Common fixed points for α-ψ-φ-contractions in generalized metric spaces*

Vincenzo La Rosa, Pasquale Vetro

Università degli Studi di Palermo, Dipartimento di Matematica e Informatica
Via Archirafi, 34, 90123 Palermo, Italy
vincenzo.larosa@math.unipa.it; vetro@math.unipa.it

Received: 22 March 2013 / **Revised:** 7 August 2013 / **Published online:** 25 November 2013

Abstract. We establish some common fixed point theorems for mappings satisfying an α-ψ-φ-contractive condition in generalized metric spaces. Presented theorems extend and generalize many existing results in the literature.

Keywords: generalized metric space, α-ψ-φ-contractive condition, contraction of integral type, fixed point, common fixed point.

1 Introduction and preliminaries

Fixed point theory is an important and actual topic of nonlinear analysis. Moreover, it's well known that the contraction mapping principle, formulated and proved in the PhD dissertation of Banach in 1920 which was published in 1922 is one of the most important theorems in classical functional analysis.

During the last four decades, this theorem has undergone various generalizations either by relaxing the condition on contractivity or withdrawing the requirement of completeness or sometimes even both. Recently, a very interesting generalization was obtained in [1] by changing the structure of the space itself. In fact, Branciari [1] introduced a concept of generalized metric space by replacing the triangle inequality by a more general inequality. As such, any metric space is a generalized metric space but the converse is not true [1]. He proved the Banach's fixed point theorem in such a space. For more, the reader can refer to [2–11].

It is also known that common fixed point theorems are generalizations of fixed point theorems. Thus, over the past few decades, there have been many researchers who have interested in generalizing fixed point theorems to coincidence point theorems and common fixed point theorems.

In this paper, we prove some common fixed point theorems for a larger class of α-ψ-φ-contractions in generalized metric spaces and improve the results obtained by Lakzian and Samet [12] and Di Bari and Vetro [13].

*This research was supported by Università degli Studi di Palermo, Local University Project R.S. ex 60%.

2 Definitions and known theorems

Let $\mathbb{R}_+$ denote the set of all positive real numbers and $\mathbb{N}$ denote the set of all positive integers.

Definition 1. Let X be a non-empty set and $d : X \times X \to [0, +\infty[$ be a mapping such that, for all $x, y \in X$ and for all distinct points $u, v \in X$ each of them different from x and y, one has

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leqslant d(x, u) + d(u, v) + d(v, y)$ (rectangular inequality).

Then (X, d) is called a generalized metric space (or shortly GMS).

We note that (iii) of Definition 1 does not ensure that d is continuous in each variable, see [10]. Also, in a GMS the notions of convergent sequence and Cauchy sequence are the same as in a standard metric space.

Definition 2. Let (X, d) be a GMS, $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) We say that $\{x_n\}$ is GMS convergent to x if and only if $d(x_n, x) \to 0$ as $n \to +\infty$. We denote this by $x_n \to x$.
(ii) We say that $\{x_n\}$ is a GMS Cauchy sequence if and only if, for each $\varepsilon > 0$, there exists a natural number $n(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m \geqslant n(\varepsilon)$.
(iii) (X, d) is called GMS complete if every GMS Cauchy sequence is GMS convergent in X.

We note that a convergent sequence in a GMS is not necessarily a Cauchy sequence, see again [10].

We denote by Ψ the set of functions $\psi : [0, +\infty[\to [0, +\infty[$ satisfying the following hypotheses:

(ψ1) ψ is continuous and nondecreasing,
(ψ2) $\psi(t) = 0$ if and only if $t = 0$.

We denote by Φ the set of functions $\varphi : [0, +\infty[\to [0, +\infty[$ satisfying the following hypotheses:

(φ1) φ is lower semi-continuous,
(φ2) $\varphi(t) = 0$ if and only if $t = 0$.

In [12], Lakzian and Samet established the following fixed point theorem involving a pair of altering distance functions in a generalized complete metric space.

