 0$ is a constant, n is a integer, with average delay $\tau = n/\alpha > 0$. Two special cases

$$f(t) = \frac{1}{\tau}\mathrm{e}^{-t/\tau} \quad (n = 1) \quad \text{and} \quad f(t) = \frac{t}{\tau^2}\mathrm{e}^{-t/\tau} \quad (n = 2) \tag{5}$$

are called weak generic kernel and strong generic kernel, respectively.

A solitary wave for Eq. (3) is a special traveling wave solution $U(x,t) = \varphi(\xi) = \varphi(x-ct)$, where $c > 0$ is speed and $\varphi(\xi)$ satisfies the following functional differential equation:

$$-(c-\alpha)\varphi' + \beta(f * \varphi)\varphi^{p-1}\varphi' + \varphi''' + \tau\varphi'' = 0 \tag{6}$$

with

$$(f * \varphi)(\xi) = \int\limits_0^{+\infty} f(s)\varphi(\xi + cs)\,\mathrm{d}s, \tag{7}$$

here $' = \mathrm{d}/\mathrm{d}\xi$ and $\lim_{\xi\to\pm\infty}\varphi(\xi) = 0$.

Taking the translation $u = \varphi/\sqrt[p]{c-\alpha}$ and ${}^{(i)} = \mathrm{d}^i/\mathrm{d}z^i$, $i = 1,2,3$, $z = \sqrt{c-\alpha}\,\xi$, Eq. (6) with Eq. (7) could be rewritten as

$$-u^{(1)} + \beta(f * u)u^{p-1}u^{(1)} + u^{(3)} + \frac{\tau}{\sqrt{c-\alpha}}u^{(2)} = 0 \tag{8}$$

with

$$(f * u)(z) = \int\limits_0^{+\infty} f(w)u\biggl(\frac{z}{\sqrt{c-\alpha}} + cw\biggr)\,\mathrm{d}w.$$

If we find a solitary solution (u, τ, c, z) to Eq. (8), then the corresponding (φ, τ, c, ξ) is our solitary wave solution to Eq. (6) and, therefore, to the original Eq. (3).

Under the assumption that the distributed delay kernel $f(t)$ is the weak generic kernel, our main idea is to change the existence problem for the functional differential Eq. (8) into the existence of a homoclinic in two-dimensional invariant manifold.

By using the linear chain trick, Eq. (6) with the weak generic delay can be transformed into a nondelay three-dimensional ordinary differential system. When the delay τ

is sufficiently small, the three-dimensional ordinary differential system is a standard singularly perturbed system. The rest part of this paper is organized as follows. At first, we briefly introduce some basic lemmas. And we will use these lemmas to establish results on solitary wave for nondelay Eq. (6). At last, by employing the geometrical singular perturbation theory, we will prove there exist a solitary wave solution for Eq. (6) with the weak generic delay and τ is sufficiently small for a particular wave speed.

2 Preliminaries

For convenience, we present some results in the paper [9, 21] and establish a lemma, which will be employed in the proof of our main theorem.

Lemma 1 (Geometric singular perturbation theorem). *For the system*

$$\begin{aligned} x'(t) &= f(x, y, \epsilon), \\ y'(t) &= \epsilon g(x, y, \epsilon), \end{aligned} \tag{9}$$

where $x \in R^n$, $y \in R^l$ and ϵ is a real parameter, f, g are C^∞ on the set $V \times I$, where $V \in R^{n+l}$ and I is an open interval, containing 0. If when $\epsilon = 0$, the system has a compact, normally hyperbolic manifold of critical points M_0, which is contained in the set $\{f(x, y, 0) = 0\}$. Then for any $0 < r < +\infty$, if $\epsilon > 0$, but sufficiently small, there exists a manifold M_ϵ:

(i) *which is locally invariant under the flow of* (9);
(ii) *which is C^r in x, y and ϵ*;
(iii) $M_\epsilon = \{(x, y)\colon x = h^\epsilon(y)\}$ *for some C^r function $h^\epsilon(y)$ and y in some compact K*;
(iv) *there exist locally invariant stable and unstable manifolds $W^s(M_\epsilon)$ and $W^u(M_\epsilon)$ that lie within $O(\epsilon)$, and are diffeomorphic to $W^s(M_0)$ and $W^u(M_0)$.*

Then it will be useful for referencing to know what the corresponding results are in the nondelay case. Because the questions, that we will address are the questions of persistence of solitary waves when the delay is small.

