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Some conditions of regularity of linear extensions of dynamical systems with respect to selected variables

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Abstract. In this paper, we dealt with the issue of the regularity of the linear extension of the dynamical system. Using the Lyapunov function with the changeable sign, conditions for the regularity of the entire examined dynamical system were given, assuming that only a certain separate part of this system was regular. Our results were generalized and improved to the wider class of dynamical systems, for which only a certain combination of components is regular.

Keywords: Green's function, linear extension of dynamical systems, linear regular extensions, weak regularity, quadratic form, Lyapunov's function, positive definite matrix.

1 Introduction

It is well known (see [1–3]) that the examination of maintenance of the invariant tori of dynamical systems is connected with the existence of the Green's function for linearized system i.e. linear extension of dynamical system. Precisely, if such a linear extension has the Green's function, then the invariant torus of heterogeneous linear extensions can be written in an explicit integral form. If there exists a unique Green's function for linear extensions, then the system is called regular. This paper is devoted to investigating the regularity of linear extensions under the condition of non–degenerated quadratic form existence whose derivative is positive definite with respect to some components. In other words, some components of the system are regular and we wonder whether the entire system will be also regular. Similar research can be found in [4].

Let us start by considering a system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = A(x)y, \tag{1}$$

with the continuous and finite on $\mathbb{R}^m$ matrix of coefficients $A(x)$, $n \times n$-dimensional, besides $y \in \mathbb{R}^n$, $x \in \mathbb{R}^m$. Let us suppose $f(x) = (f_1(x), f_2(x), \dots, f_m(x))$ is a vector function defined for all $x \in \mathbb{R}^m$ and it satisfies the Lipschitz condition.

In addition, let us assume that it fulfils all the necessary conditions to ensure that there exists a unique solution $x = x(t; x_0)$ for every $t \in \mathbb{R}$ of Cauchy's problem $\mathrm{d}x/\mathrm{d}t = f(x)$, $x|_{t=0} = x_0$ for every fixed value $x_0 \in \mathbb{R}^m$.

We will denote by $C'(\mathbb{R}^m, f)$ subspace of space $C^0(\mathbb{R}^m)$ of functions $F(x)$ such that superposition (composite function) $F(x(t; x))$ as a function of variable t is continuously differentiable, wherein

$$\dot{F}(x) \stackrel{\mathrm{df}}{=} \frac{\mathrm{d}}{\mathrm{d}t} F\bigl(x(t; x)\bigr)\Big|_{t=0}, \quad \dot{F}(x) \in C^0\bigl(\mathbb{R}^m\bigr).$$

It is known (see [5]) that the system (1) will have infinitely many different Green's functions $G_0(\tau, x)$ with the exponential estimation $\|G_0(\tau, x)\| \leqslant K \exp\{-\gamma|\tau|\}$, $K, \gamma = \mathrm{const} > 0$ if and only if the following quadratic form exists:

$$V = \bigl\langle S(x)z, z\bigr\rangle,$$

where $z \in \mathbb{R}^n$, $S(x)$ is n-dimensional matrix, $S(x) \in C'(\mathbb{R}^m, f)$, whose derivative with respect to the adjoint system (1) at normal variable y, i.e. system in the form $\mathrm{d}x/\mathrm{d}t = f(x)$, $\mathrm{d}z/\mathrm{d}t = -A^{\mathrm{T}}(x)z$, is positive definite:

$$\dot{V} = \bigl\langle \bigl[\dot{S}(x) - S(x)A^{\mathrm{T}}(x) - A(x)S(x)\bigr]z, z\bigr\rangle \geqslant \|z\|^2,$$

and wherein the matrix $S(x)$ is degenerated for some values $x = x_0$, i.e. $\det S(x_0) = 0$.

It turns out that the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = A(x)y, \qquad \frac{dz}{\mathrm{d}t} = y - A^{\mathrm{T}}(x)z, \tag{2}$$

where $z \in \mathbb{R}^n$, will always have a unique $2n \times 2n$-dimensional Green's function $\bar{G}_0(\tau, x)$ no matter if system (1) has infinitely many different functions of this kind or only one. Then the derivative of a quadratic form

$$W = 2p\langle y, z\rangle + \bigl\langle S(x)z, z\bigr\rangle$$

with respect to system (2) for sufficiently large fixed values of real-parameter $p > 0$ will be positive definite.