Theorem 1. *Let (X, d) be a Hausdorff and complete GMS and let $T : X \to X$ be a self-mapping satisfying*

$$\psi\big(d(Tx, Ty)\big) \leqslant \psi\big(d(x, y)\big) - \varphi\big(d(x, y)\big)$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\varphi : [0, +\infty[\to [0, +\infty[$ is continuous and $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Let X be a non-empty set and $T, f : X \to X$. The mappings T, f are said to be weakly compatible if they commute at their coincidence points (i.e. $Tfx = fTx$ whenever $Tx = fx$). A point $y \in X$ is called point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$. In [13], Di Bari and Vetro established the following fixed point theorem.

Theorem 2. *Let (X, d) be a Hausdorff GMS and let T and f be self-mappings on X such that $TX \subset fX$. Assume that (fX, d) is a complete GMS and that the following condition holds*:

$$\psi\big(d(Tx, Ty)\big) \leqslant \psi\big(d(fx, fy)\big) - \varphi\big(d(fx, fy)\big)$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\varphi \in \Phi$. Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

3 Main results

In this section, we prove some common fixed point results for two self-mappings satisfying an α-ψ-φ-contractive condition. For the notion of α-ψ-contractive type mappings, see Samet et al. [14]. Following [14], we introduce the notion of f-α-admissible mapping.

Definition 3. Let $T, f : X \to X$ and $\alpha : X \times X \to [0, +\infty[$. The mapping T is f-α-admissible if, for all $x, y \in X$ such that $\alpha(fx, fy) \geqslant 1$, we have $\alpha(Tx, Ty) \geqslant 1$. If f is the identity mapping, then T is called α-admissible.

Definition 4. Let (X, d) be a GMS and $\alpha : X \times X \to [0, +\infty[$. X is α-regular if, for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and $x_n \to x$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geqslant 1$ for all $k \in \mathbb{N}$.

Theorem 3. *Let (X, d) be a GMS and let T and f be self-mappings on X such that $TX \subseteq fX$ and $\alpha : X \times X \to [0, +\infty[$. Assume that (fX, d) is a complete GMS and that the following condition holds*:

$$\psi\big(\alpha(fx, fy)d(Tx, Ty)\big) \leqslant \psi\big(M(x, y)\big) - \varphi\big(M(x, y)\big) \tag{1}$$

for all $x, y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$M(x, y) = \max\big\{d(fx, fy), d(fx, Tx), d(fy, Ty)\big\}.$$

Assume also that the following conditions hold:

(i) *T is f-α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(fx_0, Tx_0) \geqslant 1$*;
(iii) *X is α-regular and, for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geqslant 1$, we have $\alpha(x_m, x_n) \geqslant 1$ for all $m, n \in \mathbb{N}$ with $m < n$*;
(iv) *either $\alpha(fu, fv) \geqslant 1$ or $\alpha(fv, fu) \geqslant 1$ whenever $fu = Tu$ and $fv = Tv$.*

Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(fx_0, Tx_0) \geqslant 1$. Define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_n = fx_{n+1} = Tx_n, \quad n \in \mathbb{N} \cup \{0\}.$$

Moreover, we assume that if $y_n = Tx_n = Tx_{n+p} = y_{n+p}$, then we choose $x_{n+p+1} = x_{n+1}$. This can be done, since $TX \subseteq fX$. In particular, if $y_n = y_{n+1}$, then y_{n+1} is a point of coincidence of T and f. Consequently, we can suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$.

By condition (ii), we have $\alpha(fx_0, Tx_0) = \alpha(fx_0, fx_1) \geqslant 1$. Since, by hypothesis, T is f-α-admissible, we obtain

$$\alpha(Tx_0, Tx_1) = \alpha(fx_1, fx_2) \geqslant 1, \qquad \alpha(Tx_1, Tx_2) = \alpha(fx_2, fx_3) \geqslant 1.$$

By induction, we get

$$\alpha(fx_n, fx_{n+1}) \geqslant 1 \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Now, by (1), we have

$$\begin{aligned} \psi\big(d(Tx_n, Tx_{n+1})\big) &\leqslant \psi\big(\alpha(fx_n, fx_{n+1}) d(Tx_n, Tx_{n+1})\big) \\ &\leqslant \psi\big(M(x_n, x_{n+1})\big) - \varphi\big(M(x_n, x_{n+1})\big), \end{aligned}$$

where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\big\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\big\} \\ &= \max\big\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\big\}. \end{aligned}$$

