

Testing the epidemic change in nearly nonstationary autoregressive processes*

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Abstract. Some tests for an epidemic type change in a first order nearly nonstationary autoregressive process are investigated. Limit distributions of the tests are found under no change. Consistency is examined under short epidemics in the mean of innovations.

Keywords: autoregressive process, epidemic change, uniform increments statistics, test power analysis.

1 Introduction

Consider random variables X_1, X_2, \dots, X_n with parameters of interest $\theta_1, \dots, \theta_n$, $n \geq 2$. Testing an epidemic type change (or changed segment) in these parameters means testing null hypothesis $\theta_1 = \theta_2 = \dots = \theta_n$ against the alternative $\theta_1 = \dots = \theta_{k^*} = \theta_{m^*+1} = \dots = \theta_n = \mu_0$ and $\theta_{k^*+1} = \dots = \theta_{m^*} = \mu_1$ for some unknown $1 < k^* < m^* \leq n$ and $\mu_0 \neq \mu_1$. Here k^* is the beginning, m^* is the end and $\ell^* = m^* - k^*$ is the length or duration of the epidemic state.

To the best of our knowledge such a problem for independent observations have been formulated for the first time by Levin and Kline [1] (we also refer to [2, Sect. 1.4]). Yao [3] have studied various test statistics in order to detect an epidemic change in the mean values of a sequence of independent normally distributed random variables. Ramanayake and Gupta [4, 5] investigated various likelihood ratio type statistics and independent random variables from an exponential family. Graiche et al. [6] investigated the changed segment problem for α -mixing random variables. For more information on this subject, see [7, Sects. 9.3, 9.4] and [8–11].

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A natural way to construct test statistic for detecting the epidemic change in the mean is to construct the uniform increments statistic:

$$T_{0,n}(X_1, \dots, X_n) = \max_{1 \leq \ell \leq n-1} \max_{1 \leq k \leq n-\ell} \left| \sum_{j=k+1}^{k+\ell} X_j - \frac{\ell}{n} \sum_{j=1}^n X_j \right|.$$

Such statistic can detect epidemic state whose the length ℓ^* is such that $n^{1/2} = o_P(\ell^*)$ (see [12]). For the shorter durations, Račkauskas and Suquet [12] have proposed the uniform increments statistics with an additional normalization. For $\alpha \in [0, 1)$, the class of statistics is defined by

$$T_{\alpha,n} = T_{\alpha,n}(X_1, \dots, X_n) = \max_{1 \leq \ell \leq n-1} \ell^{-\alpha} \max_{1 \leq k \leq n-\ell} \left| \sum_{j=k+1}^{k+\ell} X_j - \frac{\ell}{n} \sum_{j=1}^n X_j \right|.$$

For the model $X_i = \theta_i + \epsilon_i$, $i \geq 1$, where $(\epsilon_i, i \geq 1)$ is a sequence of independent identically distributed mean zero random variables, Račkauskas and Suquet [12] have shown that, for any $0 < \alpha < 1/2$, statistics $T_{\alpha,n}(X_1, \dots, X_n)$ is able to detect epidemics with duration $n^\delta = o(\ell^*)$, where $\delta = (1-2\alpha)/(2-2\alpha)$. Further, Mikosch and Račkauskas [13] have studied $T_{\alpha,n}$ for regularly varying random variables (ϵ_i) and $0 < \alpha < 1$.

Assume we are given an n -sample $y_{n,1}, \dots, y_{n,n}$ generated by

$$y_{n,k} = \phi_n y_{n,k-1} + \epsilon_k + a_{n,k}, \quad k = 1, \dots, n, \quad n \geq 1, \quad y_{n,0} = 0, \quad (1)$$

where the parameter $\phi_n \in (0, 1)$ satisfies $\phi_n \rightarrow 1$ as $n \rightarrow \infty$, $(\epsilon_k, k \geq 1)$ are i.i.d. centered, at least square integrable random variables, $(a_{n,k})$ is a sequence that will be precised later. The process (1), when $\phi_n \rightarrow 1$ as $n \rightarrow \infty$ is a nearly nonstationary first order autoregressive process with drift. Throughout the paper, the parameter ϕ_n is supposed to be known. Our aim is to propose tests for the null hypothesis

$$(H_0) \quad a_{n,1} = \dots = a_{n,n} = 0$$

against the epidemic or changed segment alternative

(H_A) there exist $1 \leq k_n^* < m_n^* \leq n$ such that

$$a_{n,k} = a_n \mathbf{1}_{\mathbb{I}_n^*}(k), \quad a_n \neq 0, \quad 1 \leq k \leq n,$$

where \mathbb{I}_n^* is the epidemics interval $\mathbb{I}_n^* = \{k_n^* + 1, \dots, m_n^*\}$ and $\mathbf{1}_{\mathbb{I}_n^*}$ denotes its indicator function.

To investigate such hypothesis, we consider the test statistics

$$\tilde{T}_{\alpha,n} = T_{\alpha,n}(y_{n,1}, \dots, y_{n,n}), \quad 0 \leq \alpha < 1/2. \quad (2)$$

To motivate such choice, rewrite the model (1) in the following way:

$$y_{n,k} - \tau_{n,k} = \phi_n (y_{n,k-1} - \tau_{n,k-1}) + \epsilon_k,$$

where

$$\tau_{n,k} = \sum_{j=1}^k \phi_n^{k-j} a_{n,j}, \quad (3)$$

$k = 1, \dots, n, n \geq 1$. Set for $n \geq 1, z_{n,0} = 0$ and

$$z_{n,k} = y_{n,k} - \tau_{n,k}, \quad k = 1, \dots, n. \quad (4)$$

Note that $(z_{n,k})$ is a nearly nonstationary first order autoregressive process

$$z_{n,k} = \phi_n z_{n,k-1} + \epsilon_k, \quad k = 1, \dots, n, n \geq 1, \quad z_{n,0} = 0.$$

So, due to (4), we have the epidemic change model, where a sequence of dependent random variables satisfying the null hypothesis is shifted by a deterministic sequence. This is the reason why statistics (2) seems very natural in this situation.

