Existence and uniqueness of solutions for a singular system of higher-order nonlinear fractional differential equations with integral boundary conditions*

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Abstract. In this paper, we study the existence and uniqueness of solutions for a singular system of nonlinear fractional differential equations with integral boundary conditions. We obtain existence and uniqueness results of solutions by using the properties of the Green's function, a nonlinear alternative of Leray–Schauder-type, Guo–Krasnoselskii's fixed point theorem in a cone and the Banach fixed point theorem. Some examples are included to show the applicability of our results.

Keywords: fractional differential equation, singular system, fractional Green's function, fixed point theorem.

1 Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various fields of sciences and engineering such as control, porous media, electromagnetic, and other fields. There are many papers deal with the existence and multiplicity of solution of nonlinear fractional differential equations (see [1–8] and the references therein).

The paper [1] considered the existence of positive solutions of singular coupled system

$$\begin{aligned} D^s u &= f(t,v), \quad 0<t<1,\\ D^p v &= g(t,u), \quad 0<t<1, \end{aligned}$$

where $0 < s, p < 1$, and $f, g : (0,1] \times [0,+\infty) \to [0,+\infty)$ are two given continuous functions, $\lim_{t\to 0+} f(t,\cdot) = +\infty, \lim_{t\to 0+} g(t,\cdot) = +\infty$ and D^s, D^p are the standard fractional Riemann–Liouville's derivatives. The existence results of positive solution

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are obtained by a nonlinear alternative of Leray–Schauder-type and Guo–Krasnoselskii's fixed point theorem in a cone.

The paper [3] considered the existence of positive solutions of singular coupled system

$$\begin{aligned} &D_{0+}^{\alpha}u(t) + f\bigl(t, v(t)\bigr) = 0, \quad 0 < t < 1, \\ &D_{0+}^{\beta}v(t) + g\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = u(1) = v(0) = v'(0) = v(1) = 0, \end{aligned}$$

where $2 < \alpha, \beta \leqslant 3$ and $f, g : (0, 1] \times [0, +\infty) \to [0, +\infty)$ are two given continuous functions, and $\lim_{t\to 0+} f(t, \cdot) = +\infty$, $\lim_{t\to 0+} g(t, \cdot) = +\infty$ and $D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the standard fractional Riemann–Liouville's derivatives. The two sufficient conditions for the existence of solutions are obtained by a nonlinear alternative of Leray–Schauder-type and Guo–Krasnoselskii's fixed point theorem in a cone.

Goodrich [5] considered the existence of a positive solution to systems of differential equations of fractional order

$$\begin{aligned} -D_{0+}^{v_1}y_1(t) &= \lambda_1 a_1(t) f\bigl(y_1(t), y_2(t)\bigr), \\ -D_{0+}^{v_2}y_2(t) &= \lambda_2 a_2(t) g\bigl(y_1(t), y_2(t)\bigr), \end{aligned}$$

subject either to the boundary conditions

$$\begin{aligned} &y_1^{(i)}(0) = 0 = y_2^{(i)}(0), \quad 0 \leqslant i \leqslant n-2, \\ &\bigl[D_{0+}^{k}y_1(t)\bigr]_{t=1} = 0 = \bigl[D_{0+}^{k}y_2(t)\bigr]_{t=1}, \quad 1 \leqslant \alpha \leqslant n-2, \end{aligned}$$

or

$$\begin{aligned} &y_1^{(i)}(0) = 0 = y_2^{(i)}(0), \quad 0 \leqslant i \leqslant n-2, \\ &\bigl[D_{0+}^{k}y_1(t)\bigr]_{t=1} = \phi_1(y), \quad \bigl[D_{0+}^{k}y_2(t)\bigr]_{t=1} = \phi_2(y), \quad 1 \leqslant \alpha \leqslant n-2, \end{aligned}$$

where $v_1, v_2 \in (n-1, n]$ for $n > 3$ and $n \in N$, the continuous functionals $\phi_1, \phi_2 : C([0, 1]) \to R$ represent nonlocal boundary conditions.

The papers [9–12] considered the existence of positive solutions of so-called $(k, n-k)$ or $(p, n-p)$ conjugate boundary value problems (BVP). For example, in [9], the authors discussed the following $(p, n-p)$ conjugate singular boundary value problem:

$$\begin{aligned} &(-1)^{(n-P)}y^{(n)} = \phi(t) f(t, y), \quad 0 < t < 1, \\ &y^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant p-1, \\ &y^{(j)}(1) = 0, \quad 0 \leqslant j \leqslant n-p-1, \end{aligned}$$

where $n \geqslant 2$, and the nonlinearity f may be singular at $y = 0$. The existence results of positive solution about singular higher order boundary value problems are established.