This implies

$$\psi\big(d(Tx_n, Tx_{n+1})\big) \leqslant \psi\big(M(x_n, x_{n+1})\big) - \varphi\big(M(x_n, x_{n+1})\big) \tag{2}$$

for all $n \in \mathbb{N}$. Now, if $M(x_n, x_{n+1}) = d(y_n, y_{n+1})$, from (2) we deduce

$$\psi\big(d(y_n, y_{n+1})\big) \leqslant \psi\big(d(y_n, y_{n+1})\big) - \varphi\big(d(y_n, y_{n+1})\big) \tag{3}$$

and, hence, $d(y_n, y_{n+1}) = 0$, which is a contradiction. Thus $M(x_n, x_{n+1}) = d(y_{n-1}, y_n) > 0$, then from (2) we get

$$\psi\big(d(y_n, y_{n+1})\big) \leqslant \psi\big(d(y_{n-1}, y_n)\big) - \varphi\big(d(y_{n-1}, y_n)\big) < \psi\big(d(y_{n-1}, y_n)\big).$$

Since ψ is nondecreasing, then $d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$ for all $n \in \mathbb{N}$, that is, the sequence of nonnegative numbers $\{d(y_n, y_{n+1})\}$ is decreasing. Hence, it converges to a nonnegative number, say $s \geqslant 0$. If $s > 0$, then letting $n \to +\infty$ in (3), we obtain $\psi(s) \leqslant \psi(s) - \varphi(s)$ which implies $s = 0$, that is

$$\lim_{n \to +\infty} d(y_n, y_{n+1}) = 0. \tag{4}$$

Suppose that $y_n \neq y_m$ for all $m \neq n$ and prove that $\{y_n\}$ is a GMS Cauchy sequence. First, we show that the sequence $\{d(y_n, y_{n+2})\}$ is bounded. Since $d(y_n, y_{n+1}) \to 0$, there exists $L > 0$ such that $d(y_n, y_{n+1}) \leqslant L$ for all $n \in \mathbb{N}$. If $d(y_n, y_{n+2}) > L$ for all $n \in \mathbb{N}$, from

$$\begin{aligned} M(x_n, x_{n+2}) &= \max\bigl\{d(fx_n, fx_{n+2}), d(fx_n, Tx_n), d(fx_{n+2}, Tx_{n+2})\bigr\} \\ &= d(y_{n-1}, y_{n+1}) \end{aligned}$$

and (iii) follows

$$\begin{aligned} \psi\bigl(d(y_n, y_{n+2})\bigr) &= \psi\bigl(d(Tx_n, Tx_{n+2})\bigr) \\ &\leqslant \psi\bigl(\alpha(fx_n, fx_{n+2})d(Tx_n, Tx_{n+2})\bigr) \\ &\leqslant \psi\bigl(M(x_n, x_{n+2})\bigr) - \varphi\bigl(M(x_n, x_{n+2})\bigr) \\ &< \psi\bigl(d(y_{n-1}, y_{n+1})\bigr). \end{aligned}$$

Thus the sequence $\{d(y_n, y_{n+2})\}$ is decreasing and, hence, is bounded. If, for some $n \in \mathbb{N}$, we have $d(y_{n-1}, y_{n+1}) \leqslant L$ and $d(y_n, y_{n+2}) > L$, then from

$$\begin{aligned} \psi\bigl(d(y_n, y_{n+2})\bigr) &= \psi\bigl(d(Tx_n, Tx_{n+2})\bigr) \\ &\leqslant \psi\bigl(\alpha(fx_n, fx_{n+2})d(Tx_n, Tx_{n+2})\bigr) \\ &\leqslant \psi\bigl(M(x_n, x_{n+2})\bigr) - \varphi\bigl(M(x_n, x_{n+2})\bigr) \\ &< \psi\bigl(M(x_n, x_{n+2})\bigr) \leqslant \psi(L), \end{aligned}$$

we get $d(y_n, y_{n+2}) < L$, a contradiction. Then $d(y_n, y_{n+2}) > L$ or $d(y_n, y_{n+2}) \leqslant L$ for all $n \in \mathbb{N}$ and in both cases the sequence $\{d(y_n, y_{n+2})\}$ is bounded. Now, if