We study limit behavior of $\tilde{T}_{\alpha,n}$ for $\alpha = 0$ (Levin and Kline statistic) and $\alpha \in (0, 1/2 - 1/p), p > 2$, (Račkauskas and Suquet statistics) trying to see how the use of extra weighting improves the detection of (relatively) short epidemics. Of course the range of detection will be smaller here than that in the case of independent samples. If $\alpha = 0$, then the innovations are required to have finite second moment. For another case, the innovations should satisfy the stronger integrability condition

$$\lim_{t \rightarrow \infty} t^p P(|\epsilon_0| > t) = 0. \quad (5)$$

In this paper, we study two types of models depending on parameterization of the coefficient ϕ_n in (1). The first type model corresponds to

$$\phi_n = e^{\gamma/n}, \quad \gamma < 0, \quad (6)$$

see [14]. The second type model corresponds to

$$\phi_n = 1 - \frac{\gamma_n}{n}, \quad \text{where } \gamma_n \rightarrow \infty, \quad \text{and } \frac{\gamma_n}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (7)$$

see [15]. As we shall see, the limit behavior of $\tilde{T}_{\alpha,n}$ differs for these two types of models.

The paper is organized as follows. Section 2 is devoted to the limit behaviour of statistics under null hypothesis. In Section 3, we show the consistency of test statistics $\tilde{T}_{\alpha,n}$. We investigate the power of the test in Section 4. Final section is devoted to some auxiliary results.

2 Limit behavior of test statistics under null hypothesis

The next two processes play the central role in this paper. We denote by $W = \{W(t), 0 \leq t \leq 1\}$ the standard Wiener process and by $U_\gamma = \{U_\gamma(t), t \in [0, 1]\}$ the following Ornstein–Uhlenbeck process:

$$U_\gamma(t) = \int_0^t e^{(t-s)\gamma} dW(s). \quad (8)$$

As usual, $C[0, 1]$ is the Banach space of continuous functions with uniform norm $\|f\|_\infty = \sup_{0 \leq t \leq 1} |f(t)|$, $f \in C[0, 1]$. For $\alpha \in [0, 1)$, the Hölder space

$$H_\alpha^0[0, 1] := \left\{ f \in C[0, 1] : \lim_{\delta \rightarrow 0} \omega_\alpha(f, \delta) = 0 \right\}$$

is a linear space endowed with the norm $\|f\|_\alpha := |f(0)| + \omega_\alpha(f, 1)$, where

$$\omega_\alpha(f, \delta) := \sup_{\substack{s, t \in [0, 1] \\ 0 < t - s < \delta}} \frac{|f(t) - f(s)|}{|t - s|^\alpha}.$$

Let us note that the spaces $H_\alpha^0[0, 1]$ and $C[0, 1]$ are isomorphic.

For any function $f \in H_\alpha^0[0, 1]$ and $0 \leq \alpha < 1/2$, set

$$T_{\alpha, \infty}(f) := \sup_{0 < t < s < 1} \frac{|f(t) - f(s) - (t - s)f(1)|}{|t - s|^\alpha}. \quad (9)$$

The functional (9) appears in the limit of our test statistics.

Throughout the paper $\xrightarrow{\mathbb{E}}$ denotes convergence in distribution in the metric space \mathbb{E} as $n \rightarrow \infty$. Accordingly, the classical convergence in distribution of a sequence of random variables is denoted by $\xrightarrow{\mathbb{R}}$ as $n \rightarrow \infty$, while convergence in probability is denoted by $\xrightarrow{\mathbb{P}}$ as $n \rightarrow \infty$.

2.1 Levin and Kline statistic

We start with Levin and Kline statistic $\tilde{T}_{0, n}$. First, we study the model (1) with the coefficient $\phi_n = e^{\gamma/n}$, $\gamma < 0$. Under the assumption of square integrability of innovations, we obtain that the limit of such statistic is a functional of an integrated Ornstein–Uhlenbeck process.

Theorem 1. *Under (H_0) , for the first type model defined by (1) and (6),*

$$n^{-3/2} \sigma^{-1} \tilde{T}_{0, n} \xrightarrow{\mathbb{R}} T_{0, \infty}(J), \quad (10)$$

where $\sigma^2 = \mathbf{E} \epsilon_1^2$ and J is an integrated Ornstein–Uhlenbeck process $J(t) = \int_0^t U_\gamma(s) ds$ with U_γ defined by (8).

