Fixed point theorems for α-contractive mappings of Meir–Keeler type and applications

Maher Berzig[a,1], **Mircea-Dan Rus**[b,2]

[a]Department of Mathematics, Tunis College of Sciences and Techniques
5 ave. Taha Hussein, Tunis, Tunisia
maher.berzig@gmail.com

[b]Department of Mathematics, Technical University of Cluj-Napoca
Memorandumului str. 28, 400114 Cluj-Napoca, Romania
rus.mircea@math.utcluj.ro

Received: 14 March 2013 / **Revised:** 29 August 2013 / **Published online:** 19 February 2014

Abstract. In this paper, we introduce the notion of α–contractive mapping of Meir–Keeler type in complete metric spaces and prove new theorems which assure the existence, uniqueness and iterative approximation of the fixed point for this type of contraction. The presented theorems extend, generalize and improve several existing results in literature. To validate our results, we establish the existence and uniqueness of solution to a class of third order two point boundary value problems.

Keywords: fixed point, α-contractive mapping of Meir–Keeler type, coupled fixed point, cyclic contraction, ordered metric space, two point boundary value problem.

1 Introduction

In [1], Meir and Keeler introduced a new contraction condition for self-maps in metric spaces and generalized the well known Banach contraction principle as follows.

Theorem 1. (*See* [1].) *Let* (X, d) *be a complete metric space and* $T : X \to X$. *Assume that, for every* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *such that*

$$x, y \in X\colon \quad \varepsilon \leqslant d(x, y) < \varepsilon + \delta(\varepsilon) \quad \Rightarrow \quad d(Tx, Ty) < \varepsilon.$$

Then T *has a unique fixed point* $x^* \in X$ *and* $T^n x \to x^*$ (*as* $n \to \infty$) *for every* $x \in X$, *where* T^n *denotes the* n*th order iterate of* T.

[1]Corresponding author.
[2]The author is grateful for the financial support provided by the Sectoral Operational Programme Human Resources Development 2007–2013 of the Romanian Ministry of Labor, Family and Social Protection through the Financial Agreement POSDRU/89/1.5/S/62557.

In another direction, Ran and Reurings [2] extended Banach's contraction principle to the setting of ordered metric spaces and obtained some interesting applications to matrix equations. Later on, the results of Ran and Reurings were extended and generalized by many authors (e.g., [3–12] and the references therein). In particular, Harjani et al. [13] unified these two directions by studying the fixed points of Meir–Keeler type contractions in ordered metric spaces.

Very recently, Samet et al. [14] took a new approach to the generalization of Banach's contraction principle and introduced the concept of α-ψ-contractive type mappings, while establishing various fixed point theorems for such mappings in the setting of complete metric spaces. In particular, this new approach contains many of the generalizations considered in [2–13] as special cases. We refer the reader to [15] for other generalization of the α-ψ-contractive type mapping.

In this context, the aim of this paper is to unify the concepts of Meir–Keeler contraction [1] and α-ψ-contractive type mapping [14] and establish some new fixed point theorems in complete metric spaces for such mappings. Several consequences of our results are presented in Section 3. We validate our results with an application to the study of the existence and uniqueness of solutions for a class of third order two point boundary value problems.

2 Main results

2.1 Preliminaries

Throughout this paper, let $\mathbb{N}$ denote the set of all non-negative integers, $\mathbb{Z}$ the set of all integers and $\mathbb{R}$ the set of all real numbers. We start by introducing the concept of α-contractive mapping of Meir–Keeler type. Subsequently, we prove some lemmas useful later.

In what follows, let (X,d) be a metric space, $T: X \to X$ and $\alpha: X \times X \to [0,+\infty)$, if not stated otherwise.

Definition 1. We say that T is an α-contractive mapping of Meir–Keeler type (with respect to d) if, for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$x,y \in X\colon\quad \varepsilon \leqslant d(x,y) < \varepsilon + \delta(\varepsilon) \quad\Longrightarrow\quad \alpha(x,y)d(Tx,Ty) < \varepsilon. \tag{1}$$

Lemma 1. *If T is an α-contractive mapping of Meir–Keeler type, then*

$$\alpha(x,y)d(Tx,Ty) < d(x,y) \quad \textit{for all } x,y \in X \textit{ with } x \neq y.$$

Proof. Fix $x,y \in X$ with $x \neq y$ and let $\varepsilon := d(x,y) > 0$. Then, by (1), $\alpha(x,y) \times d(Tx,Ty) < \varepsilon = d(x,y)$, which concludes the proof. □

Definition 2. (See [14].) We say that T is α-admissible if

$$x,y \in X\colon\quad \alpha(x,y) \geqslant 1 \quad\Longrightarrow\quad \alpha(Tx,Ty) \geqslant 1.$$

Example 1. Let $X = \mathbb{R}$. Define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} \mathrm{e}^{x-y} & \text{if } x \geqslant y, \\ 0 & \text{if } x < y. \end{cases} \tag{2}$$

Then

$$\alpha(x, y) \geqslant 1 \quad \iff \quad x \geqslant y \quad (x, y \in X),$$

hence, a mapping $T : X \to X$ is α-admissible *iff* it is nondecreasing.

Lemma 2. *Assume that T is α-admissible and α-contractive of Meir–Keeler type. Let $x, y \in X$ such that $\alpha(x, y) \geqslant 1$. Then*

$$\alpha\bigl(T^n x, T^n y\bigr) \geqslant 1 \quad \textit{for all } n \in \mathbb{N}, \tag{3}$$

the sequence $\{d(T^n x, T^n y)\}$ is nonincreasing, and

$$d(T^n x, T^n y) \to 0 \quad (\textit{as } n \to \infty).$$

Proof. Since T is α-admissible and $\alpha(x, y) \geqslant 1$, then (3) follows simply by induction on n.

Next, let $n \in \mathbb{N}$. If $T^n x \neq T^n y$, then, by (3) and Lemma 1, it follows that

$$\begin{aligned} d\bigl(T^{n+1}x, T^{n+1}y\bigr) &\leqslant \alpha\bigl(T^n x, T^n y\bigr) d\bigl(T^{n+1}x, T^{n+1}y\bigr) \\ &= \alpha\bigl(T^n x, T^n y\bigr) d\bigl(T\bigl(T^n x\bigr), T\bigl(T^n y\bigr)\bigr) < d\bigl(T^n x, T^n y\bigr). \end{aligned}$$

Else, if $T^n x = T^n y$, then $d(T^{n+1}x, T^{n+1}y) = d(T^n x, T^n y)$. Concluding, the sequence $\{d(T^n x, T^n y)\}$ is nonincreasing, hence, convergent to some $\varepsilon \geqslant 0$.

Assume that $\varepsilon > 0$, and let $p \in \mathbb{N}$ such that $\varepsilon \leqslant d(T^p x, T^p y) < \varepsilon + \delta(\varepsilon)$. Then $\alpha(T^p x, T^p y) d(T(T^p x), T(T^p y)) < \varepsilon$, and further, by (3), we get $d(T^{p+1}x, T^{p+1}y) < \varepsilon$, which is clearly not possible, hence, our assumption on ε is wrong. Concluding, we have necessarily $\varepsilon = 0$. □

Definition 3. We say that:

1. A sequence $\{x_n\}$ in X is (T, α)-orbital if $x_n = T^n x_0$ and $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$.
2. T is α-orbitally continuous if, for every (T, α)-orbital sequence $\{x_n\}$ in X such that $x_n \to x \in X$ as $n \to +\infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $Tx_{n(k)} \to Tx$ as $k \to +\infty$.
3. (X, d) is (T, α)-regular if, for every (T, α)-orbital sequence $\{x_n\}$ in X such that $x_n \to x \in X$ as $n \to +\infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geqslant 1$ for all k.
4. (X, d) is α-regular if, for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ as $n \to +\infty$ and $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geqslant 1$ for all k.