$$\lim_{n \to +\infty} d(y_n, y_{n+2}) = 0 \tag{5}$$

does not hold, then there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $d(y_{n_k}, y_{n_k+2}) \to s > 0$. From

$$d(y_{n_k-1}, y_{n_k+1}) \leqslant d(y_{n_k-1}, y_{n_k}) + d(y_{n_k}, y_{n_k+2}) + d(y_{n_k+1}, y_{n_k+2})$$

and

$$d(y_{n_k}, y_{n_k+2}) \leqslant d(y_{n_k-1}, y_{n_k}) + d(y_{n_k-1}, y_{n_k+1}) + d(y_{n_k+1}, y_{n_k+2})$$

we deduce that

$$\lim_{k \to +\infty} d(y_{n_k-1}, y_{n_k+1}) = s.$$

Now, by (1) with $x = x_{n_k}$ and $y = x_{n_k+2}$, we have

$$\begin{aligned} \psi\bigl(d(Tx_{n_k}, Tx_{n_k+2})\bigr) &\leqslant \psi\bigl(\alpha(fx_{n_k}, fx_{n_k+2})d(Tx_{n_k}, Tx_{n_k+2})\bigr) \\ &\leqslant \psi\bigl(M(x_{n_k}, x_{n_k+2})\bigr) - \varphi\bigl(M(x_{n_k}, x_{n_k+2})\bigr), \end{aligned} \tag{6}$$

where

$$
\begin{aligned}
M(x_{n_k}, x_{n_k+2}) &= \max\bigl\{d(fx_{n_k}, fx_{n_k+2}), d(fx_{n_k}, Tx_{n_k}), d(fx_{n_k+2}, Tx_{n_k+2})\bigr\} \\
&= \max\bigl\{d(y_{n_k-1}, y_{n_k+1}), d(y_{n_k-1}, y_{n_k}), d(y_{n_k+1}, y_{n_k+2})\bigr\}.
\end{aligned}
$$

This implies

$$\lim_{k\to+\infty} M(x_{n_k}, x_{n_k+2}) = s.$$

From (6) as $k \to +\infty$, we get $\psi(s) \leqslant \psi(s) - \varphi(s)$ which implies $s = 0$.

Now, if possible, let $\{y_n\}$ be not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ with $n_k > m_k \geqslant k$ such that

$$d(y_{m_k}, y_{n_k}) \geqslant \varepsilon. \tag{7}$$

Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k - m_k \geqslant 3$ and satisfying (7). Then

$$d(y_{m_k}, y_{n_k-1}) < \varepsilon. \tag{8}$$

Now, using (7), (8) and the rectangular inequality, we get

$$
\begin{aligned}
\varepsilon &\leqslant d(y_{m_k}, y_{n_k}) \\
&\leqslant d(y_{n_k}, y_{n_k-2}) + d(y_{n_k-2}, y_{n_k-1}) + d(y_{n_k-1}, y_{m_k}) \\
&< d(y_{n_k}, y_{n_k-2}) + d(y_{n_k-2}, y_{n_k-1}) + \varepsilon.
\end{aligned}
$$

Letting $k \to +\infty$ in the above inequality, using (4) and (5), we obtain

$$d(y_{m_k}, y_{n_k}) \to \varepsilon^+. \tag{9}$$

From

$$
\begin{aligned}
&d(y_{m_k}, y_{n_k}) - d(y_{m_k-1}, y_{m_k}) - d(y_{n_k-1}, y_{n_k}) \\
&\quad \leqslant d(y_{n_k-1}, y_{m_k-1}) \leqslant d(y_{n_k-1}, y_{n_k}) + d(y_{m_k}, y_{n_k}) + d(y_{m_k-1}, y_{m_k}),
\end{aligned}
$$