In traveling wave form, with $U(x, t) = \varphi(\xi)$, $\xi = x - ct$ and $c > 0$, the nondelay equation (that is, Eq. (4) with $f(t) = \delta(t)$, $\tau = 0$, i.e., generalized KdV equation (6)) reads

$$-(c - \alpha)\varphi' + \beta\varphi^p\varphi' + \varphi''' = 0, \tag{10}$$

where $' = \mathrm{d}/\mathrm{d}\xi$. Using the boundary condition at $\pm\infty$ and integrating Eq. (10) once, we yield to the equation

$$-(c - \alpha)\varphi + \frac{\beta}{p+1}\varphi^{p+1} + \varphi'' = 0.$$

Taking $u = \varphi / \sqrt[p]{c - \alpha}$, $z = \sqrt{c - \alpha}\,\xi$ and $\alpha < c$, we have

$$-u + \frac{\beta}{p+1}u^{p+1} + \ddot{u} = 0,$$

where $\cdot = \mathrm{d}/\mathrm{d}z$. We get an equivalent form

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= u - \frac{\beta}{p+1}u^{p+1}.\end{aligned} \tag{11}$$

We are now in a position to yield the existence of a solitary wave solution of the nondelay equation.

Lemma 2. *In the* (u, v) *phase plane, Eq.* (11) *has a homoclinic orbit to the critical point* $(0, 0)$. *This connection is confined to* $u > 0$.

Proof. It is easily to see that Eq. (11) has two critical points $O(0, 0)$, $N(\sqrt[p]{p + 1/\beta}, 0)$. The origin is always a saddle and $N(\sqrt[p]{p + 1/\beta}, 0)$ is center. Notice that Eq. (11) is a Hamiltonian system with the Hamiltonian function

$$H(u, v) = \frac{1}{2}v^2 - \frac{1}{2}u^2 + \frac{\beta}{p^2 + 3p + 2}u^{p+2}.$$

Consider a level curve of the form $H = k$ in the region $\{u > 0\}$, and when $k = 0$ it includes a homoclinic orbit to $(0, 0)$, i.e., $(u, \pm\sqrt{1 - 2\beta u^p/(p^2 + 3p + 2)})$, $0 < u \leqslant \sqrt{(p^2 + 3p + 2)/(2\beta)}$, which is a 1-soliton solution to mBKdV equation. So there is a homoclinic orbit to $(0, 0)$. □

3 Existence of solitary waves

In this section, we shall mainly analyze Eq. (8) for solitary waves, which in the particular case when the kernel f is the first case of the two in Eq. (5), the weak generic delay case. The corresponding calculations for the strong kernel are similar but a little complicated and will be omitted. Recall that the parameter τ measures the delay and $u^{(2)}$ means perturbation. The main result can be stated as follow.

Theorem 1. *For any* $\tau > 0$ *sufficiently small and constant* $\beta > 0$*, there exists speed* $c > \alpha$ *such that system Eq.* (8) *has solitary wave, then Eq.* (3) *has solitary wave solution.*

Proof. According to (5), we take the function $f(t)$ as

$$f(t) = \frac{1}{\tau}\mathrm{e}^{-t/\tau}, \quad \tau > 0,$$

and we define w by

$$w(z) = (f * u)(z) = \int_0^{+\infty} \frac{1}{\tau}\mathrm{e}^{-t/\tau} u\left(\frac{z}{\sqrt{c - \alpha}} + ct\right) \mathrm{d}t.$$

Differentiating with respect to z, we can obtain that

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{1}{c\sqrt{c - \alpha}\tau}(w - u).$$

Using the boundary condition at $-\infty$, Eq. (8) can be integrated once to yield the equation

$$u'' + \frac{\tau}{\sqrt{c-\alpha}}u' - u + \beta F = 0,$$

where

$$F(z) = \int_{-\infty}^{z} (f * u)u^{p-1}u' \,\mathrm{d}\xi = \int_{-\infty}^{z} u^{p-1}u'w \,\mathrm{d}\xi. \tag{12}$$

If we further denote $u' = v$, $v' = w$, then Eq. (8) with the kernel given above can be replaced by the system

$$\begin{aligned} u' &= v, \\ v' &= u - \frac{\tau}{\sqrt{c-\alpha}}v - \beta F, \\ c\sqrt{c-\alpha}\tau w' &= w - u. \end{aligned} \tag{13}$$