Proof. Consider the functionals g_n and g defined on the continuous function space $C[0, 1]$ by

$$g_n(x) := \max_{1 \leq i < j \leq n} I_0\left(x, \frac{i}{n}, \frac{j}{n}\right), \quad g(x) := \sup_{0 < s < t < 1} I_0(x, s, t), \quad (11)$$

where

$$I_0(x, s, t) := |x(t) - x(s) - (t - s)x(1)|, \quad 0 < t - s < 1.$$

By the special case of Lemma A.1 where $\alpha = 0$, the functionals g_n and g are Lipschitz on $G_0 = \{x \in C[0, 1]: x(0) = 0\}$. Note that

$$\tilde{T}_{0,n} = g_n(S_n^{\text{pl}}), \quad T_{0,\infty}(J) = g(J), \tag{12}$$

where $(S_n^{\text{pl}}(t), t \in [0, 1])$ is the polygonal line partial sums process build on the observations $(y_{n,k-1})$:

$$S_n^{\text{pl}}(t) := \sum_{k=1}^{[nt]} y_{n,k-1} + (nt - [nt])y_{n,[nt]}. \tag{13}$$

From Theorem 1 in [16],

$$n^{-3/2}\sigma^{-1}S_n^{\text{pl}} \xrightarrow{C[0,1]} J. \tag{14}$$

Note that limit theorems in [16] are proved with $\sigma^2 = 1$ for simplicity, but the results hold the same for $\sigma^2 \neq 1$ as well.

Lemma A.1 now gives

$$g_n(n^{-3/2}\sigma^{-1}S_n^{\text{pl}}) = g(n^{-3/2}\sigma^{-1}S_n^{\text{pl}}) + o_{\mathbf{P}}(1) \tag{15}$$

and the convergence (10) follows from (12), (14), (15) and the continuous mapping theorem. \square

Now we find the limit of test statistic $\tilde{T}_{0,n}$ under null hypothesis in second type model.

Theorem 2. *Under (H_0) , for the second type model defined by (1) and (7),*

$$n^{-1/2}(1 - \phi_n)\sigma^{-1}\tilde{T}_{0,n} \xrightarrow{\mathbb{R}} T_{0,\infty}(W), \tag{16}$$

where $\sigma^2 = \mathbf{E}\epsilon_1^2$.

Proof. The proof of this theorem is essentially the same as the proof of Theorem 1 using Theorem 2 in [16] instead of Theorem 1 in [16] and Lemma A.1 given below. \square

2.2 $\tilde{T}_{\alpha,n}$ statistics with $\alpha > 0$

Now we show that, for the model (1) with $\phi_n = e^{\gamma/n}$, $\gamma < 0$, the limit of $\tilde{T}_{\alpha,n}$ ($\alpha > 0$) is a functional of an integrated Ornstein–Uhlenbeck process. Here we need a stronger integrability on innovations than just a second moment.

Theorem 3. *In the first type model defined by (1) and (6), assume that (ϵ_i) satisfy condition (5) for some $p > 2$. Then under (H_0) , for any $\alpha \in (0, 1/2 - 1/p)$,*

$$n^{-3/2+\alpha}\sigma^{-1}\tilde{T}_{\alpha,n} \xrightarrow{\mathbb{R}} T_{\alpha,\infty}(J), \tag{17}$$

where $\sigma^2 = \mathbf{E}\epsilon_1^2$ and J is an integrated Ornstein–Uhlenbeck process $J(t) = \int_0^t U_\gamma(s) ds$ with U_γ defined by (8).

Proof. We use the Hölderian framework to prove this theorem. Consider the functionals g_n, g , defined on $H_\alpha^o[0, 1]$ by

$$g_n(x) := \max_{1 \leq i < j \leq n} I_\alpha \left(x, \frac{i}{n}, \frac{j}{n} \right), \quad g(x) := \sup_{0 < s < t < 1} I_\alpha(x, s, t),$$

where

$$I_\alpha(x, s, t) := \frac{|x(t) - x(s) - (t-s)x(1)|}{|t-s|^\alpha}, \quad 0 < t-s < 1.$$

By Lemma A.1, g_n and g are Lipschitz on $G_\alpha = \{x \in H_\alpha^o[0, 1]: x(0) = 0\}$. Observe that

$$n^\alpha \tilde{T}_{\alpha, n} = g_n(S_n^{\text{pl}}), \quad T_{\alpha, \infty}(J) = g(J), \quad (18)$$

where $(S_n^{\text{pl}}(t), t \in [0, 1])$ is defined by (13). From Theorem 1 in [16],

$$n^{-3/2} \sigma^{-1} S_n^{\text{pl}} \xrightarrow{H_\alpha^o[0, 1]} J. \quad (19)$$

Now from Lemma A.1 it follows that

$$g_n(n^{-3/2} \sigma^{-1} S_n^{\text{pl}}) = g(n^{-3/2} \sigma^{-1} S_n^{\text{pl}}) + o_{\mathbf{P}}(1) \quad (20)$$

and the convergence (17) follows from (18), (19), (20) and the continuous mapping theorem. \square

Further we find the limit of test statistics $\tilde{T}_{\alpha, n}$ under null hypothesis in the second type model, i.e., where coefficient ϕ_n in model (1) is defined by $\phi_n = 1 - \gamma_n/n$, $\gamma_n \rightarrow \infty$ and $\gamma_n/n \rightarrow 0$ as $n \rightarrow \infty$. The limit of these statistics is a functional depending on Wiener process. Here the requirement is not only integrability condition on innovations, but also the rate of divergence of γ_n .

