

## Numerical solution of nonlinear elliptic equation with nonlocal condition\*

Regimantas Čiupaila<sup>a</sup>, Mifodijus Sapagovas<sup>b</sup>, Olga Štikonienė<sup>c</sup>

<sup>a</sup>Vilnius Gediminas Technical University  
Saulėtekio ave. 11, LT-10223 Vilnius, Lithuania  
regimantas.ciupaila@vgtu.lt

<sup>b</sup>Institute of Mathematics and Informatics, Vilnius University  
Akademijos str. 4, LT-08663 Vilnius, Lithuania  
mifodijus.sapagovas@mii.vu.lt

<sup>c</sup>Vilnius University  
Naugarduko str. 24, LT-03225 Vilnius, Lithuania  
olga.stikoniene@mif.vu.lt

**Received:** 29 December 2012 / **Revised:** 27 August 2013 / **Published online:** 25 September 2013

**Abstract.** Two iterative methods are considered for the system of difference equations approximating two-dimensional nonlinear elliptic equation with the nonlocal integral condition. Motivation and possible applications of the problem present in the paper coincide with the small volume problems in hydrodynamics. The differential problem considered in the article is some generalization of the boundary value problem for minimal surface equation.

**Keywords:** minimal surface equation, nonlocal boundary condition, finite-difference method, iterative method.

### 1 Introduction and statement of the problem

In this paper, we investigate nonlinear elliptic equation

$$\frac{\partial}{\partial x} \left( \frac{1}{\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}} \frac{\partial u}{\partial y} \right) - H(x, y, u) + \lambda = 0 \quad (1)$$

in the domain  $(x, y) \in \Omega$  with the boundary condition

$$u|_{\Gamma} = \varphi(x, y), \quad (2)$$

\*This research was partially supported by the Research Council of Lithuania (grant No. MIP-051/2011).

where  $\Gamma$  is the contour of the domain  $\Omega$ . The parameter  $\lambda$  in equation (1) is unknown and because of that additional condition is given

$$\iint_{\Omega} u(x, y) \, dx \, dy = V. \quad (3)$$

Equation (1) is entitled as an equation of the surface with the prescribed mean curvature. When  $H = 0$ ,  $\lambda = 0$ , equation (1) is called as a minimal surface equation.

Solution of the minimal surface equation with the given boundary condition (2) is well known as a Plateau problem: provide the least-area surface through the given spatial contour. Minimal surfaces are applied in architecture, shape design, various areas of physics, chemistry, and biology [1–4].

Numerical solution of the equation of minimal surface is a difficult enough and interesting problem. The first papers on the methods of solution of the minimal surface equation are rather early (see [5, 6] and references therein). Anyway, interest in this area of numerical analysis is still strong [7–11].

Problem (1)–(3) belongs to the class of the differential equations with the nonlocal conditions, because of the presence of condition (3). Such problems are intensively investigated during the last decades.

The problem formulated in paper [12] was very close to that, investigated in our present research of problem (1)–(3). In this paper, the following problem is considered:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad 0 < x, y < 1, \quad 0 < t < T, \quad (4)$$

$$u(x, y, 0) = f(x, y), \quad (5)$$

$$u(0, y, t) = g_0(y, t), \quad u(1, y, t) = g_1(y, t), \quad u(x, 1, t) = h_1(x, t), \quad (6)$$

$$u(x, 0, t) = \mu(t)h_0(x), \quad (7)$$

$$\int_0^1 \int_0^{s(x)} u(x, y, t) \, dx \, dy = m(t), \quad (8)$$

where  $f$ ,  $g_0$ ,  $g_1$ ,  $h_0$ ,  $h_1$ ,  $s$  and  $m$  are known functions of their arguments, while the function  $\mu(t)$  is unknown. Applying the implicit difference scheme to this problem, on the upper layer  $t = t_n$  we get an elliptic difference operator with Dirichlet type boundary conditions and with one unknown parameter  $\mu(t_n)$ . As in problem (1)–(3), here, because of the unknown parameter, we get one additional overdeterminate integral condition (8), as an analogue of condition (3).

The inverse problems for parabolic equations, investigated in [13], are also close to problem (1)–(3) we investigate.

The inherent property of such problems is the presence of unknown function over time  $t$  in the equation and overdeterminate condition, analogical to condition (3) in our research.

Two-dimensional elliptic equations with the nonlocal conditions are investigated in [14–16]. Many numerical methods for different classes of partial differential equations with nonlocal conditions have been proposed (see [17–23] and references therein). The

least investigated area in the class of problems with nonlocal conditions is construction of iterative methods for the solution of systems of difference equations, substituting an elliptic equation. Only few papers [24–26] may be quoted. These difference equation systems are characterized by some specific features. For example, even for the simple one-dimensional problem

$$u'' = f(x), \quad 0 < x < 1,$$

the spectrum of eigenvalues of the matrix of the system of difference equations may be complicated enough depending on the boundary nonlocal conditions [27,28]. Negative or complex eigenvalues may emerge and the matrix itself may become defective, depending on multiple eigenvalues. This situation considerably impedes the investigation of convergence of iterative methods.