Example 2. Let d be the usual (Euclidian) distance on $\mathbb{R}$, and $\alpha : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ given by (2). Then $(\mathbb{R}, d)$ is α-regular.

Definition 4. Let $N \in \mathbb{N}$. We say that α is N-transitive (on X) if

$$x_0, x_1, \dots, x_{N+1} \in X\colon \quad \alpha(x_i, x_{i+1}) \geqslant 1 \text{ for all } i \in \{0, 1, \dots, N\} \\ \implies \quad \alpha(x_0, x_{N+1}) \geqslant 1.$$

In particular, we say that α is transitive if it is 1-transitive, i.e.,

$$x, y, z \in X\colon \quad \alpha(x, y) \geqslant 1, \quad \alpha(y, z) \geqslant 1 \quad \implies \quad \alpha(x, z) \geqslant 1.$$

The following remarks are immediate consequences of the previous definition.

Remark 1. If T is continuous, then T is α-orbitally continuous (for any α).

Remark 2. If (X, d) is α-regular, then it is also (T, α)-regular (for any T).

Remark 3. Any function $\alpha : X \times X \to [0, +\infty)$ is 0-transitive.

Remark 4. If α is N transitive, then it is kN-transitive for all $k \in \mathbb{N}$.

Remark 5. If α is transitive, then it is N-transitive for all $N \in \mathbb{N}$.

Example 3. Let $X = \mathbb{R}$. Then α defined by (2) is transitive.

Example 4. Let $N \in \mathbb{N} \setminus \{0\}$ and $\{A_1, \dots, A_N\}$ a family of nonempty sets. Let $X = \bigcup_{i=1}^N A_i$ and $R = \bigcup_{i=1}^N (A_i \times A_{i+1})$ (with $A_{N+1} := A_1$). Define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R \\ 0 & \text{otherwise.} \end{cases}$$

Then α is N-transitive, but not necessarily transitive (see, also, Corollary 7).

Definition 5. Let $x, y \in X$. A vector $\zeta = (z_0, z_1, \dots, z_n) \in X^{n+1}$ is called an α-chain (of order n) from x to y if $z_0 = x$, $z_n = y$ and, for every $i \in \{1, 2, \dots, n\}$,

$$\alpha(z_{i-1}, z_i) \geqslant 1 \quad \text{or} \quad \alpha(z_i, z_{i-1}) \geqslant 1.$$

Definition 6. We say that X is α-connected if, for every $x, y \in X$ with $x \neq y$, there exists an α-chain from x to y.

2.2 Existence and uniqueness of fixed points

Now, we are ready to present and prove the first main result of the paper.

Theorem 2. *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, +\infty)$ a N-transitive mapping (for some $N \in \mathbb{N} \setminus \{0\}$) and $T : X \to X$ an α-contractive mapping of Meir–Keeler type satisfying the following conditions*:

(A1) *T is α-admissible*;

(A2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$*;

(A3) *T is α-orbitally continuous.*

Then T has a fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$; equivalently, $x_n = T^n x_0$. Since $\alpha(x_0, Tx_0) \geqslant 1$, then by Lemma 2 we get

$$\alpha(x_n, x_{n+1}) \geqslant 1 \quad \text{for all } n \in \mathbb{N} \tag{4}$$

and

$$d(x_n, x_{n+1}) \to 0 \quad \text{as } n \to +\infty. \tag{5}$$

Fix $\varepsilon > 0$. Without any loss of generality, we may assume that $\delta(\varepsilon) \leqslant \varepsilon$. Using (5), there exists k such that

$$d(x_n, x_{n+1}) < \frac{\delta(\varepsilon)}{N} \quad \text{for all } n \geqslant k. \tag{6}$$

We introduce the set $Y \subset X$ defined by

$$\begin{aligned} Y := \bigl\{x \in X\colon &\text{ there exists } q(x) \in \{0, 1, \dots, N-1\} \text{ such that} \\ &d(x_{k+q(x)}, x) < \varepsilon + \delta(\varepsilon) \text{ and } \alpha(x_{k+q(x)}, x) \geqslant 1\bigr\}. \end{aligned}$$

Fix $x \in Y$. Our first claim is that

$$T^N x \in Y \quad \text{and} \quad q\bigl(T^N x\bigr) = q(x). \tag{7}$$

For short, let $q := q(x)$.

First, we prove that

$$d\bigl(x_{k+q}, T^N x\bigr) < \varepsilon + \delta(\varepsilon). \tag{8}$$

Using the triangle inequality and (6), we obtain

$$\begin{aligned} d\bigl(x_{k+q}, T^N x\bigr) &\leqslant \sum_{i=0}^{N-1} d(x_{k+q+i}, x_{k+q+i+1}) + d\bigl(x_{k+q+N}, T^N x\bigr) \\ &< \delta(\varepsilon) + d\bigl(T^N x_{k+q}, T^N x\bigr), \end{aligned}$$

while $\alpha(x_{k+q}, x) \geqslant 1$ leads to

$$d\bigl(T^N x_{k+q}, T^N x\bigr) \leqslant d(Tx_{k+q}, Tx) \leqslant d(x_{k+q}, x)$$

by Lemma 2. Hence, we conclude that

$$d\bigl(x_{k+q}, T^N x\bigr) < d(Tx_{k+q}, Tx) + \delta(\varepsilon) \leqslant d(x_{k+q}, x) + \delta(\varepsilon). \tag{9}$$

Clearly, if $d(x_{k+q}, x) < \varepsilon$, then (9) leads to (8), so it is enough to consider the case when $\varepsilon \leqslant d(x_{k+q}, x)$. Then $x \in Y$ leads to $\varepsilon \leqslant d(x_{k+q}, x) < \varepsilon + \delta(\varepsilon)$. Using next that T is an α-contractive mapping of Meir–Keeler type, we obtain that $\alpha(x_{k+q}, x) \times d(Tx_{k+q}, Tx) < \varepsilon$, and since $\alpha(x_{k+q}, x) \geqslant 1$, we arrive to

$$d(Tx_{k+q}, Tx) < \varepsilon. \tag{10}$$

Hence, (8) follows again by (9) and (10).

Next, we prove that

$$\alpha\bigl(x_{k+q}, T^N x\bigr) \geqslant 1. \tag{11}$$

Indeed,

$$\alpha(x_{k+q+i}, x_{k+q+i+1}) \geqslant 1 \quad \text{for all } i \in \{0, 1, \dots, N-1\} \tag{12}$$

by (4). Also, $\alpha(x_{k+q}, x) \geqslant 1$ leads by Lemma 2 to

$$\alpha\bigl(x_{k+q+N}, T^N x\bigr) \geqslant 1. \tag{13}$$

Now, using (12), (13) and the N-transitivity of α, we finally get (11).

Concluding, our first claim (7) is proven.

Our second claim is

$$x_{k+i+1} \in Y \quad \text{and} \quad q(x_{k+i+1}) = i \quad \text{for all } i \in \{0, 1, \dots, N-1\}. \tag{14}$$

Indeed, $d(x_{k+i}, x_{k+i+1}) < \delta(\varepsilon)/N < \varepsilon + \delta(\varepsilon)$ by (6), while $\alpha(x_{k+i}, x_{k+i+1}) \geqslant 1$ by (4), which proves (14).