Theorem 4. *In the second type model defined by (1) and (7), assume that (ϵ_i) satisfy condition (5) for some $p > 2$. Then, for $\alpha \in (0, 1/2 - 1/p)$, under (H_0) ,*

$$n^{-1/2+\alpha} (1 - \phi_n) \sigma^{-1} \tilde{T}_{\alpha, n} \xrightarrow{\mathbb{R}} T_{\alpha, \infty}(W) \quad (21)$$

provided that

$$\liminf_{n \rightarrow \infty} \gamma_n n^{-\alpha/(1/2-1/p)} > 0.$$

Proof. The proof of this theorem is based on the same Hölderian framework as the proof of Theorem 3 using Theorem 3 in [16] instead of Theorem 1 in [16] and Lemma A.1. \square

3 Consistency of test statistics

We investigate the consistency of the test statistics $\tilde{T}_{\alpha,n}$. The practical results are given in Corollaries 2 and 1. Proofs of these corollaries are based on the following generic result (Theorem 5) which has a broader scope. The consistency condition is expressed in terms of

$$T_{\alpha,n}(\tau_{n,1}, \dots, \tau_{n,n}) = \max_{1 \leq \ell \leq n-1} \ell^{-\alpha} \max_{1 \leq k \leq n-\ell} \left| \sum_{j=k+1}^{k+\ell} \tau_{n,j} - \frac{\ell}{n} \sum_{j=1}^n \tau_{n,j} \right|, \tag{22}$$

where the $\tau_{n,k}$'s are defined by (3).

For notational simplicity, we omit the index n in k_n^* , m_n^* and ℓ_n^* .

Theorem 5. *Consider both types of models. Assume that, for some normalizing sequence $(b_n)_{n \geq 1}$, the statistics $b_n \tilde{T}_{\alpha,n}$ is stochastically bounded under (H_0) . Then under (H_A) ,*

$$b_n \tilde{T}_{\alpha,n} \xrightarrow{\mathbf{P}} \infty \tag{23}$$

if and only if

$$b_n T_{\alpha,n}(\tau_{n,1}, \dots, \tau_{n,n}) \xrightarrow{n \rightarrow \infty} \infty. \tag{24}$$

A sufficient condition for (24) is

$$\frac{a_n b_n}{(1 - \phi_n)^{2\ell_n^* \alpha}} \left(\ell_n^* (1 - \phi_n) \left(1 - \frac{\ell_n^*}{n} \right) - (1 - \phi_n^{\ell_n^*}) \left(\phi_n - \frac{\ell_n^*}{n} \phi_n^{n-m_n^*+1} \right) \right) \xrightarrow{n \rightarrow \infty} \infty. \tag{25}$$

Proof. Recall that the process (z_n) is defined by $z_{n,k} = y_{n,k} - \tau_{n,k}$, $0 \leq k \leq n$. The key point here is that the process (z_n) satisfies (H_0) , when the process (y_n) satisfies (H_A) (if (y_n) satisfies (H_0) , then both processes are identical). Hence $b_n T_{\alpha,n}(z_{n,1}, \dots, z_{n,n})$ is stochastically bounded. Now by triangle inequality for the sequential norm $T_{\alpha,n}$:

$$\begin{aligned} & |T_{\alpha,n}(y_{n,1}, \dots, y_{n,n}) - T_{\alpha,n}(\tau_{n,1}, \dots, \tau_{n,n})| \\ & \leq T_{\alpha,n}(y_{n,1} - \tau_{n,1}, \dots, y_{n,n} - \tau_{n,n}) = T_{\alpha,n}(z_{n,1}, \dots, z_{n,n}), \end{aligned}$$

so the stochastic boundedness of $b_n T_{\alpha,n}(z_{n,1}, \dots, z_{n,n})$ gives the equivalence between (23) and (24).

Looking now for a practical sufficient condition for (24), we choose as a lower bound for $T_{\alpha,n}(\tau_{n,1}, \dots, \tau_{n,n})$ the weighted increment corresponding to the epidemics interval $(k^*, m^*]$ with length $m^* - k^* = \ell^*$. With these notations,

$$\tau_{n,k} = \sum_{j=1}^k \phi_n^{k-j} a_n \mathbf{1}_{(k^*, m^*]}(j), \quad 1 \leq k \leq n, \quad \tau_{n,0} := 0.$$

It is enough to write the proof for the case $a_n = 1$, since in the general case, all the computations made below remain valid with a_n in factor.

Let us compute $\sum_{k=1}^n \tau_{n,k}$.

$$\begin{aligned} \sum_{k=1}^n \tau_{n,k} &= \sum_{k \leq k^*} \tau_{n,k} + \sum_{k^* < k \leq m^*} \tau_{n,k} + \sum_{m^* < k \leq n} \tau_{n,k} \\ &= \underbrace{\sum_{k^* < k \leq m^*} \sum_{k^* < j \leq k} \phi_n^{k-j}}_{=:A} + \underbrace{\sum_{m^* < k \leq n} \sum_{k^* < j \leq m^*} \phi_n^{k-j}}_{=:B}. \end{aligned}$$

We compute separately the double geometric sums A and B and we obtain

$$A = \frac{1}{(1 - \phi_n)^2} (\ell^* (1 - \phi_n) - \phi_n (1 - \phi_n^{\ell^*})) \quad (26)$$

and

$$B = \frac{1}{(1 - \phi_n)^2} (\phi_n (1 - \phi_n^{\ell^*}) - \phi_n^{n-m^*+1} (1 - \phi_n^{\ell^*})). \quad (27)$$

Gathering (26) and (27), we obtain

$$\sum_{j=1}^n \tau_{n,j} = \frac{1}{(1 - \phi_n)^2} (\ell^* (1 - \phi_n) - \phi_n^{n-m^*+1} (1 - \phi_n^{\ell^*})). \quad (28)$$