Motivation of the problem investigated is the same as that of the small volume problems in hydrodynamics [1].

Let us take a liquid drop, lying on the inclined plane, making with the horizontal plane an angle  $\alpha$ . In the case where electric permeable drop of liquid (for example, mercury) is interpreted as a part of electric contact [29], usually a presumption is made that adhesion force of the drop with the plane is sufficiently strong, i.e. the drop can't roll down on the inclined plane. Technologically this situation is achievable by special cure (watering) of the part of the plane (for example,  $\Omega$  is a circle).

The drop lying on the surface of the inclined plane assumes the shape according to the principle of minimal energy. In the stationary state of the drop general energy is the minimal one, i.e.  $\min(E_T + E_P)$ , where  $E_T$  is the energy of surface tension and  $E_P$  is a potential energy.

Let us take the coordinate plane on which the axis  $x$  is directed towards an inclination of the plane, the axis  $y$  is on the same plane, and the axis  $u$  is orthogonal to the inclined plane. Therefore

$$E_P = \gamma \iint_{\Omega} u \left( \frac{u}{2} \cos \alpha - x \sin \alpha \right) dx dy, \quad (9)$$

$$E_T = \sigma \iint_{\Omega} \sqrt{1 + \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2} dx dy, \quad (10)$$

where  $\sigma$  is the coefficient of surface tension,  $\gamma$  is the coefficient of gravity. Formula (3) denominates the volume of the drop.

Euler equation for obtaining conditional minimum of the functional  $E_T + E_P$ , when the additional condition (3) is given, is as follows:

$$\begin{aligned} \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{1 + \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2}} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{\sqrt{1 + \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2}} \frac{\partial u}{\partial y} \right) \\ - \frac{\gamma \cos \alpha}{\sigma} u + \frac{\gamma x \sin \alpha}{\sigma} + \frac{\lambda}{\sigma} = 0, \end{aligned} \quad (11)$$

where  $\lambda$  is a Lagrange multiplier.

So, the boundary value problem for equation (1) with the condition (2), where  $\varphi(x) = 0$  and with nonlocal condition (3) is formulated.

Main goal of our research is to construct and investigate some iterative methods for the system of nonlinear difference equations, corresponding to the differential problem (1)–(3).

## 2 Construction of the systems of difference equations

There are many patterns to approximate nonlinear differential equation (1) in the domain  $\Omega$ , bounded by the curved line  $\Gamma$ . In the beginning, we take one of the simplest ways of approximation, concentrating our attention on the solution of the system of difference equations. Later on, in Section 4, we will provide few patterns of more precise approximation.

The domain  $\Omega$  is covered by a quadratic grid with the step  $h$  made of two types of the lines parallel to the coordinate axis. The crossing points of the lines with the coordinates  $(x_i, y_i)$  are marked as  $(i, j)$ . We create closed polygonal line by segments of the lines approximating the contour  $\Gamma$ . Entirety of the internal points  $(i, j)$ , situated inside the area framed by the polygonal line, is denoted as  $\Omega_h$ . Points of the grid  $(i, j)$ , belonging to the polygonal line, comprise the contour  $\Gamma_h$ .

Let us denote

$$\begin{aligned} \delta_x u_{ij} &= \frac{u_{i+1,j} - u_{ij}}{h}, & \delta_{\bar{x}} u_{ij} &= \frac{u_{ij} - u_{i-1,j}}{h}, \\ \delta_y u_{ij} &= \frac{u_{i,j+1} - u_{ij}}{h}, & \delta_{\bar{y}} u_{ij} &= \frac{u_{ij} - u_{i,j-1}}{h}. \end{aligned}$$

Differential equation (1) is approximated at every point of the domain  $\Omega_h$  according to regular pattern of seven points  $\{(i, j), (i \pm 1, j), (i, j \pm 1), (i - 1, j + 1), (i + 1, j - 1)\}$  (Fig. 1), by the following difference equation:

$$\delta_x(\mu(T_{ij}^2)\delta_{\bar{x}}u_{ij}) + \delta_y(\mu(T_{ij}^2)\delta_{\bar{y}}u_{ij}) - H_{ij} + \lambda_h = 0, \quad (i, j) \in \Omega_h, \tag{12}$$

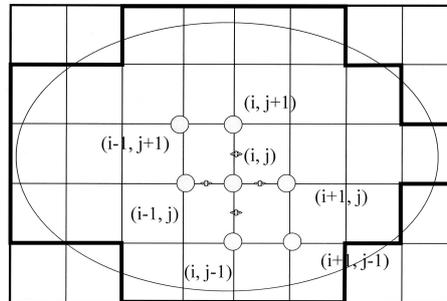
$$u_{ij} = \varphi_{ij}, \quad (i, j) \in \Gamma_h, \tag{13}$$

$$l(u_{ij}) \equiv h^2 \sum_{(i,j) \in \bar{\Omega}_h} \rho_{ij} u_{ij} = V, \tag{14}$$

where

$$\begin{aligned} \mu(T_{ij}^2) &= (1 + T_{ij}^2)^{-1/2} = (1 + (\delta_{\bar{x}}u_{ij})^2 + (\delta_{\bar{y}}u_{ij})^2)^{-1/2}, \\ \rho_{ij} &= \begin{cases} 1, & (i, j) \in \Omega_h, \\ 1/2 \text{ or } 1/4, & (i, j) \in \Gamma_h, \end{cases} \end{aligned} \tag{15}$$

according to the trapezoid rule for double integral.