Let us look at system (2) in a slightly different way. We write it in the matrix form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} A(x) & 0 \\ I_n & -A^{\mathrm{T}}(x) \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}, \tag{3}$$

and suppose that for some components of system (3) there exists a quadratic form whose derivative with respect to these components is positive definite. Therefore, let there exist a quadratic form $V = \langle S(x)z, z\rangle$, whose derivative with respect to the component of system (3)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad \frac{dz}{\mathrm{d}t} = -A^{\mathrm{T}}(x)z$$

is positive definite and a quadratic form $V = \langle y, z\rangle$, whose derivative with respect to system (3) is positive semidefinite. As mentioned before, it is clear that system (3) is regular.

2 Main results

Let us now slightly generalize system (3) and see if, on the assumption that for some components there exist appropriate quadratic forms, this new system will be regular.

Let us start with the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^n, \tag{4}$$

where $x \in \mathbb{R}^m$, $y_1 \in \mathbb{R}^{n_1}$, $y_2 \in \mathbb{R}^{n_2}$, $n = n_1 + n_2$, $A_{ij}(x)$ – matrices from $C^0(\mathbb{R}^m)$. Let us suppose as above that there exists a quadratic form V_2 with n_2-dimensional matrix $S_2(x) \in C'(\mathbb{R}^m, f)$

$$V_2 = \bigl\langle S_2(x)y_2, y_2\bigr\rangle,$$

whose derivative with respect to the component of system (4)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad \frac{\mathrm{d}y_2}{\mathrm{d}t} = A_{22}(x)y_2$$

is positive definite, i.e.

$$\dot V_2 = \bigl\langle \bigl[\dot S_2(x) + S_2(x)A_{22}(x) + A_{22}^{\mathrm{T}}(x)S_2(x)\bigr]y_2, y_2\bigr\rangle \geqslant \|y_2\|^2. \tag{5}$$

In addition, let us suppose that there exists a quadratic form V with n-dimensional non-degenerated matrix $S(x) \in C'(\mathbb{R}^m, f)$

$$V = \bigl\langle S(x)y, y\bigr\rangle,$$

whose derivative with respect to system (4) satisfies the inequality

$$\dot V = \bigl\langle \bigl[\dot S(x) + S(x)A(x) + A^{\mathrm{T}}(x)S(x)\bigr]y, y\bigr\rangle \geqslant \|y_1\|^2, \tag{6}$$

where

$$A(x) = \begin{bmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{bmatrix}.$$

Then, for sufficiently large fixed values of real-parameter $p > 0$, the derivative of square form

$$V_p = pV + V_2$$

with respect to system (4) is positive definite, hence, system (4) is regular. In fact, by calculating $\dot{V}$, we get

$$\begin{aligned}\dot{V}_p &= p\dot{V} + \dot{V}_2 \\ &\geqslant p\|y_1\|^2 + \langle \dot{S}_2(x)y_2, y_2\rangle + 2\langle S_2(x)y_2, A_{21}(x)y_1 + A_{22}(x)y_2\rangle \\ &\geqslant p\|y_1\|^2 + 2\langle S_2(x)y_2, A_{21}(x)y_1\rangle + \|y_2\|^2 \\ &\geqslant p\|y_1\|^2 - 2\|S_2(x)\|\|A_{21}(x)\|\|y_1\|\|y_2\| + \|y_2\|^2 \\ &\geqslant p\|y_1\|^2 - 2K\|y_1\|\|y_2\| + \|y_2\|^2 \geqslant \varepsilon(\|y_1\|^2 + \|y_2\|^2),\end{aligned}$$

where K is a constant and $\varepsilon = \mathrm{const} > 0$.