Inspired by the work of the above papers and many known results, in this paper, we study the existence of positive solutions of BVP (1). The existence and uniqueness results of solutions are obtained by a nonlinear alternative of Leray–Schauder-type, Guo–Krasnoselskii's fixed point theorem in a cone and the Banach fixed point theorem. We consider the singular system of nonlinear fractional differential equations with integral boundary conditions

$$\begin{aligned} &D_{0+}^{\alpha}u(t)+f\big(t,v(t)\big)=0, \quad 0<t<1,\\ &D_{0+}^{\beta}v(t)+g\big(t,u(t)\big)=0, \quad 0<t<1,\\ &u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \qquad u^{(i)}(1)=\lambda\int_0^{\eta}u(s)\,\mathrm{d}s,\\ &v(0)=v'(0)=\cdots=v^{(n-2)}(0)=0, \qquad v^{(j)}(1)=b\int_0^{c}v(s)\,\mathrm{d}s, \end{aligned} \tag{1}$$

where $n-1<\alpha,\beta\leqslant n$, $n\geqslant 3$, $0<\eta,c\leqslant 1$, $i,j\in N$, $0\leqslant i,j\leqslant n-2$ and i, j are fixed constants, $\varDelta_1-\lambda\eta^{\alpha}/\alpha>0$, $\varDelta_2-bc^{\beta}/\beta>0$,

$$\varDelta_1=\begin{cases}1, & i=0,\\ (\alpha-1)(\alpha-2)\cdots(\alpha-i), & i\geqslant 1,\end{cases}$$

$$\varDelta_2=\begin{cases}1, & j=0,\\ (\beta-1)(\beta-2)\cdots(\beta-j), & j\geqslant 1.\end{cases}$$

$f,g:(0,1]\times[0,+\infty)\to[0,+\infty)$ are two given continuous functions and singular at $t=0$ (that is, $\lim_{t\to 0+}f(t,\cdot)=+\infty, \lim_{t\to 0+}g(t,\cdot)=+\infty$), and $D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the standard fractional Riemann–Liouville's derivatives.

The paper is organized as follows. Firstly, we present some necessary definition and preliminaries, and derive the corresponding Green's function known as fractional Green's function and argue its properties. Secondly, the existence results of positive solutions are obtained by a nonlinear alternative of Leray–Schauder-type, Guo–Krasnoselskii's fixed point theorem in a cone and the Banach fixed point theorem. Finally, we construct some examples to demonstrate the application of our main results.

2 Background materials and Green's function

For the convenience of the reader, we present here the necessary definitions, lemmas and theorems from fractional calculus theory to facilitate analysis of BVP (1). These definitions, lemmas and theorems can be found in the recent literature, see [1–8].

Definition 1. The Riemann–Liouville fractional integral of order $\alpha>0$ of a function $y:(0,\infty)\to R$ is given by

$$I_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,\mathrm{d}s$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2. The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0,\infty) \to R$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n} \int\limits_{0}^{t} \frac{y(s)}{(t-s)^{\alpha-n+1}} \,\mathrm{d}s,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0,\infty)$.

From the definition of the Riemann–Liouville derivative, we can obtain the statement.

Lemma 1. (*See* [7].) *Let* $\alpha > 0$. *If we assume* $u \in C(0,1) \cap L^1(0,1)$, *then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0$$

has $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$, $C_i \in R$, $i = 1,2,\dots,N$, *as unique solutions, where* N *is the smallest integer greater than or equal to* α.

Lemma 2. (*See* [7].) *Assume that* $u \in C(0,1) \cap L^1(0,1)$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C(0,1) \cap L(0,1)$. *Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$$

for some $C_i \in R$, $i = 1,2,\dots,N$, *where* N *is the smallest integer greater than or equal to* α.

Remark 1. (See [6].) The following properties are useful for our discussion:

$$I_{0+}^{\alpha} I_{0+}^{\beta} f(t) = I^{\alpha+\beta} f(t), \quad D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t), \quad \alpha,\beta > 0.$$

In the following, we present Green's function of the fractional differential equation boundary value problem.