Fig. 1. Regular pattern of  $\Omega_h$ .

The system of equations (12)–(15) possesses the unique solution. This statement can be proved analogically as the proofs provided in [30–32], where the differential problem and a slightly changed approximation of equation (1) were considered. We have no intention to repeat these proofs, just we remind some intermediate statements, which might be useful for the investigation of iterative methods.

In the paper [30], it is proven, that there exists the unique solution  $(u(x, y), \lambda)$  for problem (1)–(3), if  $V > 0$  is considerably small. The specification for  $V$  to be considerably small positive number arises from the physical point of view, discussed in the Introduction. The solution (liquid drop in physical sense) exists, if the volume of liquid in the drop is not too large, in proportion with the given base of the drop.

Further we analyze two boundary problems without the nonlocal condition and with the known value of parameter  $\lambda$  – the problem (1), (2)

$$\begin{aligned} L(u) + \lambda &= 0, \\ u|_{\Gamma} &= \varphi \end{aligned} \quad (16)$$

and difference problem (12), (13)

$$\begin{aligned} \Delta(u_{ij}) + \lambda &= 0, \quad (i, j) \in \Omega_h, \\ u_{ij} &= \varphi_{ij}, \quad (i, j) \in \Gamma_h. \end{aligned} \quad (17)$$

In both these problems,  $\lambda$  is the solution of problem (1)–(3). In [32], the statement is proven, and we will formulate it here as Lemma 1 (see also [6]).

**Lemma 1.** *If there exists an a priori estimation*

$$\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \leq M_1 \quad (18)$$

for problem (16), then an analogous a priori estimation

$$(\delta_{\bar{x}} u_{ij})^2 + (\delta_{\bar{y}} u_{ij})^2 \leq M_2 \quad (19)$$

also exists for problem (17).

Let us take any value of the parameter  $\alpha$  and denote the solution of the problem

$$\begin{aligned} \Delta(u_{ij}) + \alpha &= 0, & (i, j) \in \Omega_h, \\ u_{ij} &= \varphi_{ij}, & (i, j) \in \Gamma_h, \end{aligned} \tag{20}$$

as  $u_{ij}(\alpha)$ . Let us denote  $\Phi(\alpha) = l(u_{ij}(\alpha)) - V$ .

The following statement is true [32].

**Lemma 2.** *There exists an interval  $[\lambda_1, \lambda_2]$ , to which value of the parameter  $\lambda_h$  of problem (12)–(14) belongs, such that*

$$\Phi'(\alpha) \geq \sigma > 0, \quad \alpha \in [\lambda_1, \lambda_2]. \tag{21}$$

### 3 Iterative methods

Lemma 1 and Lemma 2 from previous section will be used for construction and investigation of iterative methods for problem (12)–(14).

Rephrasing the statement of Lemma 2, we formulate an idea of the method of solution. Solution of system (12)–(14) can be obtained by finding solution  $\alpha^*$  of equation

$$\Phi(\alpha) = 0$$

in the interval  $[\lambda_1, \lambda_2]$  and with the help of this value  $\lambda = \alpha^*$  solve the system of nonlinear equations (12), (13).

Let us formulate the following iterative method. By solving the system of equations (12), (13) with the concrete value of  $\lambda$ , we find an interval  $[\lambda_1, \lambda_2]$  such that the inequality

$$\Phi(\lambda_1)\Phi(\lambda_2) < 0$$

is true. Afterwards the method of bisection is used.

Both of the operators applying fixed  $\lambda$  to differential problem (1), (2) and to the system of difference equations (12), (13) are considered as monotone. We write equation (1) in a following form:

$$\sum_{i=1}^2 \frac{\partial}{\partial x_i} a_i(p_1, p_2) = f(u), \tag{22}$$

where  $p_i = \partial u / \partial x_i$ . By immediate verification we get that the following inequalities are true:

$$\mu(T^2)^3 \sum_{i=1}^2 \xi_i^2 \leq \sum_{i,j=1}^2 \frac{\partial a_i}{\partial p_j} \xi_i \xi_j \leq \mu(T^2) \sum_{i=1}^2 \xi_i^2, \tag{23}$$

since  $\mu(T^2) \geq 0$ , but  $\mu(T^2) \rightarrow 0$  as  $T \rightarrow \infty$ , equation (22) is elliptic, but not uniformly elliptic. In other words, if we write the problem (22), (2) as operator equation

$$L(u) = 0, \tag{24}$$

then from condition (23) and the condition  $\partial f/\partial u \geq 0$  it follows that operator  $L$  is a monotone operator, but not a strongly monotone one. The operator for the system of difference equations has the same feature.