Now, denoting

$$C_1 = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \tag{7}$$

inequality (6) takes the following form:

$$\dot{V} = \langle [\dot{S}(x) + S(x)A(x) + A^{\mathrm{T}}(x)S(x)]y, y\rangle \geqslant \|C_1 y\|^2,$$

and the form of inequality (5) looks as follows:

$$\dot{V}_2 = \langle [\dot{\tilde{S}}(x) + \tilde{S}(x)A(x) + A^{\mathrm{T}}(x)\tilde{S}(x)]C_2 y, C_2 y\rangle \geqslant \|C_2 y\|^2,$$

where n-dimensional matrix $\tilde{S}(x)$ has the following form:

$$\tilde{S}(x) = \begin{bmatrix} 0 & 0 \\ 0 & S_2(x) \end{bmatrix}.$$

Generalizing the matrix $\tilde{S}(x)$ in the following way:

$$\tilde{S}(x) = \begin{bmatrix} S_{11}(x) & S_{12}(x) \\ S_{21}(x) & S_{22}(x) \end{bmatrix},$$

let us suppose that two following conditions are satisfied:

$$\begin{gathered}\langle [\dot{S}(x) + S(x)A(x) + A^{\mathrm{T}}(x)S(x)]y, y\rangle \geqslant \|C_1 y\|^2, \\ \langle [\dot{\tilde{S}}(x) + \tilde{S}(x)A(x) + A^{\mathrm{T}}(x)\tilde{S}(x)]C_2 y, C_2 y\rangle \geqslant \|C_2 y\|^2.\end{gathered} \tag{8}$$

The following theorem is true.

Theorem 1. *Let the two conditions* (8) *be satisfied with some $n \times n$-dimensional matrices $S(x), \tilde{S}(x) \in C'(\mathbb{R}^m, f)$ and the constant matrices from* (7)*. Then the derivative of quadratic form*

$$V_p = p\langle S(x)y, y\rangle + \langle \tilde{S}(x)y, y\rangle \tag{9}$$

with respect to system (4) *is positive definite for sufficiently large fixed values of real-parameter $p > 0$, so, system* (4) *is regular.*

Proof. By calculating the derivative of square form (9) with respect to system (4), we obtain the following estimation:

$$
\begin{aligned}
\dot{V}_p &= p\bigl\langle\bigl[\dot{S}(x)+S(x)A(x)+A^{\mathrm{T}}(x)S(x)\bigr]y,\, y\bigr\rangle \\
&\quad + \bigl\langle\bigl[\dot{\tilde{S}}(x)+\tilde{S}(x)A(x)+A^{\mathrm{T}}(x)\tilde{S}(x)\bigr]y,\, y\bigr\rangle \\
&\geqslant p\|C_1y\|^2 + \bigl\langle\bigl[\dot{\tilde{S}}(x)+\tilde{S}(x)A(x)+A^{\mathrm{T}}(x)\tilde{S}(x)\bigr](C_1+C_2)y,\, (C_1+C_2)y\bigr\rangle \\
&= p\|C_1y\|^2 + \bigl\langle\bigl[\dot{\tilde{S}}(x)+\tilde{S}(x)A(x)+A^{\mathrm{T}}(x)\tilde{S}(x)\bigr]C_1y,\, C_1y\bigr\rangle \\
&\quad + 2\bigl\langle\bigl[\dot{\tilde{S}}(x)+\tilde{S}(x)A(x)+A^{\mathrm{T}}(x)\tilde{S}(x)\bigr]C_1y,\, C_2y\bigr\rangle \\
&\quad + \bigl\langle\bigl[\dot{\tilde{S}}(x)+\tilde{S}(x)A(x)+A^{\mathrm{T}}(x)\tilde{S}(x)\bigr]C_2y,\, C_2y\bigr\rangle \\
&\geqslant p\|C_1y\|^2 - L\|C_1y\|^2 - 2L\|C_1y\|\|C_2y\| + \|C_2y\|^2 \\
&= (p-L)\|C_1y\|^2 - 2L\|C_1y\|\|C_2y\| + \|C_2y\|^2 \\
&\geqslant \frac{p-L-L^2}{p-L+1}\bigl(\|C_1y\|^2+\|C_2y\|^2\bigr),
\end{aligned}
$$

where $L = \|\dot{\tilde{S}}+\tilde{S}A+A^{\mathrm{T}}\tilde{S}\|_0 = \sup_{x\in\mathbb{R}^m}\|\dot{\tilde{S}}(x)+\tilde{S}(x)A(x)+A^{\mathrm{T}}(x)\tilde{S}(x)\|$.