Now, by (7) and (14), we can easily conclude that

$$x_n \in Y \quad \text{and} \quad q(x_n) = (n-k-1) \mod N \quad \text{for all } n \geqslant k+1. \tag{15}$$

Finally, let $m, n \geqslant k+1$ and assume that $q(x_n) \leqslant q(x_m)$ without any loss of generality. Then, by the triangle inequality, (6) and (15), it follows that

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{k+q(x_n)}) + \sum_{i=q(x_n)}^{q(x_m)-1} d\bigl(x_{k+i}, x_{k+i+1}\bigr) + d(x_{k+q(x_m)}, x_m) \\ &< 2\bigl(\varepsilon + \delta(\varepsilon)\bigr) + \bigl(q(x_m) - q(x_n)\bigr)\frac{\delta(\varepsilon)}{N} \leqslant 2\bigl(\varepsilon + \delta(\varepsilon)\bigr) + \delta(\varepsilon) \leqslant 5\varepsilon. \end{aligned}$$

Concluding, $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d), hence, convergent to some $x^* \in X$. Moreover, $\{x_n\}$ is a (T, α)-orbital sequence by (4), hence, by (A3), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $Tx_{n(k)} \to Tx^*$ as $k \to +\infty$. But $Tx_{n(k)} = x_{n(k)+1} \to x^*$ as $k \to +\infty$, hence, $Tx^* = x^*$ by the uniqueness of the limit, which concludes the proof. □

In the next theorem, we replace the α-orbital continuity of the mapping T by a regularity condition over the metric space (X, d).

Theorem 3. *In the conditions of Theorem* 2*, if* (A3) *is replaced with*

(A4) (X,d) *is* (T,α)*-regular,*

then the conclusion of Theorem 2 *holds.*

Proof. Following the proof of Theorem 2, we only have to prove that x^* is a fixed point of T. Since $\{x_n\}$ is a (T,α)-orbital sequence, then, by (A4), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n(k)}, x^*) \geqslant 1 \quad \text{for all } k \in \mathbb{N}.$$

Next, using Lemma 1, we get

$$d(Tx_{n(k)}, Tx^*) \leqslant \alpha(x_{n(k)}, x^*) d(Tx_{n(k)}, Tx^*) \leqslant d(x_{n(k)}, x^*) \quad \text{for all } k \in \mathbb{N}$$

(with equality when $x_{n(k)} = x^*$). As $x_{n(k)} \to x^*$, we obtain that $x_{n(k)+1} = Tx_{n(k)} \to Tx^*$. As $\{x_{n(k)+1}\}$ is a subsequence of $\{x_n\}$ and $x_n \to x^*$, we have $x_{n(k)+1} \to x^*$. Now, the uniqueness of the limit gives us $Tx^* = x^*$ and the proof is complete. □

To assure the uniqueness of the fixed point, we will consider the following additional assumption.

(A5) X is α-connected.

This is the purpose of the next theorem.

Theorem 4. *If adding* (A5) *to the hypotheses of Theorem* 2 (or *Theorem* 3)*, then* x^* *is the unique fixed point of* T *and* $T^n(x) \to x^*$ *(as* $n \to \infty$*) for every* $x \in X$*.*

Proof. Let $x \in X \setminus \{x^*\}$. By (A5), there exists $(x^* = z_0, z_1, \dots, z_n = x)$ an α-chain from x^* to x. Since

$$\alpha(z_{i-1}, z_i) \geqslant 1 \quad \text{or} \quad \alpha(z_i, z_{i-1}) \geqslant 1 \quad \text{for all } i \in \{1, 2, \dots, n\},$$

it follows by Lemma 2 and the symmetry of d that

$$d\big(T^n(z_{i-1}), T^n(z_i)\big) \to 0 \quad (\text{as } n \to +\infty) \text{ for all } i \in \{1, 2, \dots, n\}. \tag{16}$$

Now, since $z_0 = x^*$ is a fixed point of T, it follows that $T^n(z_0) = x^*$ for all n, which finally leads to

$$T^n z_i \to x^* \quad (\text{as } n \to +\infty) \text{ for all } i \in \{1, 2, \dots, n\},$$

using (16). Hence, $T^n x \to x^*$ (as $n \to +\infty$). In particular, if x is another fixed point of T, it follows that $x = x^*$, which is a contradiction, and the proof is concluded. □

3 Some corollaries

In this section, we will derive some corollaries from our previous theorems.

3.1 Coupled fixed point theorems for bivariate α-contractive mappings of Meir–Keeler type on complete metric spaces

The theorems obtained in the previous section allow us to derive some coupled fixed point results in complete metric spaces. First, let us recall the following definitions.

Definition 7. (See [16].) Let X be a nonempty set, and $F : X \times X \to X$ be a given mapping. A pair $(x, y) \in X \times X$ is called a coupled fixed point of F if $F(x, y) = x$ and $F(y, x) = y$.

Also, $x \in X$ is called a fixed point of F if (x, x) is a coupled fixed point, i.e., $F(x, x) = x$.

Definition 8. (See [10].) Let X be a nonempty set, and $F, G : X \times X \to X$. The symmetric composition (or the *s-composition* for short) of A and B is defined by

$$G * F : X \times X \to X, \quad (G * F)(x, y) = G\bigl(F(x, y), F(y, x)\bigr) \quad (x, y \in X).$$

Remark 6. (See [10].) The s-composition is an associative law. Also, the projection mapping

$$P_X : X \times X \to X, \quad P(x, y) = x \quad (x, y \in X)$$

is the identity element with respect to the s-composition (i.e., $F * P_X = P_X * F = F$ for all $F : X \times X \to X$). Consequently, for any $F : X \times X \to X$, one can define the functional powers (i.e., the iterates) of F with respect to the s-composition by

$$F^{n+1} = F * F^n = F^n * F \quad (n \in \mathbb{N}), \qquad F^0 = P_X.$$

We have the following result.

Corollary 1. *Let (X, d) be a complete metric space, $\alpha : (X \times X) \times (X \times X) \to [0, +\infty)$ a N-transitive mapping on $X \times X$ for some $N \in \mathbb{N} \setminus \{0\}$, and $F : X \times X \to X$ such that, for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ for which*

$$\begin{aligned} &(x, y), (u, v) \in X \times X\colon \quad \varepsilon \leqslant \frac{d(x, u) + d(y, v)}{2} < \varepsilon + \delta(\varepsilon) \\ &\quad \Longrightarrow \quad \alpha\bigl((x, y), (u, v)\bigr) d\bigl(F(x, y), F(u, v)\bigr) < \varepsilon. \end{aligned} \tag{17}$$

Suppose that:

(B1) *for all $(x, y), (u, v) \in X \times X$,*

$$\alpha\bigl((x, y), (u, v)\bigr) \geqslant 1 \quad \Longrightarrow \quad \alpha\bigl(\bigl(F(x, y), F(y, x)\bigr), \bigl(F(u, v), F(v, u)\bigr)\bigr) \geqslant 1;$$

(B2) *there exists $(x_0, y_0) \in X \times X$ such that*

$$\begin{aligned} &\alpha\bigl((x_0, y_0), \bigl(F(x_0, y_0), F(y_0, x_0)\bigr)\bigr) \geqslant 1, \\ &\alpha\bigl(\bigl(F(y_0, x_0), F(x_0, y_0)\bigr), (y_0, x_0)\bigr) \geqslant 1; \end{aligned}$$

(B3) *F is continuous.*

Then F has a coupled fixed point, that is, there exists $(x^*, y^*) \in X \times X$ such that $x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$.