Finally,

$$\begin{aligned} &A - \frac{\ell^*}{n} (A + B) \\ &= \sum_{j=k^*+1}^{k^*+\ell^*} \tau_{n,j} - \frac{\ell^*}{n} \sum_{j=1}^n \tau_{n,j} \\ &= \frac{1}{(1 - \phi_n)^2} \left(\ell^* (1 - \phi_n) \left(1 - \frac{\ell^*}{n}\right) - (1 - \phi_n^{\ell^*}) \left(\phi_n - \frac{\ell^*}{n} \phi_n^{n-m^*+1}\right) \right) \quad (29) \end{aligned}$$

and

$$b_n T_{\alpha,n}(\tau_{n,1}, \dots, \tau_{n,n}) \geq \frac{b_n a_n}{\ell^* \alpha} \left| \sum_{j=k^*+1}^{k^*+\ell^*} \tau_{n,j} - \frac{\ell^*}{n} \sum_{j=1}^n \tau_{n,j} \right|$$

which explains why (25) is a sufficient condition for (24). \square

Corollary 1. *In the first type model defined by (1) and (6), assume that, for some $p > 2$, (ϵ_i) satisfy condition (5). Let $\alpha \in (0, 1/2 - 1/p)$, then under (H_A) ,*

$$n^{-3/2+\alpha} \tilde{T}_{\alpha,n} \xrightarrow{\mathbf{P}} \infty \quad (30)$$

provided that $\ell^{*2-\alpha} n^{-3/2+\alpha} a_n \rightarrow \infty$ as $n \rightarrow \infty$, and

$$\liminf_{n \rightarrow \infty} \left| 1 + \frac{\gamma}{2} - e^{\gamma(1-m^*/n)} \right| > 0. \quad (31)$$

All this extends to the special case $\alpha = 0$, assuming that $\mathbf{E}\epsilon_1^2 < \infty$.

Remark 1. From a statistical point of view, it is useful to find for which values of the parameter γ condition (31) does not induce some extra restriction on the choice of the sequence $(m^*(n))_{n \geq 1}$. Writing $\theta_n := m^*(n)/n$, we see that (31) is not satisfied if and only if there exists some subsequence $(\theta_{n_j})_{j \geq 1}$ in $(0, 1)$ such that $e^{\gamma(1-\theta_{n_j})}$ tends to $1 + \gamma/2$. Then any θ limit of some subsequence of $(\theta_{n_j})_{j \geq 1}$ (there is at least one such θ by compactness of $[0, 1]$) must satisfy $1 + \gamma/2 = e^{\gamma(1-\theta)}$. Clearly, this equation has no solution for $\gamma \leq -2$. For $-2 < \gamma < 0$, it has a unique solution

$$\theta = 1 - \frac{1}{\gamma} \ln \left(1 + \frac{\gamma}{2} \right).$$

It is easily seen that this solution belongs to $[0, 1]$ only if $\gamma_0 \leq \gamma < 0$, where $\gamma_0 \simeq -1.5937$. From this we can conclude that if $\gamma < \gamma_0$, the condition (31) is satisfied without any extra restrictions on the choice of the sequence $(m^*(n))_{n \geq 1}$. For $\gamma_0 \leq \gamma < 0$, one can always find a sequence $(m^*(n))_{n \geq 1}$ for which (31) fails.

Remark 2. From the consistency condition, one can see that the bigger α the shorter change can be detected with the statistics. As expected, the detection is not so good as in the i.i.d. case, see [12].

Proof. We keep the notations A and B already used in the previous proof. By Theorem 3, under (H_0) , $b_n \tilde{T}_{\alpha, n}$ converges in distribution and hence is stochastically bounded for the normalization $b_n = n^{-3/2+\alpha}$. So it remains only to check condition (25). This requires an estimate for the asymptotic order of magnitude of

$$\begin{aligned} & A - \frac{\ell^*}{n}(A + B) \\ &= \frac{1}{(1 - \phi_n)^2} \left(\ell^*(1 - \phi_n) \left(1 - \frac{\ell^*}{n} \right) - \phi_n(1 - \phi_n^{\ell^*}) \left(1 - \frac{\ell^*}{n} \phi_n^{n-m^*} \right) \right). \end{aligned}$$

Using the second order expansions

$$\begin{aligned} 1 - \phi_n &= -\frac{\gamma}{n} - \frac{\gamma^2}{2n^2} + o(n^{-2}), \\ 1 - \phi_n^{\ell^*} &= -\frac{\gamma \ell^*}{n} - \frac{\gamma^2 \ell^{*2}}{2n^2} + o\left(\frac{\ell^{*2}}{n^2}\right), \end{aligned}$$

we deduce

$$\left| A - \frac{\ell^*}{n}(A + B) \right| \geq \ell^{*2} \left| \frac{1}{2} + \frac{1}{\gamma} (1 - e^{\gamma(1-m^*/n)}) \right|.$$

So the divergence (25) follows from the condition $n^{-3/2+\alpha} \ell^{*2-\alpha} a_n \rightarrow \infty$ as $n \rightarrow \infty$ and (31). □

Corollary 2. *In the second type model defined by (1) and (7), assume that, for some $p > 2$, (ϵ_i) satisfy condition (5). Let $\alpha \in (0, 1/2 - 1/p)$ and assume that*

$$\liminf_{n \rightarrow \infty} \gamma_n n^{-\alpha/(1/2-1/p)} > 0.$$

Suppose that either of the following conditions is satisfied:

1. $\ell^*(1 - \phi_n) \rightarrow \infty$, $\limsup_{n \rightarrow \infty} \ell^*/n < 1$ and $n^{-1/2+\alpha} \ell^{*1-\alpha} a_n \rightarrow \infty$ as $n \rightarrow \infty$;
2. $\ell^*(1 - \phi_n) \rightarrow c > 0$ and $n^{-1/2+\alpha} \ell^{*1-\alpha} a_n \rightarrow \infty$ as $n \rightarrow \infty$;
3. $\ell^*(1 - \phi_n) \rightarrow 0$ and $n^{-3/2+\alpha} \gamma_n \ell^{*2-\alpha} a_n \rightarrow \infty$ as $n \rightarrow \infty$.