Using

$$\|y\| = \bigl\|(C_1+C_2)y\bigr\| \leqslant \|C_1y\| + \|C_2y\|$$

and

$$\|y\|^2 \leqslant \|C_1y\|^2 + 2\|C_1y\|\|C_2y\| + \|C_2y\|^2 \leqslant 2\bigl(\|C_1y\|^2+\|C_2y\|^2\bigr),$$

we finally get the following estimation:

$$\dot{V}_p \geqslant \frac{p-L-L^2}{2(p-L+1)}\|y\|^2, \quad p > L+L^2. \qquad \square$$

Again, by making the generalization

$$C_1 = C_1(x) = \begin{bmatrix} C_{11}(x) & C_{12}(x) \\ C_{21}(x) & C_{22}(x)\end{bmatrix}, \qquad C_2 = C_2(x) = \begin{bmatrix} \tilde{C}_{11}(x) & \tilde{C}_{12}(x) \\ \tilde{C}_{21}(x) & \tilde{C}_{22}(x)\end{bmatrix},$$

we get the following theorem.

Theorem 2. *Let the two conditions be satisfied*:

$$
\begin{aligned}
&\bigl\langle\bigl[\dot{S}(x)+S(x)A(x)+A^{\mathrm{T}}(x)S(x)\bigr]y,\, y\bigr\rangle \geqslant \bigl\|C_1(x)y\bigr\|^2, \\
&\bigl\langle\bigl[\dot{\tilde{S}}(x)+\tilde{S}(x)A(x)+A^{\mathrm{T}}(x)\tilde{S}(x)\bigr]C_2(x)y,\, C_2(x)y\bigr\rangle \geqslant \bigl\|C_2(x)y\bigr\|^2,
\end{aligned} \tag{10}
$$

where $S(x)$, $\tilde{S}(x)$ are $n\times n$-dimensional matrices from $C'(\mathbb{R}^m, f)$ and $C_1(x)$, $\tilde{C}_2(x)$ are $n\times n$-dimensional matrices from $C^0(\mathbb{R}^m)$, which results in

$$C_1(x)C_2(x) \equiv C_2(x)C_1(x), \tag{11}$$

$$\det(C_1(x)+C_2(x)) \neq 0, \qquad \bigl\|\bigl(C_1(x)+C_2(x)\bigr)^{-1}\bigr\| \leqslant c = \mathrm{const} < \infty.$$

Then the derivative of quadratic form (9) *with respect to system* (4) *for sufficiently large values of parameter* $p > 0$ *will be positive definite.*

Proof. We have

$$\begin{aligned}
\dot V_p &= p\big\langle \big[\dot S(x) + S(x)A(x) + A^{\mathrm T}(x)S(x)\big] y,\ y\big\rangle \\
&\quad + \big\langle \big[\dot{\tilde S}(x) + \tilde S(x)A(x) + A^{\mathrm T}(x)\tilde S(x)\big] y,\ y\big\rangle \\
&\geqslant p\|C_1(x)y\|^2 + \big\langle \big[\dot{\tilde S}(x)+\tilde S(x)A(x)+A^{\mathrm T}(x)\tilde S(x)\big]\big(C_1(x)+C_2(x)\big)C^{-1}(x)y, \\
&\qquad \big(C_1(x)+C_2(x)\big)C^{-1}(x)y\big\rangle. \qquad (12)
\end{aligned}$$