Proof. Consider

$$D\big((x,y),(u,v)\big) := \frac{1}{2}\big(d(x,u) + d(y,v)\big) \quad \text{for all } (x,y),(u,v) \in X \times X.$$

Then, clearly, $(X \times X, D)$ is a complete metric space. Also, let $T : X \times X \to X \times X$ be defined by

$$T(x,y) = \big(F(x,y), F(y,x)\big) \quad \text{for all } (x,y) \in X \times X$$

and $\beta : (X \times X) \times (X \times X) \to [0, +\infty)$ be given by

$$\beta\big((x,y),(u,v)\big) = \min\big\{\alpha\big((x,y),(u,v)\big), \alpha\big((v,u),(y,x)\big)\big\}$$
$$\text{for all } (x,y),(u,v) \in X \times X. \tag{18}$$

First, we prove that β is N-transitive. Let $(x_i, y_i) \in X \times X$ ($i \in \{0, 1, \ldots, N+1\}$) such that $\beta((x_i, y_i), (x_{i+1}, y_{i+1})) \geqslant 1$ for all $i \in \{0, 1, \ldots, N\}$. By the definition of β, it follows that

$$\alpha\big((x_i, y_i), (x_{i+1}, y_{i+1})\big) \geqslant 1 \quad \text{and} \quad \alpha\big((y_{i+1}, x_{i+1}), (y_i, x_i)\big) \geqslant 1$$
$$\text{for all } i \in \{0, 1, \ldots, N\},$$

hence, by the N-transitivity of α, we have that

$$\alpha\big((x_0, y_0), (x_{N+1}, y_{N+1})\big) \geqslant 1 \quad \text{and} \quad \alpha\big((y_{N+1}, x_{N+1}), (y_0, x_0)\big) \geqslant 1,$$

which concludes our argument.

We claim next that T is a β-contractive mapping of Meir–Keeler type (with respect to D). Indeed, let $\varepsilon > 0$ and let $\delta(\varepsilon) > 0$ for which (17) is satisfied. If $(x,y),(u,v) \in X \times X$ are such that $\varepsilon \leqslant D((x,y),(u,v)) < \varepsilon + \delta(\varepsilon)$, then also $\varepsilon \leqslant D((v,u),(y,x)) < \varepsilon + \delta(\varepsilon)$ by the definition of D, hence,

$$\alpha\big((x,y),(u,v)\big)d\big(F(x,y), F(u,v)\big) < \varepsilon,$$
$$\alpha\big((v,u),(y,x)\big)d\big(F(v,u), F(y,x)\big) < \varepsilon$$

by (17). These two inequalities lead straight to

$$\beta\big((x,y),(u,v)\big)D\big(T(x,y), T(u,v)\big) < \varepsilon,$$

which proves our claim.

Next, it is easy to check that T is β-admissible by (B1). Moreover, (B2) ensures that $\beta((x_0, y_0), T(x_0, y_0)) \geqslant 1$, while (B3) ensures that T is continuous, hence, β-orbitally continuous.

Concluding, all the hypotheses of Theorem 2 applied to the metric space $(X \times X, D)$, the mapping T and the function β are satisfied, hence, T has a fixed point $(x^*, y^*) \in X \times X$, meaning that (x^*, y^*) is a coupled fixed point of F. The proof is now complete. □

Corollary 2. *In the conditions of Corollary* 1*, if* (B3) *is replaced with*:

(B4) *for every sequence* $\{(x_n, y_n)\}$ *in* $X \times X$ *such that* $x_n \to x \in X$, $y_n \to y \in X$ *as* $n \to +\infty$*, and*

$$\alpha\big((x_n, y_n), (x_{n+1}, y_{n+1})\big) \geqslant 1, \quad \alpha\big((y_{n+1}, x_{n+1}), (y_n, x_n)\big) \geqslant 1 \quad \textit{for all } n \in \mathbb{N},$$

there exists a subsequence $\{(x_{n(k)}, y_{n(k)})\}$ *such that*

$$\alpha\big((x_{n(k)}, y_{n(k)}), (x, y)\big) \geqslant 1, \quad \alpha\big((y, x), (y_{n(k)}, x_{n(k)})\big) \geqslant 1 \quad \textit{for all } k \in \mathbb{N};$$

then the conclusion of Corollary 1 *holds.*

Proof. Using the notations in the proof of Corollary 1, it easily follows by (B4) that $(X \times X, D)$ is β-regular, hence, (T, β)-regular. By following the proof of Corollary 1, the conclusion follows by Theorem 3 applied to the metric space $(X \times X, D)$, the mapping T and the function β. □

For the uniqueness of the coupled fixed point, we consider the following assumption.

(B5) $X \times X$ is β-connected, where β is defined by (18).

Corollary 3. *If adding condition* (B5) *to the hypotheses of Corollary* 1 (*or Corollary* 2)*, then* $x^* = y^*$*,* (x^*, x^*) *is the unique coupled fixed point of* F *and* x^* *is the unique fixed point of* F*. Moreover,* $F^n(x, y) \to x^*$ *as* $n \to \infty$ *for all* $x, y \in X$.

Proof. We use the notations in the proof of Corollary 1. Then, by Theorem 4, it follows that (x^*, y^*) is the unique fixed point of T, hence, the unique coupled fixed point of F. Since (y^*, x^*) is also a coupled fixed point of F, then $(x^*, y^*) = (y^*, x^*)$, hence, $x^* = y^*$, meaning also that x^* is the unique fixed point of F. Since $T^n(x, y) = (F^n(x, y), F^n(y, x))$ for all $n \in \mathbb{N}$ and $x, y \in X$, the proof is complete. □

We conclude this subsection with a particular form of the above corollaries, when α is represented as

$$\alpha\big((x, y), (u, v)\big) = \min\big\{\alpha_0(x, u), \alpha_0(v, y)\big\} \quad \big((x, y), (u, v) \in X \times X\big), \tag{19}$$

where $\alpha_0 : X \times X \to [0, +\infty)$. Note that, in this case, $\beta = \alpha$. We subsume the conclusions of Corollaries 1, 2 and 3 in one single result, as follows.

Corollary 4. *Let* (X, d) *be a complete metric space,* $\alpha_0 : X \times X \to [0, +\infty)$ *a* N*-transitive mapping on* $X \times X$ *for some* $N \in \mathbb{N} \setminus \{0\}$*, and* $F : X \times X \to X$ *such that, for every* $\varepsilon > 0$*, there exists* $\delta(\varepsilon) > 0$ *for which*

$$\begin{aligned} &(x, y), (u, v) \in X \times X: \quad \varepsilon \leqslant \frac{d(x, u) + d(y, v)}{2} < \varepsilon + \delta(\varepsilon) \\ &\quad \Longrightarrow \quad \min\big\{\alpha_0(x, u), \alpha_0(v, y)\big\} d\big(F(x, y), F(u, v)\big) < \varepsilon. \end{aligned}$$

Suppose that:

(C1) *for all* $(x, y), (u, v) \in X \times X$*,*

$$\alpha_0(x, u) \geqslant 1, \quad \alpha_0(v, y) \geqslant 1 \quad \Longrightarrow \quad \alpha_0\big(F(x, y), F(u, v)\big) \geqslant 1;$$

(C2) *there exists* $(x_0, y_0) \in X \times X$ *such that*

$$\alpha_0\bigl(x_0, F(x_0, y_0)\bigr) \geqslant 1, \qquad \alpha_0\bigl(F(y_0, x_0), y_0\bigr) \geqslant 1.$$

If either

(C3) F *is continuous,*

or

(C4) *for every sequence* $\{(x_n, y_n)\}$ *in* $X \times X$ *such that* $x_n \to x \in X$, $y_n \to y \in X$ *as* $n \to +\infty$, *and*

$$\alpha_0(x_n, x_{n+1}) \geqslant 1, \quad \alpha_0(y_{n+1}, y_n) \geqslant 1 \quad \textit{for all } n \in \mathbb{N},$$

there exists a subsequence $\{(x_{n(k)}, y_{n(k)})\}$ *such that*

$$\alpha_0\bigl(x_{n(k)}, x\bigr) \geqslant 1, \quad \alpha_0(y, y_{n(k)}) \geqslant 1 \quad \textit{for all } k \in \mathbb{N};$$

then F *has a coupled fixed point, that is, there exists* $(x^*, y^*) \in X \times X$ *such that* $x^* = F(x^*, y^*)$ *and* $y^* = F(y^*, x^*)$.