Iterative methods for the systems of elliptic type difference equations with a strongly monotone operator are well investigated in [33].

One of the results on the convergence of iterative methods is provided in following lemma.

**Lemma 3.** *If the Gâteaux derivative  $A'$  of operator  $A$  for the system of difference equations  $A(u) = f$  is a self-adjoint operator, and the following inequalities:*

$$\gamma_1(B(u-v), u-v) \leq (A(u) - A(v), u-v) \leq \gamma_2(B(u-v), u-v), \quad (25)$$

are true,  $\gamma_1 > 0$ ,  $B$  is a positive definite operator, then the iterative method

$$Bu^{n+1} = Bu^n - \tau(A(u^n) - f) \quad (26)$$

converges in the norm  $\|u\|_B = (Bu, u)^{1/2}$  as  $0 < \tau < 2/\gamma_2$ .

Operator  $B$  for elliptic type difference equations is chosen as a five-point difference equivalent for the Laplace operator.

For the system of equations (1), (2), as  $\lambda$  is fixed and  $\partial H/\partial u \geq 0$ , all the conditions of Lemma considered are met, except one ( $\gamma_1 > 0$ ).

For the system of difference equations (12), (13) we get

$$\gamma_1 = \mu(T_{ij}^2)^3 = (1 + T_{ij}^2)^{-3/2},$$

where  $T_{ij}^2 = (\delta_x u_{ij})^2 + (\delta_y u_{ij})^2$ . From a priori estimation (19) it follows the condition

$$\gamma_1 \geq (1 + M_2^2)^{-3/2} > 0 \quad (27)$$

for the system of equations (12), (13). But we cannot guarantee that a priori estimation (19) will hold with the same constant  $M_2$  in every step of iterative method (26). So, the iterative method may not converge.

Thus if differential equation (1) is a type of equation for a minimal surface, iterative method (26) is modified in the following way. Idea of modification is described in the paper [31], where operators with unbounded nonlinearity are investigated.

Assume that a priori estimation (19), i.e.  $T_{ij}^2 \leq M_2$ , is true for the system of equations (12), (13).

Let us define a new function  $\tilde{\mu}(T_{ij}^2)$

$$\tilde{\mu}(T_{ij}^2) = \begin{cases} \mu(T_{ij}^2) & \text{if } T_{ij}^2 \leq M_2, \\ \mu(M_2)^2 & \text{if } T_{ij}^2 > M_2. \end{cases} \quad (28)$$

Instead of problem (12), (13) let us consider the following problem:

$$\delta_x(\tilde{\mu}(T_{ij}^2)\delta_x u_{ij}) + \delta_y(\tilde{\mu}(T_{ij}^2)\delta_y u_{ij}) - H_{ij} + \lambda_h = 0, \quad (i, j) \in \Omega_h, \quad (29)$$

$$u_{ij} = \varphi_{ij}, \quad (i, j) \in \Gamma_h, \quad (30)$$

or, in short,

$$\tilde{A}(u_{ij}) = f. \tag{31}$$

Let us write iterative method for solution of this system of equations analogically as for an iterative method (26)

$$Bu^{n+1} = Bu^n - \tau(\tilde{A}(u^n) - f). \tag{32}$$

**Theorem 1.** *If there exists unique solution of system (12), (13) with the a priori estimation  $T_{ij} \leq M_2$ , then the solution of system (29), (30) also exists, it is unique and coincident with the solution of system (12), (13). Iterative method (32), as  $0 < \tau < 2/\gamma_2$ , converges to the solution of system (12), (13).*

*Proof.* The operator  $\tilde{A}$  of system (29), (30), taking into account the definition (28) of  $\tilde{\mu}(T_{ij}^2)$ , is continuous and the inequalities (25) for this operator are true with the constant  $\gamma_1 > 0$ , defined by formula (27). So, the operator  $\tilde{A}$  is strongly monotone in the vector space  $R_N$ , where  $N$  is the number of points in the domain  $\Omega_h$ . Consequently, the system of equations (29), (30) possesses unique solution in the space  $R_N$ .

Further, if the solution of problem (12), (13) exists in the same vector space  $R_N$  with the a priori estimation  $T_{ij}^2 \leq M_2$ , then it is also the solution of problem (29), (30). Thus, after solving the system of equations (29), (30), we also have the solution of the system (12), (13).

Since inequalities (25) are true for the operator  $\tilde{A}$  with  $\gamma_1 > 0$ , iterative method (32) converges as all the conditions of Lemma 3 are fulfilled.