Taking into account the last element of inequality (12)

$$\begin{aligned}
&\big\langle \big[\dot{\tilde S}(x) + \tilde S(x)A(x) + A^{\mathrm T}(x)\tilde S(x)\big]\big(C_1(x) + C_2(x)\big)C^{-1}(x)y, \\
&\big(C_1(x) + C_2(x)\big)C^{-1}(x)y\big\rangle \\
&\quad = \big\langle \big[\dot{\tilde S}(x) + \tilde S(x)A(x) + A^{\mathrm T}(x)\tilde S(x)\big]C_1(x)C^{-1}(x)y,\ C_1(x)C^{-1}(x)y\big\rangle \\
&\qquad + \big\langle \big[\dot{\tilde S}(x) + \tilde S(x)A(x) + A^{\mathrm T}(x)\tilde S(x)\big]C_1(x)C^{-1}(x)y,\ C_2(x)C^{-1}(x)y\big\rangle \\
&\qquad + \big\langle \big[\dot{\tilde S}(x) + \tilde S(x)A(x) + A^{\mathrm T}(x)\tilde S(x)\big]C_2(x)C^{-1}(x)y,\ C_1(x)C^{-1}(x)y\big\rangle \\
&\qquad + \big\langle \big[\dot{\tilde S}(x) + \tilde S(x)A(x) + A^{\mathrm T}(x)\tilde S(x)\big]C_2(x)C^{-1}(x)y,\ C_2(x)C^{-1}(x)y\big\rangle,
\end{aligned}$$

let us estimate each component of this element.

Therefore, we get

$$\begin{aligned}
&\big\langle \big[\dot{\tilde S}(x) + \tilde S(x)A(x) + A^{\mathrm T}(x)\tilde S(x)\big]C_1(x)C^{-1}(x)y,\ C_1(x)C^{-1}(x)y\big\rangle \\
&\quad \geqslant -M\|C_1(x)y\|^2,
\end{aligned}$$

where

$$\begin{aligned}
M &= \big\|\dot{\tilde S} + \tilde S A + A^{\mathrm T}\tilde S\big\|_0 \big\|C^{-1}\big\|_0^2 \\
&= \sup_{x\in\mathbb R^m} \big\|\dot{\tilde S}(x) + \tilde S(x)A(x) + A^{\mathrm T}(x)\tilde S(x)\big\| \sup_{x\in\mathbb R^m} \big\|\big(C_1(x) + C_2(x)\big)^{-1}\big\|^2.
\end{aligned}$$

Then

$$\begin{aligned}
&\big\langle \big[\dot{\tilde S}(x) + \tilde S(x)A(x) + A^{\mathrm T}(x)\tilde S(x)\big]C_2(x)C^{-1}(x)y,\ C_1(x)C^{-1}(x)y\big\rangle \\
&\quad \geqslant -M\big\|C_1(x)y\big\|\big\|C_2(x)y\big\|.
\end{aligned}$$

By using (10), (11), we get

$$\begin{aligned}
&\big\langle \big[\dot{\tilde S}(x) + \tilde S(x)A(x) + A^{\mathrm T}(x)\tilde S(x)\big]C_2(x)C^{-1}(x)y,\ C_2(x)C^{-1}(x)y\big\rangle \\
&\quad \geqslant \big\|C_2(x)C^{-1}(x)y\big\|^2 = \big\|C^{-1}(x)C_2(x)y\big\|^2 \geqslant \frac{1}{\|C\|_0^2}\big\|C_2(x)y\big\|^2.
\end{aligned}$$

Thus, by returning to the estimation of derivative $\dot{V}_p$, we obtain

$$\begin{aligned}\dot{V}_p &\geqslant (p-M)\|C_1(x)y\|^2 - 2M\|C_1(x)y\|\|C_2(x)y\| + \frac{1}{\|C\|_0^2}\|C_2(x)y\|^2\\ &\geqslant \frac{\gamma(p-M)-M^2}{p-M+\gamma}\bigl(\|C_1(x)y\|^2 + \|C_2(x)y\|^2\bigr), \quad \gamma = \frac{1}{\|C\|_0^2}.\end{aligned}$$