Additionally, if

(C5) X *is* α_0*-connected,*

then $x^* = y^*$, (x^*, x^*) *is the unique coupled fixed point of* F, x^* *is the unique fixed point of* F *and* $F^n(x, y) \to x^*$ *as* $n \to \infty$ *for all* $x, y \in X$.

Proof. It checks easily that the hypotheses of Corollaries 1, 2 and 3 are satisfied with α defined by (19). ☐

3.2 Fixed point theorems for $\mathcal{R}$-contractive mappings of Meir–Keeler type on a metric space endowed with a N-transitive binary relation

Existence of fixed point in a metric space endowed with an arbitrary binary relation has been introduced recently in [12] by Samet and Turinici. Very recently, Berzig in [17] presented some new results for contractions in a metric space endowed with an arbitrary binary relation.

In this section, we translate easily the notions and results in Section 2 to the setting of metric spaces endowed with a N-transitive binary relation.

In what follows, let (X, d) be a metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to X$. We first start with some terminology that is symmetrical to that in Section 2.

Definition 9. We say that T is a $\mathcal{R}$-contractive mapping of Meir–Keeler type (with respect to d) if, for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$x, y \in X\colon \quad x\mathcal{R}y,\ \varepsilon \leqslant d(x, y) < \varepsilon + \delta(\varepsilon) \quad \Longrightarrow \quad d(Tx, Ty) < \varepsilon.$$

Definition 10. We say that T is $\mathcal{R}$-preserving if

$$x, y \in X\colon \quad x\mathcal{R}y \quad \Longrightarrow \quad Tx\mathcal{R}Ty.$$

Definition 11. We say that a sequence $\{x_n\}$ in X is $(T,\mathcal{R})$-orbital if $x_n = T^n x_0$ and $x_n \mathcal{R} x_{n+1}$ for all $n \in \mathbb{N}$.

Definition 12. We say that T is $\mathcal{R}$-orbitally continuous if, for every $(T,\mathcal{R})$-orbital sequence $\{x_n\}$ in X such that $x_n \to x \in X$ as $n \to +\infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $Tx_{n(k)} \to Tx$ as $k \to +\infty$.

Remark 7. Clearly, if T is continuous, then T is $\mathcal{R}$-orbitally continuous (for any $\mathcal{R}$).

Definition 13. We say that (X,d) is $(T,\mathcal{R})$-regular if, for every $(T,\mathcal{R})$-orbital sequence $\{x_n\}$ in X such that $x_n \to x \in X$ as $n \to +\infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \mathcal{R} x$ for all k.

Definition 14. We say that (X,d) is $\mathcal{R}$-regular if, for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ as $n \to +\infty$ and $x_n \mathcal{R} x_{n+1}$ for all n, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \mathcal{R} x$ for all k.

Remark 8. Clearly, if (X,d) is $\mathcal{R}$-regular, then it is also $(T,\mathcal{R})$-regular (for any T).

Definition 15. Let $N \in \mathbb{N}$. We say that $\mathcal{R}$ is N-transitive (on X) if

$$x_0, x_1, x_2, \dots, x_N, x_{N+1} \in X\colon \quad x_i \mathcal{R} x_{i+1} \text{ for all } i \in \{0,1,\dots,N\} \quad \implies \quad x_0 \mathcal{R} x_{N+1}.$$

In particular, for $N = 1$ we recover the usual transitivity property.

Definition 16. Let $x, y \in X$. A vector $\zeta = (z_0, z_1, \dots, z_n) \in X^{n+1}$ is called a $\mathcal{R}$-chain (of order n) from x to y if $z_0 = x$, $z_n = y$ and

$$z_{i-1} \mathcal{R} z_i \quad \text{or} \quad z_i \mathcal{R} z_{i-1} \quad \text{for every } i \in \{1,2,\dots,n\}.$$

Definition 17. We say that X is $\mathcal{R}$-connected if, for every $x, y \in X$ with $x \neq y$, there exists a $\mathcal{R}$-chain from x to y.

The main results in Section 2 translate to the setting of metric spaces endowed with an arbitrary binary relation as follows.

Corollary 5. *Let (X,d) be a complete metric space, $\mathcal{R}$ a N-transitive binary relation over X (for some $N \in \mathbb{N} \setminus \{0\}$) and $T : X \to X$ a $\mathcal{R}$-contractive mapping of Meir–Keeler type. Assume that*:

(D1) *T is $\mathcal{R}$-preserving*;

(D2) *there exists $x_0 \in X$ such that $x_0 \mathcal{R} T x_0$.*

If either

(D3) *T is continuous,*

or

(D4) *(X,d) is $(T,\mathcal{R})$-regular,*

then T has a fixed point $x^* \in X$.

Additionally, if

(D5) X *is* $\mathcal{R}$*-connected,*

then x^* *is the unique fixed point of* T *and* $T^n(x) \to x^*$ (*as* $n \to \infty$) *for every* $x \in X$.

Proof. Define the mapping $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x\mathcal{R}y, \\ 0 & \text{otherwise.} \end{cases}$$

The conclusions then follows directly from Theorems 2, 3 and 4. □

The following result is a consequence of Corollary 4 for bivariate $\mathcal{R}$-contractive mappings of Meir–Keeler type.

Corollary 6. *Let* (X,d) *be a complete metric space,* $\mathcal{R}$ *a* N*-transitive binary relation over* X (*for some* $N \in \mathbb{N} \setminus \{0\}$), *and* $F : X \times X \to X$ *such that for, every* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *for which*;

$$\begin{aligned} &x,y,u,v \in X\colon \quad x\mathcal{R}y, \quad v\mathcal{R}u, \quad \varepsilon \leqslant \frac{d(x,u)+d(y,v)}{2} < \varepsilon + \delta(\varepsilon) \\ &\quad\Longrightarrow \quad d\bigl(F(x,y), F(u,v)\bigr) < \varepsilon. \end{aligned}$$

Suppose that:

(E1) *for all* $x,y,u,v \in X$,

$$x\mathcal{R}y, \quad v\mathcal{R}u \quad \Longrightarrow \quad F(x,y)\mathcal{R}F(u,v);$$

(E2) *there exists* $(x_0,y_0) \in X \times X$ *such that*

$$x_0\mathcal{R}F(x_0,y_0), \qquad F(y_0,x_0)\mathcal{R}y_0.$$

If either

(E3) F *is continuous,*

or

(E4) *for every sequence* $\{(x_n,y_n)\}$ *in* $X \times X$ *such that* $x_n \to x \in X$, $y_n \to y \in X$ *as* $n \to +\infty$, *and* $x_n\mathcal{R}x_{n+1}$, $y_{n+1}\mathcal{R}y_n$ *for all* $n \in \mathbb{N}$, *there exists a subsequence* $\{(x_{n(k)}, y_{n(k)})\}$ *such that* $x_{n(k)}\mathcal{R}x$, $y\mathcal{R}y_{n(k)}$ *for all* $k \in \mathbb{N}$,

then F *has a coupled fixed point* $(x^*,y^*) \in X \times X$.

Additionally, if

(E5) X *is* $\mathcal{R}$*-connected,*

then $x^* = y^*$, (x^*,x^*) *is the unique coupled fixed point of* F, x^* *is the unique fixed point of* F *and* $F^n(x,y) \to x^*$ *as* $n \to \infty$ *for all* $x,y \in X$.