The theorem is proven. □

Summarizing the previous statements we form an algorithm for solution of the system of nonlinear equations with integral condition (12)-(14) as an iterative method:

1. *Internal iteration.* We solve the system of equations (12) with  $\lambda_h^{n+1}$  and boundary condition (13) by iterative method (32):

$$\Delta u_{ij}^{k+1} = \Delta u_{ij}^k - \tau \{ \delta_x (\tilde{\mu}(T_{ij}^2)^k \delta_x u_{ij}^k) + \delta_y (\tilde{\mu}(T_{ij}^2)^k \delta_y u_{ij}^k) - H_{ij}^k + \lambda_h^{n+1} \}, \quad (i, j) \in \Omega_h, \tag{33}$$

$$u_{ij}^{k+1} = \varphi_{ij}, \quad (i, j) \in \Gamma_h, \tag{34}$$

where

$$\begin{aligned} \Delta u_{ij}^k &= \delta_x \delta_{\bar{x}} u_{ij}^k + \delta_y \delta_{\bar{y}} u_{ij}^k, \\ \tilde{\mu}(T_{ij}^2)^k &= \begin{cases} \{1 + (\delta_{\bar{x}} u_{ij}^k)^2 + (\delta_{\bar{y}} u_{ij}^k)^2\}^{-1/2} & \text{if } (T_{ij}^2)^k \leq M_2, \\ \{1 + M_2\}^{-1/2} & \text{if } (T_{ij}^2)^k > M_2, \end{cases} \\ H_{ij}^k &= H(x_i, y_j, u_{ij}^k). \end{aligned}$$

2. *External iteration.* Having found  $u_{ij}^{n+1} = \lim_{k \rightarrow \infty} u_{ij}^k$ , we calculate

$$\Phi(\lambda_h^{n+1}) = h^2 \sum_{(i,j) \in \bar{\Omega}_h} \rho_{ij} u_{ij}^{n+1} - V \quad (35)$$

and verify the condition

$$\Phi(\lambda_h^{n+1})\Phi(\lambda_h^n) < 0. \quad (36)$$

Aim of external iteration is to find the interval  $(\lambda_h^n, \lambda_h^{n+1})$ , in which condition (36) is true, and afterwards dividing the interval into two parts by the method of bisection, we search for the solution  $u_{ij}^{n+1}$  until it will satisfy the desirable accuracy

$$\max_{(ij)} \frac{|u_{ij}^{n+1} - u_{ij}^n|}{|u_{ij}^{n+1}|} \leq \varepsilon. \quad (37)$$

When function  $H(x, y, u)$  is defined as in equation (11) and the domain  $\Omega$  is supposed to be the circle of radius  $R$ , then exact value of  $\lambda$  of problem (1)–(3) belongs to interval  $[-(\gamma R/\sigma) \sin \alpha, (\gamma R/\sigma) \sin \alpha]$ . So,  $\lambda_h^0$  could be chosen from this particular interval.

## 4 Other methods of constructing systems of difference equations

The iterative method for solution of the system of difference equations (12)–(14), defined by formulas (33)–(36) may be applied when instead of the problem (12) we take another approximation of the equation (1).

### 4.1 The method of finite elements

The method of finite elements for investigation of the equation of a minimal surface is theoretically considered in the paper [6]. It is one of the first papers validating the finite elements methods for nonlinear nonuniform elliptic equations. On the base of this research, the method of finite elements [31, 32] is investigated for problem (1)–(3). It is proven that operator  $A$  of the system of equations, gained by the method of finite elements satisfies conditions (25). The a priori estimation is valid for the solution of this system analogically as of (18), and the derivative of operator  $A$  is a self-adjoint operator. So, Lemma 3 is valide for the system of equations and it is possible to solve the system mentioned by the iterative method described in Section 3.

### 4.2 Differential equation in a system of polar coordinates

When the domain  $\Omega$  is the circle of radius  $R$ , it is reasonable to write equation (1) in the system of polar coordinates. The system of difference equations thereby was theoretically investigated in [32], and a priori estimation (19) is valid for the solution of the problem. So, it is possible to apply the iterative method described in Section 3.

The advantage of this method of approximation in comparison of the approximation of (12), (13) is more precise accuracy [32].

### 4.3 Increased accuracy difference approximation

If the domain  $\Omega$  is not a circle, an increased accuracy approximation of equation (1) might be obtained by approximating the derivatives on the non-uniform grid. Let us use the same seven-point pattern (Fig. 1), if all the points belong to the domain  $\bar{\Omega} = \Omega \cup \Gamma$ . But the difference equation we take is a different one:

$$\delta_x(\mu(T_{i-1/2,j}^2)\delta_x u_{ij}) + \delta_y(\mu(T_{i,j-1/2}^2)\delta_y u_{ij}) - H_{ij} + \lambda_h = 0, \quad (38)$$

where

$$T_{i-1/2,j}^2 = \left(\frac{u_{ij} - u_{i-1,j}}{h}\right)^2 + \left(\frac{u_{i-1,j+1} - u_{i-1,j}}{2h} + \frac{u_{ij} - u_{i,j-1}}{2h}\right)^2, \quad (39)$$

$$T_{i,j-1/2}^2 = \left(\frac{u_{i+1,j-1} - u_{i,j-1}}{2h} + \frac{u_{ij} - u_{i-1,j}}{2h}\right)^2 + \left(\frac{u_{ij} - u_{i,j-1}}{h}\right)^2. \quad (40)$$

Difference equation (38) approximates differential equation (1) in the point  $(i, j)$  with the accuracy  $O(h^2)$ .