letting $k \to +\infty$, we obtain

$$d(y_{m_k-1}, y_{n_k-1}) \to \varepsilon. \tag{10}$$

From (1) with $x = x_{n_k}$ and $y = x_{m_k}$, we get

$$
\begin{aligned}
\psi\bigl(d(Tx_{m_k}, Tx_{n_k})\bigr) &\leqslant \psi\bigl(\alpha(fx_{m_k}, fx_{n_k}) d(Tx_{m_k}, Tx_{n_k})\bigr) \\
&\leqslant \psi\bigl(M(fx_{m_k}, fx_{n_k})\bigr) - \varphi\bigl(M(fx_{m_k}, fx_{n_k})\bigr),
\end{aligned}
$$

where

$$
\begin{aligned}
M(fx_{m_k}, fx_{n_k}) &= \max\bigl\{d(fx_{m_k}, fx_{n_k}), d(fx_{n_k}, Tx_{n_k}), d(fx_{m_k}, Tx_{m_k})\bigr\} \\
&= \max\bigl\{d(y_{n_k-1}, y_{m_k-1}), d(y_{n_k-1}, y_{n_k}), d(y_{m_k-1}, y_{m_k})\bigr\}.
\end{aligned}
$$

Now, using the continuity of ψ and the lower semi-continuity of φ as $k \to +\infty$, we obtain

$$\psi(\varepsilon) \leqslant \psi(\varepsilon) - \varphi(\varepsilon),$$

which implies that $\varepsilon = 0$, a contradiction with $\varepsilon > 0$. Hence, $\{y_n\}$ is a GMS Cauchy sequence. Since (fX, d) is GMS complete, there exists $z \in fX$ such that $y_n \to z$. Let $y \in X$ be such that $fy = z$. Since X is α-regular there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\alpha(y_{n_k-1}, fy) \geqslant 1$ for all $k \in \mathbb{N}$. If $fy \neq Ty$, applying inequality (1) with $x = x_{n_k}$, we obtain

$$\begin{aligned}\psi\big(d(Tx_{n_k}, Ty)\big) &\leqslant \psi\big(\alpha(fx_{n_k}, fy)d(Tx_{n_k}, Ty)\big)\\ &\leqslant \psi\big(M(fx_{n_k}, fy)\big) - \varphi\big(M(fx_{n_k}, fy)\big),\end{aligned}$$

where

$$\begin{aligned}M(fx_{n_k}, fy) &= \max\big\{d(fx_{n_k}, fy), d(fx_{n_k}, Tx_{n_k}), d(fy, Ty)\big\}\\ &= \max\big\{d(y_{n_k-1}, fy), d(y_{n_k-1}, y_{n_k}), d(fy, Ty)\big\}.\end{aligned}$$

Now, from

$$d(y_{n_k-1}, fy), d(y_{n_k-1}, y_{n_k}) \to 0 \quad \text{as } k \to +\infty,$$

for k great enough, we deduce $M(fx_{n_k}, fy) = d(fy, Ty)$. On the other hand,

$$d(fy, Ty) \leqslant d(fy, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k}) + d(Tx_{n_k}, Ty)$$

implies

$$d(fy, Ty) \leqslant \liminf_{k\to+\infty} d(Tx_{n_k}, Ty).$$

Since ψ is continuous and nondecreasing, for k great enough, we get

$$\psi\big(d(fy, Ty)\big) \leqslant \liminf_{k\to+\infty} \psi\big(d(Tx_{n_k}, Ty)\big) \leqslant \psi\big(d(fy, Ty)\big) - \varphi\big(d(fy, Ty)\big)$$

which implies $d(fy, Ty) = 0$, that is, $z = fy = Ty$ and so z is a point of coincidence for T and f.

Suppose that there exist $n, p \in \mathbb{N}$ such that $y_n = y_{n+p}$. We prove that $p = 1$, then $fx_{n+1} = Tx_n = Tx_{n+1} = y_{n+1}$ and so y_{n+1} is a point of coincidence of T and f. Assume $p > 1$, this implies that $d(y_{n+p-1}, y_{n+p}) > 0$. Using (3), we obtain

$$\begin{aligned}\psi\big(d(y_n, y_{n+1})\big) &= \psi\big(d(y_{n+p}, y_{n+p+1})\big)\\ &\leqslant \psi\big(d(y_{n+p-1}, y_{n+p})\big) - \varphi\big(d(y_{n+p-1}, y_{n+p})\big)\\ &< \psi\big(d(y_{n+p-1}, y_{n+p})\big).\end{aligned}$$

Since the sequence $d(y_n, y_{n+1})$ is decreasing, we deduce

$$\psi\big(d(y_n, y_{n+1})\big) < \psi\big(d(y_n, y_{n+1})\big),$$

a contradiction and, hence, $p = 1$. We deduce that T and f have a point of coincidence. The uniqueness of the point of coincidence is a consequence of the conditions (1) and (iv), and so we omit the details.