Proof. Define the mapping $\alpha_0 : X \times X \to [0, +\infty)$ by

$$\alpha_0(x, y) = \begin{cases} 1 & \text{if } x\mathcal{R}y, \\ 0 & \text{otherwise.} \end{cases}$$

The conclusions then follows directly from Corollary 4. □

3.3 Fixed point results for cyclic contractive mappings of Meir–Keeler type

In this section, we obtain some fixed point results for cyclic α-contractions of Meir–Keeler type. We start by recalling the result obtained by Kirk, Srinivasan and Veeramani in [7] for cyclic contractive mappings.

Theorem 5. (*See* [7].) *Let (X, d) be a complete metric space, $\{A_1, A_2, \dots, A_N\}$ a family of nonempty and closed subsets of X and $T : X \to X$. Suppose that the following conditions hold*:

(F1) $T(A_i) \subseteq A_{i+1}$ *for all* $i \in \{1, 2 \dots, N\}$ (*where* $A_{N+1} = A_1$);

(F2) *there exists* $k \in (0, 1)$ *such that*

$$d(Tx, Ty) \leqslant kd(x, y) \quad \text{for all } x \in A_i,\ y \in A_{i+1},\ i \in \{1, 2 \dots, N\}.$$

Then $\bigcap_{i=1}^N A_i$ is non-empty and T has a unique fixed point in $\bigcap_{i=1}^N A_i$.

The aim of our next result is to weaken the contraction condition (F2) by considering the following condition of Meir–Keeler type:

(F3) for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\begin{aligned} &x \in A_i,\ y \in A_{i+1},\ i \in \{1, 2, \dots, N\}: \quad \varepsilon \leqslant d(x, y) < \varepsilon + \delta(\varepsilon) \\ &\quad \Longrightarrow \quad d(Tx, Ty) < \varepsilon. \end{aligned}$$

Corollary 7. *Let (X, d) be a complete metric space, $\{A_1, A_2, \dots, A_N\}$ a family of nonempty and closed subsets of X and $T : X \to X$. Suppose that* (F1) *and* (F3) *hold.*

Then $\bigcap_{i=1}^N A_i$ is non-empty and T has a fixed point $x^* \in \bigcap_{i=1}^N A_i$. Moreover, x^* is the unique fixed point of T in $\bigcup_{i=1}^N A_i$ and $T^n(x) \to x^*$ for all $x \in \bigcup_{i=1}^N A_i$.

Proof. Let $Y := \bigcup_{i=1}^N A_i$. Then Y is a closed part of X. Hence, (Y, d) is a complete metric space. Moreover, the restriction $T|_Y$ of T to Y is a self-map of Y by (F1); for convenience, we write T instead of $T|_Y$.

Define the mapping $\alpha : Y \times Y \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R := \bigcup_{i=1}^N (A_i \times A_{i+1}), \\ 0 & \text{otherwise.} \end{cases}$$

We check that the conditions in Theorem 3 are satisfied for the complete metric space (Y, d), the mappings α and T.

First, define $A_{i+kN} := A_i$ for all $i \in \{1, 2, \dots, N\}$ and $k \in \mathbb{Z}$. Then (F1) extends to

$$T(A_i) \subseteq A_{i+1} \quad \text{for all } i \in \mathbb{Z}.$$

We check that α is N-transitive (see also Example 4). Indeed, let $x_0, x_1, \dots, x_{N+1} \in Y$ such that $\alpha(x_k, x_{k+1}) \geqslant 1$ (i.e., $(x_k, x_{k+1}) \in R$) for all $k \in \{0, 1, \dots, N\}$. This means that there exists $i \in \{1, \dots, N\}$ such that

$$x_0 \in A_i, \quad x_1 \in A_{i+1}, \dots, \quad x_k \in A_{i+k}, \dots, \quad x_{N+1} \in A_{i+N+1} = A_{i+1},$$

hence, $(x_0, x_{N+1}) \in A_i \times A_{i+1} \subseteq R$, which finally leads to $\alpha(x_0, x_{N+1}) \geqslant 1$.

Clearly, T is α-contractive of Meir–Keeler type by (F3).

We claim next that T is α-admissible, i.e., (A1) is satisfied. Indeed, let $x, y \in Y$ such that $\alpha(x, y) \geqslant 1$. Hence, there exists $i \in \{1, 2 \dots, N\}$ such that $x \in A_i, y \in A_{i+1}$. Then, by (F1), $(Tx, Ty) \in (A_{i+1}, A_{i+2}) \subseteq R$, hence, $\alpha(Tx, Ty) \geqslant 1$.

Now, let $x_0 \in A_1$ arbitrary. Then $Tx_0 \in A_2$, hence, $\alpha(x_0, Tx_0) \geqslant 1$ which concludes (A2).

Next, we prove (A4), by showing that (Y, d) is α-regular, so let $\{x_n\}$ be a sequence in Y such that

$$x_n \to x \in Y \quad \text{as } n \to \infty \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geqslant 1 \quad \text{for all } n \in \mathbb{N}.$$

It follows that there exist $i, j \in \{1, \dots, N\}$ such that

$$x_n \in A_{i+n} \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad x \in A_j,$$

hence,

$$x_{(j-i-1+N)+kN} \in A_{j-1+(k+1)N} = A_{j-1} \quad \text{for all } k \in \mathbb{N}.$$

By letting

$$n(k) := (j - i - 1 + N) + kN \quad \text{for all } k \in \mathbb{N},$$

note that $j - i - 1 + N \geqslant 0$, and we conclude that the subsequence $\{x_{n(k)}\}$ satisfies

$$(x_{n(k)}, x) \in A_{j-1} \times A_j \subseteq R \quad \text{for all } k \in \mathbb{N},$$

hence, $\alpha(x_{n(k)}, x) \geqslant 1$ for all k, which proves our claim.

Now, all the conditions in Theorem 3 (for (Y, d), α and T) are satisfied, hence, there exists a fixed point $x^* \in Y$ of T. Clearly, $x^* \in \bigcap_{i=1}^N A_i$, since

$$x^* \in A_k \quad \text{for some } k \in \{1, 2, \dots, N\}$$

and

$$x^* \in A_i \quad \Longrightarrow \quad x^* = Tx^* \in A_{i+1} \quad \text{for all } i.$$

Moreover, it is straightforward to check that Y is α-connected, i.e., (A5) is satisfied. Indeed, if $x, y \in Y$ ($x \neq y$) with $x \in A_i$, $y \in A_j$ ($i, j \in \{1, 2, \dots, N\}$), then let $z_0 := x$, $z_k \in A_{k+i}$ arbitrary for every $k \in \{1, 2, \dots, N + j - i - 1\}$ and $z_{N+j-i} := y$. Note that $N + j - i \geqslant 1$. Then $(z_{k-1}, z_k) \in R$ (i.e., $\alpha(z_{k-1}, z_k) \geqslant 1$) for every $k \in \{1, 2, \dots, N + j - i\}$, hence, $(z_0, z_1, \dots, z_{N+j-i})$ is a α-chain from x to y.

Now, the rest of the conclusion follows by Theorem 4. □

4 Some consequences in ordered metric spaces

Clearly, the initial result of Meir and Keeler (Theorem 1) follows as a particular case of our Theorems 3 and 4, by simply choosing $\alpha(x, y) = 1$ for all $x, y \in X$. In what follows, we will also show that several fixed point and coupled fixed point results in ordered metric spaces can be easily deduced (and improved) from our theorems.

4.1 Fixed point results in ordered metric spaces

Let X be a nonempty set. Recall that a binary relation $\preccurlyeq$ over X is called a partial order if it is reflexive, transitive and anti-symmetric. If $\preccurlyeq$ is a partial order over X, then $x, y \in X$ are called comparable (subject to $\preccurlyeq$) if $x \preccurlyeq y$ or $y \preccurlyeq x$. Also, X is called $\preccurlyeq$-connected if, for every $x, y \in X$, there exist $z_0, z_1, \ldots, z_n \in X$ such that $z_0 = x$, $z_n = y$ and z_{i-1}, z_i are comparable for every $i \in \{1, 2, \ldots, n\}$.