If at least one of the points of the pattern is outside the limits of the domain  $\bar{\Omega}$ , equation (1) is approximated on the irregular grid, depending on the irregular pattern chosen. A typical case is showed in Fig. 2. For this case, the difference equation is taken as follows:

$$\begin{aligned} & h^{-2} \{ \mu(\tilde{T}_{i-1/2,j}^2) \tilde{u}_{i-1,j} - (\mu(\tilde{T}_{i-1/2,j}^2) + \mu(\tilde{T}_{i+1/2,j}^2)) u_{ij} + \mu(\tilde{T}_{i+1/2,j}^2) u_{i+1,j} \} \\ & + h^{-2} \{ \mu(\tilde{T}_{i,j-1/2}^2) u_{i,j-1} - (\mu(\tilde{T}_{i,j-1/2}^2) + \mu(\tilde{T}_{i,j+1/2}^2)) u_{ij} + \mu(\tilde{T}_{i,j+1/2}^2) \tilde{u}_{i,j+1} \} \\ & - H_{ij} + \lambda_h = 0, \end{aligned} \quad (41)$$

where the expressions  $\tilde{T}_{i-1/2,j}^2, \tilde{T}_{i+1/2,j}^2, \tilde{T}_{i,j-1/2}^2, \tilde{T}_{i,j+1/2}^2$  are defined by replacing the derivatives  $\partial u / \partial x, \partial u / \partial y$  by the differences on the non-uniform grid, depending on the irregular pattern.

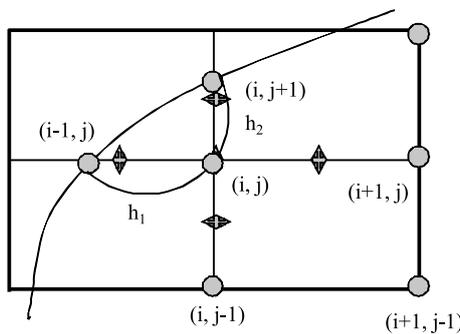


Fig. 2. Irregular pattern of  $\Omega_h$ .

For example,

$$\tilde{T}_{i-1/2,j}^2 = \left( \frac{u_{ij} - \tilde{u}_{i-1,j}}{h_1} \right)^2 + \left( \frac{u_{ij} - u_{i,j+1}}{h} \right)^2, \quad (42)$$

$$\tilde{T}_{i+1/2,j}^2 = \left( \frac{u_{i+1,j} - u_{ij}}{h} \right)^2 + \left( \frac{\tilde{u}_{i,j+1} - u_{i,j}}{2h_2} + \frac{u_{i+1,j} - u_{i+1,j-1}}{2h} \right)^2, \quad (43)$$

etc. In the general case, the approximation error of equation (41) is  $O(h)$ .

Let us define the norm

$$\|u\| = \left( h^2 \sum_{(i,j) \in \Omega_h} u_{ij}^2 \right)^{1/2}.$$

The approximation error  $R(u)$  of equations (38) and (41) in this norm is

$$\|R(u)\| = O(h^{3/2}).$$

The accuracy of approximation  $O(h^{1+\delta})$ ,  $\delta > 0$  guarantees, that there exists an a priori estimation for the solution of this system [31], analogically as for (19). Only the difference is that the derivative of the operator in this case is non self-adjoint operator. In this case, the convergence of iterative method (32) is proved by a more complicated technique as shown in the proof of Lemma 3 (see [31]).

## 5 One-stage iterative method

The typical feature of a two-stage iterative method, described in Section 3, with the external and internal iteration is that the nonlocal condition (14) is not fulfilled on every step of iteration.

From the practical point of view it seemed rather rational to modify the iterative method, so that at every step of iteration, the approximation  $u_{ij}^n$  would exactly satisfy not only boundary condition (13), but also nonlocal condition (14).

We consider problem (12)–(14), in short

$$A(u) + \lambda_h = 0, \quad (44)$$

$$u|_{\Gamma} = \varphi, \quad (45)$$

$$l(u) = V. \quad (46)$$

We construct an iterative method

$$Bu^{n+1} = Bu^n - \tau(A(u^n) + \lambda_h^{n+1}), \quad (47)$$

$$u^{n+1}|_{\Gamma} = \varphi, \quad (48)$$

$$l(u^{n+1}) = V, \quad (49)$$

where operator  $B$  is defined as in (33).

The main difference from the method proposed in Section 3 is that in the previously proposed iterative method (33) for one concrete value  $\lambda_h^{n+1}$ , the iterative method was aimed to obtain the value  $u^{n+1}$ . For the method proposed, only one iteration with the value  $\lambda_h^{n+1}$  is realised based on the same method. In other words, the system of nonlinear difference equations with the unknown parameter in the equation and the integral condition is solved by the iterative method, at each step of which the system of linear equations with the unknown parameter and the integral condition should be solved.

Now we propose the algorithm, based on the idea of superposition of the linear problem solution for the realization of one step of iteration. I.e. how to obtain  $u^{n+1}$  and  $\lambda_h^{n+1}$ , when we know  $u^n$ .