So, by using

$$\begin{aligned}\|y\| &= \|C^{-1}(C_1+C_2)y\| \leqslant \|C^{-1}C_1y\| + \|C^{-1}C_2y\|\\ &\leqslant \|C^{-1}\|\bigl(\|C_1y\| + \|C_2y\|\bigr)\end{aligned}$$

and

$$\begin{aligned}\|y\|^2 &\leqslant \|C^{-1}\|\bigl(\|C_1y\|^2 + 2\|C_1y\|\|C_2y\| + \|C_2y\|^2\bigr)\\ &\leqslant 2\|C^{-1}\|\bigl(\|C_1y\|^2 + \|C_2y\|^2\bigr),\end{aligned}$$

we finally get the following estimation:

$$\dot{V}_p \geqslant \frac{\gamma(p-M)-M^2}{2(p-M+\gamma)\|C^{-1}\|}\|y\|^2, \quad p > M + M^2\|C\|_0.$$

□

Remark 1. If condition (11) is not satisfied, then a quadratic form, whose derivative with respect to system (4) would be positive definite, cannot exist.

Now, let us consider the generalization of system (4) in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad \begin{aligned}\frac{\mathrm{d}y_1}{\mathrm{d}t} &= A_{11}(x)y_1 + A_{12}(x)y_2 + A_{13}(x)y_3,\\ \frac{\mathrm{d}y_2}{\mathrm{d}t} &= A_{21}(x)y_1 + A_{22}(x)y_2 + A_{23}(x)y_3,\\ \frac{\mathrm{d}y_3}{\mathrm{d}t} &= A_{31}(x)y_1 + A_{32}(x)y_2 + A_{33}(x)y_3,\end{aligned} \tag{13}$$

where $y_i \in \mathbb{R}^{n_i}$, $n_1 + n_2 + n_3 = n$, A_{ij} are the matrices of appropriate dimensions from $C^0(\mathbb{R}^m)$ and let us suppose that for the following component of system (13):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad \frac{\mathrm{d}y_3}{\mathrm{d}t} = A_{33}(x)y_3,$$

there exists $n_3 \times n_3$-dimensional matrix $S_{33} \in C'(\mathbb{R}^m, f)$, which results in

$$\bigl\langle \bigl[\dot{S}_{33}(x) + S_{33}(x)A_{33}(x) + A_{33}^{\mathrm{T}}(x)S_{33}(x)\bigr]y_3, y_3\bigr\rangle \geqslant \|y_3\|^2. \tag{14}$$

Let us denote

$$\hat{A}(x) = \begin{bmatrix} A_{22}(x) & A_{23}(x)\\ A_{32}(x) & A_{33}(x)\end{bmatrix}, \quad \hat{y} = \begin{bmatrix} y_2\\ y_3\end{bmatrix},$$

and let us suppose that for system (13) component in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad \frac{\mathrm{d}\hat{y}}{\mathrm{d}t} = \hat{A}(x)\hat{y},$$

there exists a square matrix $\hat{S}(x) \in C'(\mathbb{R}^m, f)$ of dimension $n_2 + n_3$, which results in

$$\left\langle \left[\dot{\hat{S}}(x) + \hat{S}(x)\hat{A}(x) + \hat{A}^{\mathrm{T}}(x)\hat{S}(x)\right]\hat{y},\, \hat{y}\right\rangle \geqslant \|y_2\|^2. \tag{15}$$

Finally, it denotes

$$\hat{A}(x) = \begin{bmatrix} A_{11}(x) & A_{12}(x) & A_{13}(x) \\ A_{21}(x) & A_{22}(x) & A_{23}(x) \\ A_{31}(x) & A_{32}(x) & A_{33}(x) \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Let us also suppose that for system (13), there exists n-dimensional square matrix $S_1(x) \in C'(\mathbb{R}^m, f)$, which results in

$$\left\langle \left[\dot{S}_1(x) + S_1(x)A(x) + A^{\mathrm{T}}(x)S_1(x)\right]y, y\right\rangle \geqslant \|y_1\|^2. \tag{16}$$