In [13], Harjani et al. obtained several fixed point results in partially ordered sets for mappings satisfying some contraction condition of Meir–Keeler type. The main results in [13] for the case of nondecreasing mappings can be summarized as follows.

Theorem 6. (*See* [13].) *Let (X, d) be a complete metric space, $\preccurlyeq$ a partial order over X and $T : X \to X$ such that, for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ for which*

$$x, y \in X\colon \quad x \preccurlyeq y, \quad \varepsilon \leqslant d(x, y) < \varepsilon + \delta(\varepsilon) \quad \Longrightarrow \quad d(Tx, Ty) < \varepsilon.$$

Assume that:

(G1) *T is nondecreasing (subject to $\preccurlyeq$)*;

(G2) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$.*

If either

(G3) *T is continuous,*

or

(G4) *for every nondecreasing sequence $\{x_n\}$ in X such that $x_n \to x \in X$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preccurlyeq x$ for all $k \in \mathbb{N}$,*

then T has a fixed point.

In addition, if

(G5) *for every $x, y \in X$, there exists $z \in X$ which is comparable to x and y,*

then the fixed point of T is unique.

As it can be easily seen, this result follows straight from Corollary 5, with $\mathcal{R}$ being the partial order $\preccurlyeq$. Moreover, (G5) can be replaced by the weaker assumption

(G5a) X is $\preccurlyeq$-connected.

Also, if x^* is the unique fixed point of T, then $T^n(x) \to x^*$ (as $n \to \infty$) for every $x \in X$. This follows by Corollary 5 and its an extension of the conclusion in Theorem 6.

4.2 Coupled fixed point results in ordered metric spaces

In [11], Samet studied the coupled fixed points of mixed strict monotone mappings that satisfied a contraction condition of Meir–Keeler type, thereby extending the previous work of Bhaskar and Lakshmikantham [6]. In what follows we present an extension of the results of Samet [11]; in this direction, we do not require that the mixed monotone property be strict and we also weaken other assumptions. We also improve the conclusion.

First, recall the following definition.

Definition 18. (See [16].) Let $(X, \preccurlyeq)$ be a partially ordered set. A mapping $F : X \times X \to X$ is said to have the mixed monotone property if

$$x_1, x_2, y_1, y_2 \in X\colon \quad x_1 \preccurlyeq x_2, \quad y_1 \succcurlyeq y_2 \quad \Longrightarrow \quad F(x_1, y_1) \preccurlyeq F(x_2, y_2).$$

Our extension of the main results in [11] follows straight from Corollary 6, with $\mathcal{R}$ being the partial order $\preccurlyeq$, and can be stated as follows.

Theorem 7. *Let (X, d) be a complete metric space, $\preccurlyeq$ a partial order over X and $F : X \times X \to X$ such that, for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ for which*:

$$x, y, u, v \in X\colon \quad x \preccurlyeq u, \quad y \succcurlyeq v, \quad \varepsilon \leqslant \frac{1}{2}\bigl[d(x, u) + d(y, v)\bigr] < \varepsilon + \delta(\varepsilon)$$
$$\Longrightarrow \quad d\bigl(F(x, y), F(u, v)\bigr) < \varepsilon.$$

Suppose that:

(H1) *F has the mixed monotone property*;

(H2) *there exist $x_0, y_0 \in X$ such that $x_0 \preccurlyeq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$.*

If either

(H3) *F is continuous,*

or

(H4) *$(X, d, \preccurlyeq)$ has the following property*: *if $\{x_n\}$ is a nondecreasing* (*respectively, nonincreasing*) *sequence in X such that $x_n \to x$, then $x_n \preccurlyeq x$* (*respectively, $x_n \succcurlyeq x$*) *for all n,*

then F has a coupled fixed point $(x^*, y^*) \in X \times X$.

In addition, if

(H5) *X is $\preccurlyeq$-connected,*

then $x^* = y^*$, (x^*, x^*) is the unique coupled fixed point of F, x^* is the unique fixed point of F and $F^n(x, y) \to x^*$ as $n \to \infty$ for all $x, y \in X$.

5 Application to a third order two point boundary value problem

We study the existence and uniqueness of solution to the third order differential equation

$$x'''(t) + f\bigl(t, x(t)\bigr) = 0, \quad t \in (0, 1), \tag{20}$$

where $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, with the boundary value conditions

$$x(0) = x(1) = x''(0) = 0. \tag{21}$$

This problem is equivalent to finding a solution $x \in C([0,1], \mathbb{R})$ to the integral equation

$$x(t) = \int_0^1 G(t,s) f\big(s, x(s)\big)\, \mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} \frac{1}{2}(1-t)(t-s^2), & 0 \leqslant s \leqslant t \leqslant 1, \\ \frac{1}{2}t(1-s)^2, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Clearly, $G(t,s) \geqslant 0$ for all $t, s \in [0,1]$. Also, we can verify easily that

$$\int_0^1 G(t,s)\, \mathrm{d}s = \frac{t-t^3}{6} \leqslant \frac{\sqrt{3}}{27} \quad \text{for all } t \in [0,1]. \tag{22}$$

Let Φ be the set of all nondecreasing functions $\varphi : [0,+\infty) \to [0,+\infty)$ such that, for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ with

$$\varepsilon \leqslant t < \varepsilon + \delta(\varepsilon) \quad \Longrightarrow \quad \varphi(t) < \varepsilon.$$

Let $\xi : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi \in \Phi$. We consider the following assumptions:

(J1) There exists $N \in \mathbb{N} \setminus \{0\}$ such that

$$\begin{aligned} &a_0, a_1, \ldots, a_{N+1} \in [0,1]: \quad \xi(a_i, a_{i+1}) \geqslant 0 \quad \text{for all } i \in \{0,1,\ldots,N\} \\ &\quad \Longrightarrow \quad \xi(a_0, a_{N+1}) \geqslant 0. \end{aligned}$$

(J2) For every $a, b \in \mathbb{R}$,

$$\xi(a,b) \geqslant 0 \quad \Longrightarrow \quad \big|f(t,a) - f(t,b)\big| \leqslant 9\sqrt{3}\varphi\big(|a-b|\big) \quad \text{for all } t \in [0,1].$$

(J3) For every $x, y \in C([0,1])$,

$$\begin{aligned} &\inf_{t\in[0,1]} \xi\big(x(t), y(t)\big) \geqslant 0 \\ &\quad \Longrightarrow \quad \inf_{t\in[0,1]} \xi\left(\int_0^1 G(t,s) f\big(s,x(s)\big)\, \mathrm{d}s, \int_0^1 G(t,s) f\big(s,y(s)\big)\, \mathrm{d}s\right) \geqslant 0. \end{aligned}$$

(J4) There exists $x_0 \in C([0,1])$ such that

$$\inf_{t\in[0,1]} \xi\left(x_0(t), \int_0^1 G(t,s) f\big(s, x_0(s)\big)\, \mathrm{d}s\right) \geqslant 0.$$

(J5) For every $x, y \in C([0,1])$, there exist $z_0, z_1, \ldots, z_n \in C([0,1])$ such that $z_0 = x$, $z_n = y$ and, for every $i \in \{1, 2, \ldots, n\}$,

$$\inf_{t\in[0,1]} \xi\bigl(z_{i-1}(t), z_i(t)\bigr) \geqslant 0 \quad \text{or} \quad \inf_{t\in[0,1]} \xi\bigl(z_i(t), z_{i-1}(t)\bigr) \geqslant 0.$$

Theorem 8. *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be continuous and assume that there exist $\xi : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi \in \Phi$ such that* (J1)–(J4) *are satisfied. Then equation* (20) *with the boundary conditions* (21) *has solution. In addition, if* (J5) *is satisfied, then the solution is unique.*

Proof. Let $X := C([0,1])$ be endowed with the metric

$$d(u, v) = \max_{t\in[0,1]} \bigl|u(t) - v(t)\bigr|, \quad u, v \in X.$$

It is well known that (X, d) is a complete metric space. Define the mapping $T : X \to X$ by

$$(Tx)(t) = \int_0^1 G(t,s) f\bigl(s, x(s)\bigr)\,\mathrm{d}s \quad (x \in X, t \in [0,1]).$$

The problem reduces to the fixed point problem for T.