We solve two linear problems (without the integral condition)

$$\begin{aligned} Bv &= -\tau, \\ v|_{\Gamma} &= 0 \end{aligned} \quad (50)$$

and

$$\begin{aligned} Bw^{n+1} &= Bu^n - \tau A(u^n), \\ w^{n+1}|_{\Gamma} &= \varphi. \end{aligned} \quad (51)$$

Then we choose  $\lambda_h^{n+1}$ , so that the function

$$u^{n+1} = \lambda_h^{n+1}v + w^{n+1} \quad (52)$$

would be the solution of the problem (47)–(48).

Putting solution (52) into boundary condition (48), we get the identity.

Substituting the same expression to equation (47), we get the identity as well. At the same time, from expression (49) we derive

$$\lambda_h^{n+1} = \frac{V - l(w^{n+1})}{l(v)}. \quad (53)$$

Note that from the maximum principle for problem (50) it follows that  $v_{ij} \geq 0$  for all  $(i, j)$ . Besides, as  $\tau > 0$ ,  $v_{ij} \neq 0$ . So,  $l(v) > 0$ , and division by  $l(v)$  is always possible.

The convergence of method (47)–(49) remains open. However, if the method converges, then as the limit we always obtain the solution to problem (44)–(46).

## 6 Numerical example

In order to verify the efficiency of the iterative method described in Section 5, we carried out the numerical experiment by calculating the shape of the drop of mercury. The calculation was performed with the real data, matching the drop of mercury as a part of contact in a precision vibro-mechanics [1]:  $K = g\rho/\sigma$ ,  $g = 9.8 \text{ m/s}^2$ ,  $\rho = 13.55 \cdot 10^3 \text{ kg/m}^3$ ,  $\sigma = 0.462 \text{ N/m}$ ,  $a = 3.5 \cdot 10^{-10} \text{ m}$ . Practical calculations confirmed the effectiveness of iterative method (47)–(49).

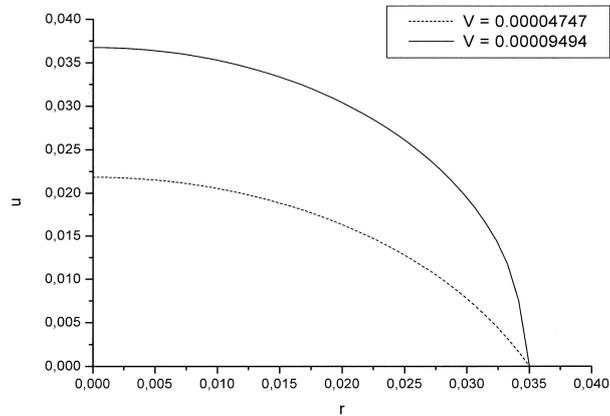


Fig. 3. The form of the ridge of the drop for two values of  $V$ .

Potential energy was taken much smaller than the energy of surface tension for concrete data, the drop practically is symmetrical (Fig. 3). Equation (1) then may be simplified as an equation of the drop lying on a horizontal plane:

$$\frac{1}{r} \frac{d}{dr} \left( \frac{r}{\sqrt{1 + \left(\frac{du}{dr}\right)^2}} \frac{du}{dr} \right) = Ku + \lambda = 0, \quad 0 < r < a, \quad (54)$$

$$u'(0) = 0, \quad u(a) = 0, \quad (55)$$

$$2\pi \int_0^a ru \, dr = V. \quad (56)$$

Note that calculating by iterative method (47)–(49), number of iterations depends not only on the accuracy chosen, but also on order of the system of equations, i.e. on the step  $h$ . The number of iterations of the internal iterative method (33), (34) of a two-stage method does not depend on  $h$ , however the general number of iterations is always rather high, because of two stages of the method.

## References

1. O. Frei, B. Rasch, *Finding Form: Towards an Architecture of the Minimal*, Axel Menges, Stuttgart, 1995.
2. S. Hyde, S. Andersson, K. Larsson, Z. Blum, T. Landh, S. Lidin, B.W. Ninham, *The Language of Shape. The Role of Curvature in Condensed Matter: Physics, Chemistry and Biology*, Elsevier, Amsterdam, 1997.
3. G. Shapiro, *Geometric Partial Differential Equations and Image Analysis*, Cambridge Univ. Press, New York, 2001.