Then under (H_A) ,

$$n^{-1/2+\alpha} (1 - \phi_n) \tilde{T}_{\alpha, n} \xrightarrow{\mathbf{P}} \infty. \quad (32)$$

The conclusion extends to the special case $\alpha = 0$, under the same assumptions provided that (5), is replaced by $\mathbf{E}\epsilon_1^2 < \infty$.

Proof. By Theorem 4, under (H_0) , $b_n \tilde{T}_{\alpha, n}$ converges in distribution and hence is stochastically bounded for the normalization $b_n = n^{-1/2+\alpha} (1 - \phi_n)$. So it remains only to check condition (25) in the three considered cases.

Case 1. If $\ell^*(1 - \phi_n) \rightarrow \infty$, noting that

$$|(1 - \phi_n^{\ell^*}) (\phi_n - \ell^* n^{-1} \phi_n^{n-m^*+1})| \leq 1$$

and recalling that $\limsup \ell^*/n < 1$, we immediately see that, for n large enough, there is some positive constant c such that

$$\left| A - \frac{\ell^*}{n} (A + B) \right| \geq \frac{c\ell^*}{1 - \phi_n}.$$

Then the divergence (25) follows clearly from the condition

$$n^{-1/2+\alpha} \ell^{*1-\alpha} a_n \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Case 2. If $\ell^*(1 - \phi_n) \rightarrow c > 0$, this implies in particular that ℓ^*/n tends to zero and

$$1 - \phi_n^{\ell^*} \xrightarrow[n \rightarrow \infty]{} 1 - e^{-c}.$$

By strict convexity of the exponential function, $e^{-c} \geq 1 - c$ with equality only if $c = 0$, hence $c - 1 + e^{-c} > 0$ since $c > 0$ and

$$\left| A - \frac{\ell^*}{n} (A + B) \right| \sim \frac{c - 1 + e^{-c}}{(1 - \phi_n)^2} \sim \frac{(c - 1 + e^{-c})\ell^*}{c(1 - \phi_n)}.$$

Again the divergence (25) follows from the condition

$$n^{-1/2+\alpha} \ell^{*1-\alpha} a_n \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Case 3. Assume finally that $\ell^*(1 - \phi_n) \rightarrow 0$ (this implies in particular that $\ell^* = o(n)$). Then in (29) the term $\ell^*(1 - \phi_n)$ is equivalent to $(1 - \phi_n^{\ell^*})$. By second order expansion, we find that

$$1 - \phi_n^{\ell^*} = \frac{\ell^* \gamma_n}{n} + \frac{\ell^{*2} \gamma_n^2}{2n^2} (1 + o(1)).$$

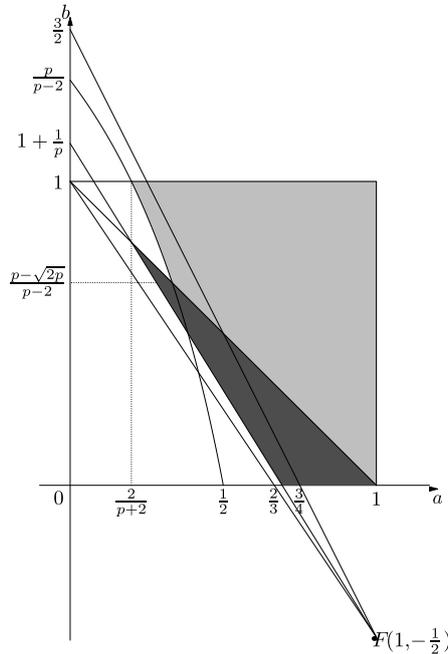


Fig. 1. Detection area in the space of parameters ($\ell^* \asymp n^a, \gamma_n \asymp n^b$) for Corollary 2.

This leads by elementary computation to

$$A - \frac{\ell^*}{n}(A + B) \sim -\frac{\ell^{*2}}{2},$$

so the the divergence (25) follows from the condition

$$n^{-3/2+\alpha} \gamma_n \ell^{*2-\alpha} a_n \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

□

Remark 3. The graphical interpretation presented in Fig. 1 may provide a better understanding of the results in Corollary 2. Assume for simplicity that $a_n = 1, \ell^* \asymp n^a$ (that is there are positive constants c_1 and c_2 such that, for n large enough, $c_1 n^a \leq \ell^* \leq c_2 n^a$) and that $\phi_n \asymp n^b$ for some $0 < a, b < 1$. For a given value of p in condition (5), what are the pairs (a, b) for which Corollary 2 allows detection of an epidemics of length $\ell^* \asymp n^a$, subject to an admissible choice of α ? The set of solutions is represented by the shadowed area of the unit square. The light grey part above the diagonal corresponds to Cases 1 and 2, that is $\lim_{n \rightarrow \infty} \ell^*(1 - \phi_n)$ belongs to $(0, \infty]$. The triangular darker grey area corresponds to the case where $\ell^*(1 - \phi_n)$ tends to 0. One can remark that, when p tends to infinity, the whole shadowed area converges to the trapezoid with upper basis the upper side of the unit square and lower basis the segment $[2/3, 1]$ on the horizontal axis.

4 Test power analysis

Here we perform the test power analysis. For this, we present the results of experiments in the Tables 1 and 2. We computed empirical power on size-adjusted (not nominal size) basis, i.e., replaced the nominal value of significance level by the value of empirical distribution function for p -values under null hypothesis. For more details on size power curves, see [17].