Let $\alpha : X \times X \to [0, \infty)$ be defined by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } \xi(x(t), y(t)) \geqslant 0 \text{ for all } t \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to observe that α is N-transitive by (J1), T is α-admissible by (J3) and $\alpha(x_0, Tx_0) \geqslant 1$ by (J4). Also, it follows in a standard fashion that T is continuous, hence, we omit this proof.

Now, using (J2), (22) and the fact that φ is nondecreasing, it follows that, for all $x, y \in X$ with $\alpha(x,y) \geqslant 1$,

$$\begin{aligned} \bigl|(Tx)(t) - (Ty)(t)\bigr| &\leqslant \int_0^1 G(t,s)\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,\mathrm{d}s \\ &\leqslant 9\sqrt{3}\left(\int_0^1 G(t,s)\,\mathrm{d}s\right)\varphi\bigl(d(x,y)\bigr) \leqslant \varphi\bigl(d(x,y)\bigr), \end{aligned}$$

hence,

$$d\bigl(Tx, Ty\bigr) \leqslant \varphi\bigl(d(x,y)\bigr) \quad \text{for all } x, y \in X \text{ with } \alpha(x,y) \geqslant 1.$$

This clearly leads to

$$\alpha(x,y) d\bigl(Tx, Ty\bigr) \leqslant \varphi\bigl(d(x,y)\bigr) \quad \text{for all } x, y \in X. \tag{23}$$

Now, let $\varepsilon > 0$. Since $\varphi \in \Phi$, there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leqslant a < \varepsilon + \delta(\varepsilon) \quad \Longrightarrow \quad \varphi(a) < \varepsilon. \tag{24}$$

Let $x, y \in X$ with $\varepsilon \leqslant d(x,y) < \varepsilon + \delta(\varepsilon)$. Then, by (23) and (24), it follows that

$$\alpha(x,y)d\bigl(Tx,Ty\bigr) \leqslant \varphi(d(x,y)) < \varepsilon.$$

Hence, we conclude that T is α-contractive mapping of Meir–Keeler type.

Now, we can apply Theorem 2 and obtain the existence of a fixed point of T, hence, the existence of a solution to (20)–(21). In addition, (J5) ensures that X is α-connected and the uniqueness of the solution follows by Theorem 4. The proof is now complete. □

Corollary 8. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous and assume there exists $\varphi \in \Phi$ such that the following conditions are satisfied*:

(K1) $0 \leqslant f(t,b) - f(t,a) \leqslant 9\sqrt{3}\varphi(b-a)$ *for all* $t \in [0,1]$ *and* $a, b \in \mathbb{R}$ *with* $a \leqslant b$;

(K2) *there exists* $x_0 \in C([0,1])$ *such that, for all* $t \in [0,1]$, *we have*

$$x_0(t) \leqslant \int\limits_0^1 G(t,s)f\bigl(s,x_0(s)\bigr)\,\mathrm{d}s.$$

Then (20)–(21) *has a unique solution.*

Proof. Consider the mapping $\xi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\xi(a,b) = b - a$ $(a, b \in \mathbb{R})$. Then the result follows straight from Theorem 8. Indeed, ξ clearly satisfies (J1), while (J2) and (J3) follow by (K1). Condition (K2) ensures (J4), while (J5) follows easily, by noting that, for every $x, y \in C([0,1])$, the function

$$z : [0,1] \to \mathbb{R}, \quad z(t) = \max\bigl\{x(t), y(t)\bigr\} \quad (t \in [0,1])$$

satisfies

$$z \in C\bigl([0,1]\bigr), \quad \inf_{t\in[0,1]} \xi\bigl(x(t), z(t)\bigr) \geqslant 0, \quad \inf_{t\in[0,1]} \xi\bigl(y(t), z(t)\bigr) \geqslant 0.$$ □

Remark 9. Condition (K2) can be replaced by

(K2a) there exists $x_0 \in C([0,1])$ such that, for all $t \in [0,1]$, we have

$$x_0(t) \geqslant \int\limits_0^1 G(t,s)f\bigl(s,x_0(s)\bigr)\,\mathrm{d}s,$$

while all the other conditions and conclusions remain unchanged. In this case, the proof follows similarly, by letting $\xi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\xi(a,b) = a - b$ $(a, b \in \mathbb{R})$.

References

1. A. Meir, E. Keeler, A theorem on contraction mappings, *J. Math. Anal. Appl.*, **28**:326–329, 1969.

2. A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. Am. Math. Soc.*, **132**:1435–1443, 2004.

3. R.P. Agarwal, M.A. El-Gebeily, D. O'Regan, Generalized contractions in partially ordered metric spaces, *Appl. Anal.*, **87**:1–8, 2008.
4. M. Berzig, B. Samet, An extension of coupled fixed point's concept in higher dimension and applications, *Comput. Appl. Math.*, **63**:1319–1334, 2012.
5. M. Berzig, B. Samet, Positive solutions to periodic boundary value problems involving nonlinear operators of Meir–Keeler-type, *Rend. Circ. Mat. Palermo*, **61**(2):279–296, 2012.
6. T.G. Bhaskar, V. Lakshmikantham, Fixed point theory in partially ordered metric spaces and applications, *Nonlinear Anal., Theory Methods Appl.*,**65**(7):1379–1393, 2006.
7. W.A. Kirk, P.S. Srinivasan, P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, *Fixed Point Theory*, **4**(1):79–89, 2003.
8. J.J. Nieto, R.R. López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order*, **22**:223–239, 2005.
9. J.J. Nieto, R.R. López, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, *Acta Math. Sin., Engl. Ser.*, **23**(12):2205–2212, 2007.
10. M.D. Rus, Fixed point theorems for generalized contractions in partially ordered metric spaces with semi-monotone metric, *Nonlinear Anal., Theory Methods Appl.*,**74**(5):1804–1813, 2011.
11. B. Samet, Coupled fixed point theorems for a generalized Meir–Keeler contraction in partially ordered metric spaces, *Nonlinear Anal., Theory Methods Appl.*,**72**(12):4508–4517, 2010.
12. B. Samet, M. Turinici, Fixed point theorems for on a metric space endowed with an arbitrary binary relation and applications, *Commun. Math. Anal.*, **13**(2):82–97, 2012.
13. J. Harjani, B. López, K. Sadarangani, A fixed point theorem for Meir–Keeler contractions in ordered metric spaces, *Fixed Point Theory Appl.*, **1**:1–8, 2011.
14. B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Anal., Theory Methods Appl.*,**75**(4):2154–2165, 2012.
15. K. Erdal, B. Samet, Generalized α-ψ contractive type mappings and related fixed point theorems with applications, *Abstr. Appl. Anal.*, **2012**, 793486, 17 pp., 2012.
16. D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, *Nonlinear Anal., Theory Methods Appl.*,**11**(5):623–632, 1987.
17. M. Berzig, Coincidence and common fixed point results on metric spaces endowed with an arbitrary binary relation and applications, *J. Fixed Point Theory Appl.*, **12**:221–228, 2013.