From system (12) and Eq. (13), one has

$$\begin{aligned} \frac{\partial F}{\partial u} &= \frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}u} = \frac{u^{p-1}u'w}{u'} = u^{p-1}w, \\ \frac{\partial F}{\partial w} &= \frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}w} = \frac{u^{p-1}u'w}{w'} = c\sqrt{c-\alpha}\tau\frac{u^{p-1}vw}{w-u}, \\ \frac{\partial F}{\partial v} &= \frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}z}{dv} = \frac{u^n vw}{u - \tau/\sqrt{c-\alpha} - \beta F} \end{aligned}$$

with the boundary condition $F(-\infty) = 0$ implies that $F(z) = F(u, v, w, \tau)$.

Note that if $\tau = 0$, then

$$F(u, v, w, 0) = \int_{-\infty}^{z} u^p u' \,\mathrm{d}\xi = \frac{u^{p+1}}{p+1}.$$

System (13) reduces to the following ordinary differential equation:

$$u'' = u - \beta F = u - \frac{\beta u^{p+1}}{p+1}.$$

For $\tau > 0$, system (13) defines a system of ODEs whose solutions evolve in the three-dimensional (u, v, w) phase space. In this phase space, there are critical points at

$$O(0,0,0), \qquad B\left(\sqrt[p]{\frac{p+1}{\beta}}, 0, \sqrt[p]{\frac{p+1}{\beta}}\right).$$

The linearized matrix of (13) at critical point $O(0, 0, 0)$ is

$$J(u, v, w) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\tau/\sqrt{c-\alpha} & 0 \\ -1/(c\sqrt{c-\alpha}\tau) & 0 & 1/(c\sqrt{c-\alpha}\tau) \end{pmatrix}.$$

The eigenvalues λ of this matrix satisfy

$$\tau c\sqrt{c-\alpha}\lambda^3 + \left(c\tau^2 - 1\right)\lambda^2 - \left(c\tau\sqrt{c-\alpha} + \frac{\tau}{\sqrt{c-\alpha}}\right)\lambda + 1 = 0.$$

This equation has two real positive roots $1/(c\sqrt{c-\alpha}\tau)$, $(\sqrt{\tau^2+4(c-\alpha)} - \tau)/(2\sqrt{c-\alpha})$ and one negative root $-(\sqrt{\tau^2+4(c-\alpha)} - \tau)/(2\sqrt{c-\alpha})$. A solitary wave solution of the original equation will exist if among the solutions of system (13), there exists a homoclinic orbit to the critical point $O(0,0,0)$.

We want to show that for $\tau > 0$ but very small, the positive branch of the one-dimensional stable manifold of $O(0,0,0)$ for system (13), $W^s_\tau(O)$, connect to the origin.

When $\tau = 0$, system (13) does not define a dynamical system in R^3. By introducing a new independent variable η defined by $z = \tau\eta$, the flow system (13) becomes the following fast system:

$$\begin{aligned}
&\dot{u} = \tau v,\\
&\dot{v} = \tau\left(u - \frac{\tau}{\sqrt{c-\alpha}}v - \beta F\right),\\
&c\sqrt{c-\alpha}\dot{w} = w - u,
\end{aligned}\tag{14}$$

since the time scale z is slow and η is fast, where $\cdot = \mathrm{d}/\mathrm{d}\eta$. If $\tau > 0$ and sufficiently small, flow system (13) and fast system (14) are equivalent.

Consider the slow system (13), for $\tau = 0$, then the flow of that system is confined to the set

$$M_0 = \left\{(u,v,w) \in R^3\colon\ w = u\right\},$$

which is, therefore, a two-dimensional invariant manifold for system (13). If M_0 is normally hyperbolic, then for sufficiently small $\tau > 0$, Lemma 1 provides us with a two-dimensional invariant manifold M_τ for system (13), studying the system (13) reduced to this manifold, the dimensionality is reduced back to 2 and the existence of the homoclinic orbit we are seeking can be established.