4. P.-G. De Gennes, F. Brochard-Wyart, D. Quere, *Capillarity and Wetting Phenomena: Drops, Bubbles, Pearls, Waves*, Springer-Verlag, New York, 2004.
5. P. Concus, Numerical solution of the minimal surface equation, *Math. Comput.*, **21**(99):340–350, 1967.
6. C. Johnsen, V. Thomee, Error estimation for a finite element approximation of a minimal surface, *Math. Comput.*, **29**(130):343–349, 1975.
7. T. Cecil, *A numerical method for computing minimal surfaces in arbitrary dimension*, *J. Comput. Phys.* **206**:650–660, 2005.
8. G. Dziuk, J.E. Hutchinson, Finite element approximations to surfaces of prescribed variable mean curvature, *Numer. Math.*, **102**(4):611–648, 2006.
9. H. Harbrecht, On the numerical solution of Plateau's problem, *Appl. Numer. Math.*, **59**:2785–2800, 2009.
10. Q. Pan, G. Xu, Construction of minimal subdivision surface with a given boundary, *Comput. Aided Design*, **43**:374–380, 2011.
11. Ø. Trasdahl, M. Renquist, High order numerical approximation of minimal surfaces, *J. Comput. Phys.*, **230**:4795–4810, 2011.
12. J.R. Cannon, Y. Lin, A. Matheson, The solution of the diffusion equation in two spaces variables subject to the specification of mass, *Appl. Anal.*, **50**:1–15, 1993.
13. M. Dehghan, An inverse problem of finding a source parameter in a semilinear parabolic equation, *Appl. Math. Modelling*, **25**:743–754, 2001.
14. Y. Wang, Solutions to nonlinear elliptic equations with a nonlocal condition, *Electron. J. Differ. Equ.*, **2002**(5):1–16, 2002.
15. N. Gordeziani, P. Natalini, P.E. Ricci, Finite-difference methods for solution of nonlocal boundary value problems, *Int. J. Comput. Math. Appl.*, **50**:1333–1344, 2005.
16. M.P. Sapagovas, A difference method of increased order of accuracy for the Poisson equation with nonlocal conditions, *Differ. Equ.*, **44**(7):1018–1028, 2008.
17. J. Martin-Vaquero, J. Vigo-Agiuar, On the numerical solution of the heat conduction equations subject to nonlocal conditions, *Appl. Numer. Math.*, **59**(10):2507–2514, 2009.
18. F. Ivanauskas, T. Meškauskas, M. Sapagovas, Stability of difference schemes for two-dimensional parabolic equations with non-local boundary conditions, *Appl. Math. Comput.*, **215**(7):2716–2732, 2009.
19. J. Jachimavičienė, Ž. Jesevičiūtė, M. Sapagovas, The stability of finite-difference schemes for a pseudoparabolic equation with nonlocal conditions, *Numer. Funct. Anal. Optim.*, **30**(9–10):988–1001, 2009.
20. A. Ashyralyev, O. Yildirim, A note on the second order of accuracy stable difference schemes for the nonlocal boundary value hyperbolic problem, *Abstr. Appl. Anal.*, Article ID 846582, 29 pp., 2012.

21. O. Gercek, A. Ashyralyev, On a difference scheme of the second order of accuracy for elliptic-parabolic equations, *Bound. Value Probl.*, **2012**(80), 13 pp., 2012.
22. J. Martin-Vaquero, B.A. Wade, On efficient numerical methods for an initial- boundary value problem with nonlocal boundary conditions, *Appl. Math. Modelling*, **36**(8):3411–3418, 2012.
23. M. Sapagovas, K. Jakubėlienė, Alternating direction method for two-dimensional parabolic equation with nonlocal integral condition, *Nonlinear Anal. Model. Control*, **17**(1):91–98, 2012.
24. M. Sapagovas, O. Štikonienė, A fourth-order alternating-direction method for difference schemes with nonlocal condition, *Lith. Math. J.*, **49**(3):309–317, 2009.
25. M. Sapagovas, A. Štikonas, O. Štikonienė, Alternating direction method for the Poisson equation with variable weight coefficient in an integral condition, *Differ. Equ.*, **47**(8):1176–1187, 2011.
26. M. Sapagovas, O. Štikonienė, Alternating-direction method for a mildly nonlinear elliptic equation with nonlocal integral conditions, *Nonlinear Anal. Model. Control*, **16**(2):220–230, 2011.
27. M.P. Sapagovas, A.D. Štikonas, On the structure of the spectrum of a differential operator with a nonlocal condition, *Differ. Equ.*, **41**(7):961–969, 2005.
28. A. Skučaitė, K. Skučaitė-Bingelė, S. Pečiulytė, A. Štikonas, Investigation of the spectrum for the Sturm-Liouville problem with one integral boundary condition, *Nonlinear Anal. Model. Control*, **15**(4):501–512, 2010.
29. K. Ragulskis, M. Sapagovas, R. Čiupaila, A. Jurkulnevičius, Numerical experiment in stationary problems of liquid metal contact, *Vibrotechnika*, **4**(57):105–111, 1986 (in Russian).
30. M.P. Sapagovas, Investigation of nonclassical drop equation, *Differ. Equ.*, **19**(7):1271–1276, 1983.
31. M. Sapagovas, Numerical methods for the solution of the equation of a surface with prescribed mean curvature, *Lith. Math. J.*, **23**(3):321–326, 1983.
32. M.P. Sapagovas, Numerical methods for the two-dimensional problem with a nonlocal condition, *Differ. Equ.*, **20**(7):926–933, 1984.
33. A.A. Samarskii, E.S. Nikolaev, *The Methods for Solution of Difference Equations*, Nauka, Moscow, 1978 (in Russian).