Now, taking the notations

$$S_3(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_3(x) \end{bmatrix}, \quad S_2(x) = \begin{bmatrix} 0 & 0 \\ 0 & \hat{S}(x) \end{bmatrix},$$

$$C_1 = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{n_3} \end{bmatrix}, \tag{17}$$

inequalities (14), (15), (16) can be written in the following form:

$$\left\langle \left[\dot{S}_1(x) + S_1(x)A(x) + A^{\mathrm{T}}(x)S_1(x)\right]y,\, y\right\rangle \geqslant \|C_1 y\|^2, \tag{18}$$

$$\left\langle \left[\dot{S}_2(x) + S_2(x)A(x) + A^{\mathrm{T}}(x)S_2(x)\right](C_2 + C_3)y,\, (C_2 + C_3)y\right\rangle \geqslant \|C_2 y\|^2, \tag{19}$$

$$\left\langle \left[\dot{S}_3(x) + S_3(x)A(x) + A^{\mathrm{T}}(x)S_3(x)\right]C_3 y,\, C_3 y\right\rangle \geqslant \|C_3 y\|^2. \tag{20}$$

Then the following theorem is true.

Theorem 3. *Let there exist three n-dimensional symmetrical matrices $S_j(x) \in C'(\mathbb{R}^m, f)$, $j = 1, 2, 3$, that satisfy inequalities* (18)–(20) *with matrices* (17). *Then the derivative of quadratic form*

$$V_p = p_1\left\langle S_1(x)y, y\right\rangle + p_2\left\langle S_2(x)y, y\right\rangle + \left\langle S_3(x)y, y\right\rangle \tag{21}$$

with respect to system (13) *for sufficiently large values of parameters $p_1, p_2 > 0$ will be positive definite.*

Outline of proof. Considering the quadratic form

$$V = \big\langle S(x, p_1, p_2)y, y\big\rangle$$

with parameters p_1, p_2, where

$$S(x, p_1, p_2) = p_1 S_1(x) + p_2 S_2(x) + S_3(x) = p_1 S_1(x) + S(x, p_2),$$

it can be shown that the derivative of this form is positive definite for sufficiently large values of p_1, p_2.

Analogically, as it was before, the following theorem is true.

Theorem 4. *Let there exist three symmetrical n-dimensional matrices $S(x)$, $\bar{S}(x)$, $\tilde{S}(x) \in C'(\mathbb{R}^m, f)$ that satisfy the following inequalities*:

$$\begin{aligned}
&\big\langle \big[\dot{S}(x) + S(x)A(x) + A^{\mathrm{T}}(x)S(x)\big]y,\, y\big\rangle \geqslant \big\|C_1(x)y\big\|^2,\\
&\big\langle \big[\dot{\bar{S}}(x) + \bar{S}(x)A(x) + A^{\mathrm{T}}(x)\bar{S}(x)\big]\big(C_2(x) + C_3(x)\big)y, \big(C_2(x) + C_3(x)\big)y\big\rangle\\
&\quad \geqslant \big\|C_2(x)y\big\|^2,\\
&\big\langle \big[\dot{\tilde{S}}(x) + \tilde{S}(x)A(x) + A^{\mathrm{T}}(x)\tilde{S}(x)\big]C_3(x)y, C_3(x)y\big\rangle \geqslant \big\|C_3(x)y\big\|^2
\end{aligned}$$

for some $n \times n$-dimensional matrices $C_i(x) \in C^0(\mathbb{R}^m)$, $i = 1, 2, 3$, such that

$$C_i(x)C(x) \equiv C(x)C_i(x), \quad i = 1, 2, 3,$$
$$\det C(x) \neq 0, \qquad \big\|C^{-1}(x)\big\| \leqslant c = \mathrm{const} < \infty,$$

where $C(x) = C_1(x) + C_2(x) + C_3(x)$. Then the derivative of quadratic form

$$V_p = p^3\big\langle S(x)y, y\big\rangle + p\big\langle \bar{S}(x)y, y\big\rangle + \big\langle \tilde{S}(x)y, y\big\rangle$$

with respect to system (13) *for sufficiently large values of parameter $p > 0$ will be positive definite.*

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