In order to check that the invariant manifold M_0 is a normally hyperbolic in the sense of Fenichel [9, 21], we need to check that the linearization of the fast system (14), restricted to M_0, has precisely $\dim M_0$ eigenvalues on the imaginary axis, with the remainder of the spectrum being hyperbolic. The linearization of Eq. (14) restricted to M_0 is given by the following matrix:

$$\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ -1/(c\sqrt{c-\alpha}) & 0 & 1/(c\sqrt{c-\alpha}) \end{pmatrix},$$

which has three eigenvalues $0, 0, 1/(c\sqrt{c-\alpha})$. Thus M_0 is normally hyperbolic.

According to geometric singular perturbation theory, there exist a locally invariant two-dimensional manifold M_τ with $\tau > 0$ but sufficiently small, under the flow of system (13), which can be written in the form

$$M_\tau = \left\{(u,v,w) \in R^3\colon\ w = u + g(u,v,\tau)\right\},$$

where the function g is a smooth function defined on a compact domain, and it satisfies $g(u, v, 0) = 0$.

By substituting into slow system (13), g must satisfy

$$c\sqrt{c-\alpha}\tau\left[v + \frac{\partial g}{\partial u}v + \frac{\partial g}{\partial v}\left(u - \frac{\tau}{\sqrt{c-\alpha}}v - \beta F\right)\right] = g. \tag{15}$$

Since τ is small, we attempt solutions of this partial differential equation in the form of regular perturbation series in τ. From Eq. (15), we can see when $\tau = 0$, g is zero, then let g be

$$g(u, v, \tau) = \tau g_1(u, v) + \tau^2 g_2(u, v) + \cdots. \tag{16}$$

Substituting Eq. (16) into Eq. (15) and comparing powers of τ yields, which implies

$$g_1(u, v) = c\sqrt{c-\alpha}v, \qquad g_2(u, v) = c^2(c-\alpha)\left(u - \frac{u^{p+1}}{p+1}\right).$$

We would study the flow of system (13) restricted to M_τ and show that it has a solitary solution. The slow system (13) restricted to M_τ is, therefore, given by

$$\begin{aligned} u' &= v, \\ v' &= u - \frac{\beta}{p+1}u^{p+1} - \tau\left(\frac{v}{\sqrt{c-\alpha}} + \beta c\sqrt{c-\alpha}G\right) + O\left(\tau^2\right), \end{aligned} \tag{17}$$

where $G(z) = \int_{-\infty}^{z} u^{p-1}v^2 \,\mathrm{d}\xi$, by the same way, we have $G(z) = G(u, v, \tau)$. It is easily verified that for $\tau > 0$, system (17) still has critical point $(u, v) = (0, 0)$.

For convenience, we consider delay parameter τ and wave speed c as variables, then system (17) is equivalent to

$$\begin{aligned} u' &= v, \\ v' &= u - \frac{\beta}{p+1}u^{p+1} - \tau\left(\frac{v}{\sqrt{c-\alpha}} + \beta c\sqrt{c-\alpha}G\right) + O\left(\tau^2\right), \\ \tau' &= 0, \\ c' &= 0. \end{aligned} \tag{18}$$

So we can study the flow in $(u, v, \tau, c) \in R^4$. We seek homoclinic orbits for system (18) with small τ. The critical point 0 can, in reference to system (18), be construed as a surface of critical point, say S, parameterized by c, τ, i.e., critical point $(u, v) = (u(c, \tau), v(c, \tau)) = (0, 0)$. This in turn spawns an unstable manifold $W_\tau^u(S)$ and stable manifold $W_\tau^s(S)$ which meet in the curve at $\tau = 0$, namely, the homoclinic orbit found already in Lemma 2. Furthermore, by Lemma 1, $W_\tau^u(S)$ and $W_\tau^s(S)$ must still cross hyperplane $v = 0$. In the set $v = 0$, we parameterize W^u and W^s, respectively, near the intersection away from the critical point 0 as $u = h^-(c, \tau)$ and $u = h^+(c, \tau)$.

We next define

$$d(c, \tau) = h^-(c, \tau) - h^+(c, \tau),$$

and observe that zeroes of d render homoclinic orbits. From Lemma 2, there are homoclinic orbits independently of c when $\tau = 0$, we have that $d(c,0) = 0$, and thus let $d(c,\tau) = \tau\bar{d}(c,\tau)$. Then we have

$$\bar{d}(c,0) = M(c) := \left(\frac{\partial h^-}{\partial \tau} - \frac{\partial h^+}{\partial \tau}\right)\bigg|_{\tau=0}.$$

If there exists a (unique) value of $c = c(\tau)$ for τ small, near to $c = c(0)$, such that $\bar{d}(c,\tau) = 0$, that means if at $c = c(0)$,

$$M(c) = 0, \qquad M'(c) \neq 0 \tag{19}$$

hold, then it is a simple application of the Implication function theorem to see that there is a curve of homoclinic orbits exists.