Lemma 3. *Given* $y \in C[0,1]$, *the problem*

$$\begin{aligned} &D_{0+}^{\alpha} u(t) + y(t) = 0, \\ &u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ &u^{(i)}(1) = \lambda \int\limits_{0}^{\eta} u(s) \,\mathrm{d}s, \end{aligned} \tag{2}$$

where $\alpha > 2$, $n-1 < \alpha \leqslant n$, $0 < t < 1$, $\eta \in (0,1]$, $i \in N$, $0 \leqslant i \leqslant n-2$ *and* i *is a fixed constant,* $\Delta_1 - (\lambda/\alpha)\eta^{\alpha} > 0$,

$$\Delta_1 = \begin{cases} 1, & i = 0, \\ (\alpha-1)(\alpha-2)\cdots(\alpha-i), & i \geqslant 1, \end{cases}$$

$i \in N$ and i is a fixed constant, is equivalent to

$$u(t) = \int_0^1 G_1(t,s) y(s)\,\mathrm{d}s,$$

$$G_1(t,s) = \begin{cases} \frac{\Delta_1 t^{\alpha-1}(1-s)^{\alpha-1-i} - \frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1} - (\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)(t-s)^{\alpha-1}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}, & 0 \leqslant s \leqslant t \leqslant 1,\ s \leqslant \eta, \\ \frac{\Delta_1 t^{\alpha-1}(1-s)^{\alpha-1-i} - (\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)(t-s)^{\alpha-1}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}, & 0 \leqslant \eta \leqslant s \leqslant t \leqslant 1, \\ \frac{\Delta_1 t^{\alpha-1}(1-s)^{\alpha-1-i} - \frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}, & 0 \leqslant t \leqslant s \leqslant \eta \leqslant 1, \\ \frac{\Delta_1 t^{\alpha-1}(1-s)^{\alpha-1-i}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}, & 0 \leqslant t \leqslant s \leqslant 1,\ \eta \leqslant s. \end{cases} \tag{3}$$

Here, $G_1(t,s)$ is called the Green's function of BVP (2). *Obviously, $G_1(t,s)$ is continuous on* $[0,1] \times [0,1]$.

Proof. We may apply Lemma 2 to reduce (2) to an equivalent integral equation

$$u(t) = -I_{0+}^\alpha y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$$

for some $C_1, C_2, \dots, C_n \in R$. Consequently, the general solution of (2) is

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,\mathrm{d}s + C_1 t^{\alpha-1} C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}.$$

By $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, one gets that $C_2 = C_3 = \cdots = C_n = 0$. Then we have

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s + C_1 t^{\alpha-1},$$

$$u^{(i)}(t) = -\int_0^t \frac{\Delta_1 (t-s)^{\alpha-1-i}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s + \Delta_1 C_1 t^{\alpha-1-i}.$$

On the other hand, $u^{(i)}(1) = \lambda \int_0^\eta u(s)\,\mathrm{d}s$ combining with

$$u^{(i)}(1) = -\int_0^1 \frac{\Delta_1 (1-s)^{\alpha-1-i}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s + \Delta_1 C_1,$$

$$\int_0^\eta u(s)\,\mathrm{d}s = -\frac{1}{\Gamma(\alpha)}\int_0^\eta\int_0^x (x-s)^{\alpha-1}y(s)\,\mathrm{d}s\,\mathrm{d}x + C_1\int_0^\eta s^{\alpha-1}\,\mathrm{d}s$$
$$= -\frac{1}{\Gamma(\alpha)}\int_0^\eta\int_s^\eta (x-s)^{\alpha-1}y(s)\,\mathrm{d}x\,\mathrm{d}s + C_1\int_0^\eta s^{\alpha-1}\,\mathrm{d}s$$
$$= -\frac{1}{\Gamma(\alpha)}\int_0^\eta \frac{(\eta-s)^\alpha}{\alpha}y(s)\,\mathrm{d}s + \frac{C_1\eta^\alpha}{\alpha},$$

yields

$$C_1 = \int_0^1 \frac{\Delta_1(1-s)^{\alpha-1-i}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}y(s)\,\mathrm{d}s - \int_0^\eta \frac{\lambda(\eta-s)^\alpha}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\alpha\Gamma(\alpha)}y(s)\,\mathrm{d}s.$$

Therefore, the unique solution of the problem (2) is

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,\mathrm{d}s + \int_0^1 \frac{\Delta_1(1-s)^{\alpha-1-i}t^{\alpha-1}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}y(s)\,\mathrm{d}s$$
$$- \int_0^\eta \frac{\lambda(\eta-s)^\alpha t^{\alpha-1}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\alpha\Gamma(\alpha)}y(s)\,\mathrm{d}s.$$