Now, if z is the point of coincidence of T and f as T and f are weakly compatible, we deduce that $fz = Tz$ and so $z = fz = Tz$. Consequently, z is the unique common fixed point of T and f. □

From Theorem 3, if we choose $f = I_X$ the identity mapping on X, we deduce the following corollary.

Corollary 1. *Let (X, d) be a complete GMS, let T be a self-mapping on X and $\alpha : X \times X \to [0, +\infty[$. Assume that the following condition holds*:

$$\psi\big(\alpha(x, y)d(Tx, Ty)\big) \leqslant \psi\big(M(x, y)\big) - \varphi\big(M(x, y)\big) \tag{11}$$

for all $x, y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$M(x, y) = \max\big\{d(x, y), d(x, Tx), d(y, Ty)\big\}.$$

Assume also that the following conditions hold:

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$*;
(iii) *X is α-regular and, for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geqslant 1$, we have $\alpha(x_m, x_n) \geqslant 1$ for all $m, n \in \mathbb{N}$ with $m < n$*;
(iv) *either $\alpha(u, v) \geqslant 1$ or $\alpha(v, u) \geqslant 1$ whenever $u = Tu$ and $v = Tv$.*

Then T has a unique fixed point.

From Theorem 3, if the function $\alpha : X \times X \to [0, +\infty[$ is such that $\alpha(x, y) = 1$ for all $x, y \in X$, we deduce the following theorem.

Theorem 4. *Let (X, d) be a GMS and let T and f be self-mappings on X such that $TX \subseteq fX$. Assume that (fX, d) is a complete GMS and that the following condition holds*:

$$\psi\big(d(Tx, Ty)\big) \leqslant \psi\big(M(x, y)\big) - \varphi\big(M(x, y)\big) \tag{12}$$

for all $x, y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi$ and

$$M(x, y) = \max\big\{d(fx, fy), d(fx, Tx), d(fy, Ty)\big\}.$$

Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Let X be a non-empty set. If (X, d) is a GMS and $(X, \preceq)$ is a partially ordered set, then $(X, d, \preceq)$ is called a partially ordered GMS. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. Let $(X, \preceq)$ be a partially ordered set and $T, f : X \to X$ be two mappings. T is called an f-nondecreasing mapping if $Tx \preceq Ty$ whenever $fx \preceq fy$ for all $x, y \in X$.

From Theorem 3, in the setting of partially ordered GMS spaces, we obtain the following theorem.

Theorem 5. *Let $(X, d, \preceq)$ be a partially ordered GMS and let T and f be self-mappings on X such that $TX \subseteq fX$. Assume that (fX, d) is a complete GMS and that the following condition holds*:

$$\psi\big(d(Tx, Ty)\big) \leqslant \psi\big(M(x, y)\big) - \varphi\big(M(x, y)\big) \tag{13}$$

for all $x, y \in X$ such that $fx \preceq fy$, where $\psi \in \Psi$ and $\varphi \in \Phi$ with $\psi(t) - \varphi(t) \geqslant 0$ for all $t \geqslant 0$, and

$$M(x, y) = \max\big\{d(fx, fy), d(fx, Tx), d(fy, Ty)\big\}.$$

Assume also that the following conditions hold:

(i) *T is f-nondecreasing*;
(ii) *there exists $x_0 \in X$ such that $fx_0 \preceq Tx_0$*;
(iii) *if $\{x_n\} \subset X$ is such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$*;
(iv) *for all $u, v \in X$ such that $fu = Tu$ and $fv = Tv$, then fu and fv are comparable.*

Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Proof. Define the mapping $\alpha : X \times X \to [0, +\infty[$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in fX \text{ and } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