Now we consider the Eq. (17), for which an intersection of $W^u(S)$ and $W^s(S)$ is sought. We need to establish the existence of $M(c)$ defined in (19).

In fact the variational equation for system (18), the differential form with $\tau = 0$ can be calculated as

$$\begin{aligned}
\mathrm{d}u' &= \mathrm{d}v,\\
\mathrm{d}v' &= \mathrm{d}u - \beta u^p \mathrm{d}u - \left(\frac{v}{\sqrt{c-\alpha}} + \beta c\sqrt{c-\alpha}G\right)\mathrm{d}\tau,\\
\mathrm{d}\tau' &= 0,\\
\mathrm{d}c' &= 0.
\end{aligned}$$

For the tangent spaces $\varPi^{\pm}(0)$ of the invariant manifold $W^u_\tau(S)$ and $W^s_\tau(S)$, there are three tangent vectors to W^u and W^s at $z = 0$ that are easily found (when $\tau = 0$, $v(0) = 0$)

$$\begin{aligned}
\eta_1 &= \left(\frac{\partial h^{\pm}}{\partial \tau}, 0, 1, 0\right),\\
\eta_2 &= \left(v, u - \frac{\beta}{p+1}u^{p+1}, 0, 0\right) = (v, \delta, 0, 0),\\
\eta_3 &= (0, 0, 0, 1),
\end{aligned}$$

where the u of η_2 satisfies $u > \sqrt[p]{(p+1)/\beta}$, i.e., $\delta < 0$. It can be checked that

$$\mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}c(\eta_1, \eta_2, \eta_3) = \sum_\pi (-1)^{\mathrm{sgn}\,\pi}\,\mathrm{d}u(\eta_{\pi(1)})\,\mathrm{d}v(\eta_{\pi(2)})\,\mathrm{d}c(\eta_{\pi(3)}) = \delta\frac{\partial h^{\pm}}{\partial \tau},$$

where π is a permutation of $(1,2,3)$. Then we can see that the equation for the form $\mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}c$ can be calculated as $(\tau = 0)$

$$\begin{aligned}
(\mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}c)' &= \mathrm{d}u' \wedge \mathrm{d}v \wedge \mathrm{d}c + \mathrm{d}u \wedge \mathrm{d}v' \wedge \mathrm{d}c + \mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}c'\\
&= \mathrm{d}u \wedge dv' \wedge \mathrm{d}c
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{d}u \wedge \left[\mathrm{d}u - \beta u^p \mathrm{d}u - \left(\frac{v}{\sqrt{c-\alpha}} + \beta c\sqrt{c-\alpha}G\right)\mathrm{d}\tau\right] \wedge \mathrm{d}c \\
&= -\left(\frac{v}{\sqrt{c-\alpha}} + \beta c\sqrt{c-\alpha}G\right)\mathrm{d}u \wedge \mathrm{d}\tau \wedge \mathrm{d}c.
\end{aligned}
$$

Similarly, the form $\mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}c$, when applied to the tangent space $\Pi^{\pm}(z)$ at $\eta_i z$, $i = 1, 2, 3$, can actually be calculated. Since

$$
\begin{aligned}
&\eta_1 z = (*, *, 1, 0), \\
&\eta_2 z = \left(v, u - \frac{\beta}{p+1}u^{p+1}, 0, 0\right), \\
&\eta_3 z = (*, *, 0, 1),
\end{aligned}
$$

we can get $\mathrm{d}u \wedge \mathrm{d}\tau \wedge \mathrm{d}c(\eta_1 z, \eta_2 z, \eta_3 z) = -v$. It follows that

$$
(\mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}c)' = \beta c\sqrt{c-\alpha}Gv + \frac{1}{\sqrt{c-\alpha}}v^2.
$$

So letting $p^{\pm}(z) = \mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}c(\Pi^{\pm}(z)$, we obtain

$$
(p^{\pm})' = \beta c\sqrt{c-\alpha}Gv + \frac{1}{\sqrt{c-\alpha}}v^2. \tag{20}
$$