For different values of parameters γ , γ_n , α , k^* , ℓ^* and a_n , we compute $N = 1000$ realizations of test statistics with the sample size n . Innovations have been generated as standard normally distributed random variables. For the limit distribution, we compute $N = 5000$ realizations of test statistics with the sample size $n = 5000$. We approximate the values of the standard Wiener process by

$$W\left(\frac{k}{5000}\right) = n^{-1/2} \sum_{j=1}^k \epsilon(j), \quad k = 1, \dots, 5000,$$

where $\epsilon(j)$ are generated as standard normally distributed random variables. The Ornstein–Uhlenbeck process have been approximated by the the following discretization:

$$S_j = S_{j-1}e^{\gamma/n} + \sqrt{\frac{1 - e^{2\gamma/n}}{-2\gamma}} \cdot \epsilon_j, \quad \epsilon_j \sim \mathfrak{N}(0, 1), \quad j = 1, 2, \dots, n, \quad (33)$$

and $S_0 = 0$. For more details about (33), see [18]. Using values generated by (33), we approximate the integrated Ornstein–Uhlenbeck process by

$$J\left(\frac{k}{5000}\right) = n^{-1} \sum_{j=1}^k S_j, \quad k = 1, \dots, 5000.$$

Next, we define the basic parameter set for the first type model

$$\gamma = -2, \quad a_n = 1, \quad n = 1000, \quad \frac{\ell^*}{n} = 0.05, \quad \frac{k^*}{n} = 0.4, \quad y_{n,0} = 0.$$

Further modifying the separate parameters we compute the empirical size-power. We always keep all these parameters except one (indicated in the first column in both tables) which we allow to vary. Note that, in order to compute the test power, we need to compute the empirical p -values. Usually, the estimate of empirical p -value is $\hat{p} = s/N$, where s is the number of values (limit process) that are greater than or equal to the observed value (statistics), N is the number of values. Nevertheless, the previous formula is biased due to the finite sampling. Davison and Hinkley [19, p. 141] suggested to correct the bias with such formula $\hat{p} = (s + 1)/(N + 1)$. One can observe, that these two formulas are essentially the same when the number of replications N is large, but we use unbiased estimate in this computations.

As one can see in Table 1 the test power is almost the same for all α . The test power increases with the length of epidemics, but it has no big difference with increasing α .

Table 1. Empirical power at the size-adjusted significance level 0.05 for the first type model.

Parameters	$\alpha = 0$	$\alpha = 2/32$	$\alpha = 6/32$	$\alpha = 12.5/32$
$\ell^*/n = 0.035$	0.442	0.440	0.446	0.421
$\ell^*/n = 0.050$	0.758	0.757	0.767	0.752
$\ell^*/n = 0.100$	1.000	1.000	1.000	1.000
$k^*/n = 0.2$	0.591	0.589	0.615	0.653
$k^*/n = 0.4$	0.758	0.757	0.767	0.752
$k^*/n = 0.8$	0.587	0.616	0.697	0.784
$a_n = 0.8$	0.554	0.549	0.556	0.534
$a_n = 1$	0.758	0.757	0.767	0.752
$a_n = 1.2$	0.907	0.908	0.920	0.914
$n = 500$	0.388	0.404	0.408	0.409
$n = 1000$	0.758	0.757	0.767	0.752
$n = 2000$	0.979	0.982	0.980	0.983
$\gamma = -2$	0.758	0.757	0.767	0.752
$\gamma = -12$	0.677	0.728	0.822	0.896
$\gamma = -100$	0.748	0.833	0.967	0.998

Table 2. Empirical power at the size-adjusted significance level 0.05 for the second type model.

Parameters	$\alpha = 0$	$\alpha = 2/32$	$\alpha = 6/32$	$\alpha = 10/32$
$\ell^*/n = 0.035$	0.373	0.441	0.675	0.909
$\ell^*/n = 0.050$	0.758	0.859	0.974	0.996
$\ell^*/n = 0.065$	0.980	0.990	0.999	1.000
$k^*/n = 0.2$	0.780	0.875	0.980	0.999
$k^*/n = 0.4$	0.758	0.859	0.974	0.996
$k^*/n = 0.8$	0.783	0.877	0.981	0.998
$a_n = 0.8$	0.478	0.565	0.780	0.929
$a_n = 1$	0.758	0.859	0.974	0.996
$a_n = 1.2$	0.949	0.985	0.999	1.000
$n = 500$	0.422	0.480	0.676	0.813
$n = 1000$	0.758	0.859	0.974	0.996
$n = 2000$	0.997	1.000	1.000	1.000
$\gamma_n = n/\ln(n)$	0.754	0.847	0.970	0.995
$\gamma_n = \ln^{2.5}(n)$	0.758	0.844	0.972	0.995
$\gamma_n = n^{3/4}$	0.758	0.859	0.974	0.996

Note that, for the first type model, the location of epidemics makes the difference. The biggest power is for the epidemics in the middle of the observations. For this model, the test can detect the epidemic change best when $a_n = 1$ or bigger, for the smaller changes, it has a lower power. Naturally, the test power increases with the number of observations and α . Further the bigger is γ , the bigger is test power. That is the test power increases when the ratio γ/n increases.

The basic parameter set for the second type model ($\phi_n = 1 - \gamma_n/n$) are

$$\gamma_n = n^{3/4}, \quad a_n = 1, \quad n = 1000, \quad \frac{\ell^*}{n} = 0.05, \quad \frac{k^*}{n} = 0.4, \quad y_{n,0} = 0.$$

For the second type model (Table 2), the test power for all parameter values is the lowest, when $\alpha = 0$ and increases with the α . For this model, detection of epidemic changes becomes better with the increasing length of epidemics, but the test detects short epidemic change very good for the bigger α . Note that the test power does not depend on the place of epidemics. Also, it detects quite good even small changes as $a_n = 0.8$. The test power increases when the number of observations and α are increasing. The test power does not vary to much depending on γ_n .