The reader can show easily that T is an f-α-admissible mapping. Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to +\infty$. By the definition of α, we have

$$x_n, x_{n+1} \in fX \quad \text{and} \quad x_n \preceq x_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

Since fX is complete, we deduce that $x \in fX$. By (iii), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$ and so $\alpha(x_{n_k}, x) \geqslant 1$ for all $k \in \mathbb{N}$ and so X is α-regular. Moreover, $\alpha(x_m, x_n) \geqslant 1$ for all $m, n \in \mathbb{N}$ with $m < n$. Hence, (iii) of Theorem 3 holds. The same considerations show that (ii) and (iv) of this theorem imply (ii) and (iv) of Theorem 3. Thus the hypotheses (i)–(iv) of Theorem 3 are satisfied. Also the contractive condition (1) is satisfied, since $\alpha(fx, fy) = 1$ for all $x, y \in X$ such that $fx \preceq fy$. Otherwise $\psi(\alpha(fx, fy)d(Tx, Ty)) = 0$ and so condition (1) holds. By Theorem 3, T and f have a unique common fixed point. □

Now, from Theorem 3, we can derive many interesting fixed point results in generalized metric spaces. Denote by Λ the set of functions $\gamma : [0, +\infty[\to [0, +\infty[$ Lebesgue integrable on each compact subset of $[0, +\infty[$ such that, for every $\varepsilon > 0$, we have

$$\int_0^{\varepsilon} \gamma(s)\,\mathrm{d}s > 0.$$

As the function $\psi : [0,+\infty[\to [0,+\infty[$ defined by $\psi(t) = \int_0^t \gamma(s)\,\mathrm{d}s$ belongs to Ψ, we obtain the following theorem.

Theorem 6. *Let (X,d) be a GMS and let T and f be self-mappings on X such that $TX \subseteq fX$ and $\alpha : X \times X \to [0,+\infty[$. Assume that (fX,d) is a complete GMS and that the following condition holds*:

$$\int_0^{\alpha(fx,fy)d(Tx,Ty)} \gamma(s)\,\mathrm{d}s \leqslant \int_0^{M(x,y)} \gamma(s)\,\mathrm{d}s - \int_0^{M(x,y)} \delta(s)\,\mathrm{d}s$$

for all $x, y \in X$, where $\gamma, \delta \in \Lambda$ and

$$M(x,y) = \max\bigl\{d(fx,fy), d(fx,Tx), d(fy,Ty)\bigr\}.$$

Assume also that the following conditions hold:

(i) *T is f-α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(fx_0,Tx_0) \geqslant 1$*;
(iii) *X is α-regular and, for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n,x_{n+1}) \geqslant 1$, we have $\alpha(x_m,x_n) \geqslant 1$ for all $m,n \in \mathbb{N}$ with $m < n$*;
(iv) *either $\alpha(fu,fv) \geqslant 1$ or $\alpha(fv,fu) \geqslant 1$ whenever $fu = Tu$ and $fv = Tv$.*

Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Taking $\delta(s) = (1-k)\gamma(s)$ for $k \in [0,1[$ in Theorem 6, we obtain the following result.

Theorem 7. *Let (X,d) be a GMS and let T and f be self-mappings on X such that $TX \subseteq fX$ and $\alpha : X \times X \to [0,+\infty[$. Assume that (fX,d) is a complete GMS and that the following condition holds*:

$$\int_0^{\alpha(fx,fy)d(Tx,Ty)} \gamma(s)\,\mathrm{d}s \leqslant k \int_0^{M(x,y)} \gamma(s)\,\mathrm{d}s$$

for all $x, y \in X$, where $k \in [0,1[$. Assume also that the following conditions hold:

(i) *T is f-α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(fx_0,Tx_0) \geqslant 1$*;
(iii) *X is α-regular and, for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n,x_{n+1}) \geqslant 1$, we have $\alpha(x_m,x_n) \geqslant 1$ for all $m,n \in \mathbb{N}$ with $m < n$*;
(iv) *either $\alpha(fu,fv) \geqslant 1$ or $\alpha(fv,fu) \geqslant 1$ whenever $fu = Tu$ and $fv = Tv$.*

Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Example 1. Let $X = [0,1]$ and $A = \{1/2, 1/3, 1/4\}$. Define the generalized metric d on X as follows:

$$d\left(\frac{1}{2},\frac{1}{3}\right) = d\left(\frac{1}{4},\frac{1}{5}\right) = \frac{3}{5}, \qquad d\left(\frac{1}{2},\frac{1}{5}\right) = d\left(\frac{1}{3},\frac{1}{4}\right) = \frac{2}{5},$$
$$d\left(\frac{1}{2},\frac{1}{4}\right) = d\left(\frac{1}{5},\frac{1}{3}\right) = \frac{6}{5}, \qquad d(x,y) = |x-y| \quad \text{otherwise}.$$

Clearly, (X,d) is a complete GMS. Let $T : X \to X$ and $\psi, \varphi : [0,+\infty[\to [0,+\infty[$ be defined by

$$Tx = \begin{cases} \frac{1}{4} & \text{if } x \in A, \\ 1-x & \text{if } x \in [0,1] \setminus A, \end{cases} \qquad \psi(t) = t \quad \text{and} \quad \varphi(t) = \frac{t}{5}.$$

Finally, consider $\alpha : X \times X \to [0,+\infty[$ given by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in A \text{ or } x = y, \\ 0 & \text{otherwise}. \end{cases}$$

Then T and α satisfy all the conditions of Corollary 1 and, hence, T has a unique fixed point on X, that is, $x = 1/4$.

We note that if X is endowed with the standard metric $d(x,y) = |x-y|$ for all $x,y \in X$, then there do not exist $\psi, \varphi : [0,+\infty[\to [0,+\infty[$, where $\psi \in \Psi$ and $\varphi \in \Phi$, such that

$$\psi\big(d(Tx,Ty)\big) \leqslant \psi\big(M(x,y)\big) - \varphi\big(M(x,y)\big)$$

for all $x,y \in X$.

Acknowledgment. The authors would like to thank the editor and reviewers for their helpful comments.

References

1. A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, *Publ. Math. Debrecen*, **57**:31–37, 2000.
2. J. Ahmad, M. Arshad, C. Vetro, On a theorem of Khan in a generalized metric space, *International Journal of Analysis*, **2013**, Article ID 852727, 6 pp., 2013.
3. A. Azam, M. Arshad, Kannan fixed point theorem on generalized metric spaces, *J. Nonlinear Sci. Appl.*, **1**:45–48, 2008.
4. P. Das, A fixed point theorem in a generalized metric space, *Soochow J. Math.*, **33**:33–39, 2007.
5. P. Das, B.K. Lahiri, Fixed point of contractive mappings in generalized metric spaces, *Math. Slovaca*, **59**:499–504, 2009.

6. A. Fora, A. Bellour, A. Al-Bsoul, Some results in fixed point theory concerning generalized metric spaces, *Mat. Vesnik*, **61**:203–208, 2009.
7. B.K. Lahiri, P. Das, Fixed point of a Ljubomir Ćirić's quasi-contraction mapping in a generalized metric space, *Publ. Math. Debrecen*, **61**:589–594, 2002.
8. D. Miheţ, On Kannan fixed point principle in generalized metric spaces, *J. Nonlinear Sci. Appl.*, **2**:92–96, 2009.
9. B. Samet, A fixed point theorem in a generalized metric space for mappings satisfying a contractive condition of integral type, *Int. J. Math. Anal.*, **3**:1265–1271, 2009.
10. B. Samet, Discussion on: A fixed point theorem of Banach–Caccioppoli-type on a class of generalized metric spaces by A. Branciari, *Publ. Math. Debrecen*, **76**:493–494, 2010.
11. I.R. Sarma, J.M. Rao, S.S. Rao, Contractions over generalized metric spaces, *J. Nonlinear Sci. Appl.*, **2**:180–182, 2009.
12. H. Lakzian, B. Samet, Fixed point for (ψ, φ)-weakly contractive mappings in generalized metric spaces, *Appl. Math. Lett.*, **25**:902–906, 2012.
13. C. Di Bari, P. Vetro, Common fixed points in generalized metric spaces, *Appl. Math. Comput.*, **218**:7322–7325, 2012.
14. B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Anal.*, **75**:2154–2165, 2012.