It is easy to check that $p^{\pm} \to 0$ as $z \to \pm\infty$. Eq. (20) can then be easily solved to render

$$
p^{\pm} = \int_{\pm\infty}^{z} \left(\beta c\sqrt{c-\alpha}Gv + \frac{1}{\sqrt{c-\alpha}}v^2\right)\mathrm{d}\xi,
$$

which shows that $p^{\pm}(0) = \delta(\partial h^{\pm}/\partial\tau)$. Then we get

$$
\delta\frac{\partial h^-}{\partial\tau} = \int_{-\infty}^{0} \left(\beta c\sqrt{c-\alpha}Gv + \frac{1}{\sqrt{c-\alpha}}v^2\right)\mathrm{d}\xi \tag{21}
$$

and

$$
\delta\frac{\partial h^+}{\partial\tau} = -\int_{0}^{+\infty} \left(\beta c\sqrt{c-\alpha}Gv + \frac{1}{\sqrt{c-\alpha}}v^2\right)\mathrm{d}\xi. \tag{22}
$$

From Eq. (21) and Eq. (22), we obtain that

$$
\delta M(c) = \delta\frac{\partial h^-}{\partial\tau} - \delta\frac{\partial h^+}{\partial\tau} = \int_{-\infty}^{+\infty} \left(\beta c\sqrt{c-\alpha}Gv + \frac{1}{\sqrt{c-\alpha}}v^2\right)\mathrm{d}\xi. \tag{23}
$$

Substituting $G(z)=\int_{-\infty}^{z}u^{p-1}v^2\,\mathrm{d}\xi$ into Eq. (23), we get

$$
\begin{aligned}
\delta M(c) &= \beta c\sqrt{c-\alpha}\int_{-\infty}^{+\infty} v\left(\int_{-\infty}^{z} u^{p-1}v^2\,\mathrm{d}\xi\right)\mathrm{d}\xi + \frac{1}{\sqrt{c-\alpha}}\int_{-\infty}^{+\infty} v^2\,\mathrm{d}\xi \\
&= \beta c\sqrt{c-\alpha}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{z} u^{p-1}v^2\,\mathrm{d}\xi\right)\mathrm{d}u + \frac{1}{\sqrt{c-\alpha}}\int_{-\infty}^{+\infty} v^2\,\mathrm{d}\xi \\
&= \beta c\sqrt{c-\alpha}\left(u\int_{-\infty}^{z} u^{p-1}v^2\,\mathrm{d}\xi\bigg|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} u^p v^2\,\mathrm{d}\xi\right) + \frac{1}{\sqrt{c-\alpha}}\int_{-\infty}^{+\infty} v^2\,\mathrm{d}\xi \\
&= \beta c\sqrt{c-\alpha}\left(-\int_{-\infty}^{+\infty} u^p\dot{u}^2\,\mathrm{d}\xi\right) + \frac{1}{\sqrt{c-\alpha}}\int_{-\infty}^{+\infty} \dot{u}^2\,\mathrm{d}\xi \\
&= \beta c\sqrt{c-\alpha}\left[\int_{-\infty}^{+\infty}(1-u^p)\dot{u}\dot{u}\,\mathrm{d}\xi - \int_{-\infty}^{+\infty}\dot{u}^2\,\mathrm{d}\xi\right] + \frac{1}{\sqrt{c-\alpha}}\int_{-\infty}^{+\infty} \dot{u}^2\,\mathrm{d}\xi \\
&= \beta c\sqrt{c-\alpha}\left[\int_{-\infty}^{+\infty}\dot{u}d\left(u-\frac{u^{p+1}}{p+1}\right)\right] + \left(\frac{1}{\sqrt{c-\alpha}} - \beta c\sqrt{c-\alpha}\right)\int_{-\infty}^{+\infty} \dot{u}^2\,\mathrm{d}\xi \\
&= \beta c\sqrt{c-\alpha}\left[\left(u-\frac{u^{p+1}}{p+1}\right)\dot{u}\bigg|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\left(u-\frac{u^{p+1}}{p+1}\right)\ddot{u}\,\mathrm{d}\xi\right] \\
&\quad + \left(\frac{1}{\sqrt{c-\alpha}} - \beta c\sqrt{c-\alpha}\right)\int_{-\infty}^{+\infty} \dot{u}^2\,\mathrm{d}\xi \\
&= \beta c\sqrt{c-\alpha}\left[\left(\frac{1}{\beta c(c-\alpha)}-1\right)\int_{-\infty}^{+\infty}\dot{u}^2\,\mathrm{d}\xi - \int_{-\infty}^{+\infty}\ddot{u}^2\,\mathrm{d}\xi\right],
\end{aligned}
$$