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Appendix. Tools

The next results help us to prove the limiting behaviour of the test statistics under null hypothesis.

Lemma A.1. *Suppose $\alpha \in [0, 1/2 - 1/p]$, $p > 2$. Consider the functionals g_n and g defined on the Hölder space $H_\alpha^0[0, 1]$ by*

$$g_n(x) := \max_{1 \leq i, j \leq n} I_\alpha\left(x, \frac{i}{n}, \frac{j}{n}\right), \quad g(x) := \sup_{0 < s < t < 1} I_\alpha(x, s, t), \quad (\text{A.1})$$

where

$$I_\alpha(x, s, t) := \frac{|x(t) - x(s) - (t-s)x(1)|}{|t-s|^\alpha}, \quad 0 < t - s < 1. \quad (\text{A.2})$$

Then g_n and g are Lipschitz on $G_\alpha = \{x \in H_\alpha^0[0, 1]: x(0) = 0\}$ with the same constant $C = 2$, if $\alpha \in (0, 1/2 - 1/p]$. Also, g_n and g are Lipschitz on $G_0 = \{x \in C[0, 1]: x(0) = 0\}$ with the same constant $C = 2$, if $\alpha = 0$.

Further, for any tight sequence of random elements $(\eta_n)_{n \geq 0}$ in $C[0, 1]$ or $H_\alpha^0[0, 1]$, it holds

$$g_n(\eta_n) = g(\eta_n) + o_{\mathbf{P}}(1). \quad (\text{A.3})$$

Proof. Here we shall give an unified proof for the cases $\alpha = 0$ and $\alpha \in (0, 1/2 - 1/p]$. Since the spaces $(C, \|\cdot\|_\infty)$ and $(H_0^0, \|\cdot\|_0)$ are isomorphic, thus putting $\alpha = 0$ in the proof gives the special case of g_n and g being Lipschitz on $C[0, 1]$. To show that $q = I(\cdot, s, t)$ is Lipschitz, we shall use Lemma A.3. Clearly, q satisfies conditions (i) and (ii) of this lemma. We shall check the condition (iii)

$$q(x) = I_\alpha(x, s, t) \leq \frac{|x(t) - x(s)|}{|t-s|^\alpha} + |t-s|^{1-\alpha}|x(1)| \leq 2\|x\|_\alpha. \quad (\text{A.4})$$

Let us introduce the closed subspace $G_\alpha = \{x \in H_\alpha^0[0, 1]: x(0) = 0\}$. From (A.4) we see that the functional $q = I_\alpha(\cdot, s, t)$ satisfies on G_α the Condition (iii) of Lemma A.3 with the constant $C = 2$. It follows by Lemma A.3 that g_n as well as g are Lipschitz on G_α with the same constant C . In fact, the sequence $(g_n)_{n \geq 2}$ is equicontinuous on G_α .

Further, to check the pointwise convergence on G_α of g_n to g , it is enough to show that, for each $x \in G_\alpha$, the function $(s, t) \mapsto I_\alpha(x, s, t)$ can be extended by continuity to the compact set $K = \{(s, t) \in [0, 1]^2: 0 \leq s \leq t \leq 1\}$. From (A.4) we get $0 \leq I_\alpha(x, s, t) \leq \omega_\alpha(x, t-s) + |x(1)||t-s|^{1-\alpha}$ which allows the continuous extension along diagonal by putting $I_\alpha(x, s, s) := 0$. The definition of $I_\alpha(x, s, t)$ allows us continuous extension at the point $(0, 1)$ putting $I_\alpha(x, 0, 1) := 0$.

Suppose that the sequence $(\eta_n)_{n \geq 1}$ is tight on $C[0, 1]$ or on $H_\alpha^0[0, 1]$. As the pointwise convergence of (g_n) is established, using Lemma A.2 we obtain that

$$g_n(\eta_n) = g(\eta_n) + o_{\mathbf{P}}(1). \quad \square$$

The two following lemmas one can find in [12].

Lemma A.2. *Let (η_n) be a tight sequence of random elements in separable Banach space B and g_n, g be continuous functionals $B \rightarrow \mathbb{R}$. Assume that g_n converges pointwise to g on B and that (g_n) is equicontinuous. Then*

$$g_n(\eta_n) = g(\eta_n) + o_{\mathbf{P}}(1).$$

Lemma A.3. *Let $(\mathcal{B}, \|\cdot\|)$ be a vector normed space and $q : \mathcal{B} \rightarrow \mathbb{R}$ such that:*

- (i) *q is subadditive: $q(x+y) \leq q(x) + q(y)$, $x, y \in \mathcal{B}$;*
- (ii) *q is symmetric: $q(-x) = q(x)$, $x \in \mathcal{B}$;*
- (iii) *for some constant C , $q(x) \leq C\|x\|$, $x \in \mathcal{B}$.*

Then q satisfies the Lipschitz condition

$$|q(x+y) - q(x)| \leq C\|y\|, \quad x, y \in \mathcal{B}. \quad (\text{A.5})$$

If \mathcal{F} is any set of functionals q fulfilling (i), (ii) and (iii) with the same constant C , then (i), (ii) and (iii) are satisfied by $g(x) := \sup\{q(x): q \in \mathcal{F}\}$ which therefore satisfies (A.5).

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