where u comes from the underlying, already known, homoclinic orbit of Lemma 2, and $\delta\neq 0$, we obtain that

$$
M(c)=\frac{1}{\delta}\beta c\sqrt{c-\alpha}\left[\left(\frac{1}{\beta c(c-\alpha)}-1\right)\int_{-\infty}^{+\infty}\dot{u}^2\,\mathrm{d}\xi - \int_{-\infty}^{+\infty}\ddot{u}^2\,\mathrm{d}\xi\right].
$$

It is clear then that Eq. (19) at a unique value of c. □

Remark 1. The perturbation term u_{xx} is necessary in Eq. (3) for the existence of homoclinic. Otherwise if Eq. (3) does not have the perturbation term u_{xx}, the equation corresponding to Eq. (8) would be

$$-u^{(1)} + \beta(f * u)u^{p-1}u^{(1)} + u^{(3)} = 0,$$

which is equivalent to

$$\begin{aligned}
&u' = v, \\
&v' = u - \beta F, \\
&c\sqrt{c-\alpha}\tau w' = w - u,
\end{aligned}$$

where $F(z) = \int_{-\infty}^{z} u^{p-1}u'w \,\mathrm{d}\xi$. The $M(c)$ function is

$$M(c) = -\frac{1}{\delta}\beta c\sqrt{c-\alpha}\left[\int_{-\infty}^{+\infty} \dot{u}^2 \,\mathrm{d}\xi + \int_{-\infty}^{+\infty} \ddot{u}^2 \,\mathrm{d}\xi\right].$$

Then homoclinic cannot exist in this case, so the perturbation term u_{xx} is necessary.

Remark 2. We extended the results in [18] since Eqs. (2) and (3) are more generalized. In fact, choosing $\alpha = 0$, $\beta = 1$, $p = n + 1$, where $n \in Z^+ \cup \{0\}$, then Eq. (2) would be

$$U_t + U^{n+1}U_x + U_{xxx} = 0,$$

and the corresponding to Eq. (3) would be

$$U_t + (f * U)U^n U_x + U_{xxx} + \tau U_{xx} = 0,$$

which was the equations discussed in [18]. Obviously, the results given in [18] is not available to our following example.

Example. Taking $\alpha = -1, \beta = 6, p = n + 1$, Eq. (3) would be

$$U_t - U_x + 6(f * U)U^n U_x + U_{xxx} + \tau U_{xx} = 0, \tag{24}$$

and the corresponding to Eq. (8) would be

$$-u^{(1)} + 6(f * u)u^n u^{(1)} + u^{(3)} + \frac{\tau}{\sqrt{c+1}}u^{(2)} = 0,$$

which is equivalent to

$$\begin{aligned}
&\dot{u} = v, \\
&\dot{v} = u - \frac{\tau}{\sqrt{c+1}}v - F, \\
&c\sqrt{c+1}\tau\dot{w} = w - u,
\end{aligned}$$

where $F(z) = \int_{-\infty}^{z} u\dot{u}w \,\mathrm{d}\xi$. The $M(c)$ function is

$$M(c) = \frac{1}{\delta}c\sqrt{c+1}\left[\left(\frac{1}{6c(c+1)} - 1\right)\int_{-\infty}^{+\infty} \dot{u}^2 \,\mathrm{d}\xi - \int_{-\infty}^{+\infty} \ddot{u}^2 \,\mathrm{d}\xi\right].$$

From Theorem 1, Eq. (24) exists the homoclinic orbit, i.e., Eq. (24) has solitary wave solution.

4 Conclusion

In this work, we establish the existence result of solitary wave solutions for a generalized KdV–mKdV equation with time delay under assumption that the distributed delay kernel $f(t)$ is the weak generic kernel. We could obtain the similar result for the strong generic kernel. Our methods are geometrical singular perturbation theory and the linear chain trick. Furthermore, we investigate the traveling wave solutions for the generalized KdV–mKdV equation by using (G'/G)-expansion method. We will leave this for future work.

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