For $t \leqslant \eta$, one has

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,\mathrm{d}s + \left(\int_0^t + \int_t^\eta + \int_\eta^1\right)\frac{\Delta_1(1-s)^{\alpha-1-i}t^{\alpha-1}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}y(s)\,\mathrm{d}s$$
$$- \left(\int_0^t + \int_t^\eta\right)\frac{\lambda(\eta-s)^\alpha t^{\alpha-1}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\alpha\Gamma(\alpha)}y(s)\,\mathrm{d}s$$
$$= \int_0^t \frac{\Delta_1 t^{\alpha-1}(1-s)^{\alpha-1-i} - \frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1} - (\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)(t-s)^{\alpha-1}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}y(s)\,\mathrm{d}s$$
$$+ \int_t^\eta \frac{\Delta_1 t^{\alpha-1}(1-s)^{\alpha-1-i} - \frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}y(s)\,\mathrm{d}s$$
$$+ \int_\eta^1 \frac{\Delta_1 t^{\alpha-1}(1-s)^{\alpha-1-i}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}y(s)\,\mathrm{d}s$$
$$= \int_0^1 G_1(t,s)y(s)\,\mathrm{d}s.$$

For $t \geqslant \eta$, one has

$$\begin{aligned} u(t) &= -\left(\int_0^\eta + \int_\eta^t\right) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s \\ &\quad + \left(\int_0^\eta + \int_\eta^t + \int_t^1\right) \frac{\Delta_1 (1-s)^{\alpha-1-i} t^{\alpha-1}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} y(s)\,\mathrm{d}s \\ &\quad - \int_0^\eta \frac{\lambda(\eta-s)^\alpha t^{\alpha-1}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\alpha\Gamma(\alpha)} y(s)\,\mathrm{d}s \\ &= \int_0^\eta \frac{\Delta_1 t^{\alpha-1}(1-s)^{\alpha-1-i} - \frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1} - (\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)(t-s)^{\alpha-1}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} y(s)\,\mathrm{d}s \\ &\quad + \int_\eta^t \frac{(\alpha-1)t^{\alpha-1}(1-s)^{\alpha-1-i} - (\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)(t-s)^{\alpha-1}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} y(s)\,\mathrm{d}s \\ &\quad + \int_t^1 \frac{\Delta_1 t^{\alpha-1}(1-s)^{\alpha-1-i}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} y(s)\,\mathrm{d}s \\ &= \int_0^1 G_1(t,s) y(s)\,\mathrm{d}s. \end{aligned}$$

The proof is complete. □

Lemma 4. *The function $G(t,s)$ defined by* (3) *satisfies*:

(a1) $G_1(t,s) > 0$ *for all* $t, s \in (0,1)$;

(a2) $(\Delta_1 - \lambda\eta^\alpha/\alpha)\Gamma(\alpha)G_1(t,s) \geqslant (\lambda\eta^\alpha/\alpha)(1-\eta)^i s(1-s)^{\alpha-1-i}t^{\alpha-1}$ *for all* $t, s \in [0,1]$;

(a3) $(\Delta_1 - \lambda\eta^\alpha/\alpha)\Gamma(\alpha)G_1(t,s) \leqslant n(\Delta_1 - \lambda\eta^\alpha/\alpha + \lambda\eta^{\alpha-1}/\alpha)s(1-s)^{\alpha-1-i}$ *for all* $t, s \in [0,1]$.

Proof. For $s \leqslant t$, $s \leqslant \eta$,

$$\begin{aligned} &\left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)\Gamma(\alpha)G_1(t,s) \\ &\quad = \Delta_1 t^{\alpha-1}(1-s)^{\alpha-1-i} - \frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1} - \left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(t-s)^{\alpha-1} \\ &\quad \geqslant \Delta_1 t^{\alpha-1}(1-s)^{\alpha-1} - \frac{\lambda}{\alpha}\eta^\alpha\Big(1-\frac{s}{\eta}\Big)^\alpha t^{\alpha-1} - \left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(t-s)^{\alpha-1} \end{aligned}$$

$$
\begin{aligned}
&\geqslant \varDelta_1 t^{\alpha-1}(1-s)^{\alpha-1} - \frac{\lambda}{\alpha}\eta^\alpha(1-s)^\alpha t^{\alpha-1} - \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(t-s)^{\alpha-1} \\
&= \left[\varDelta_1 - \frac{\lambda}{\alpha}\eta^\alpha(1-s)\right](1-s)^{\alpha-1}t^{\alpha-1} - \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(t-s)^{\alpha-1} \\
&= \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)\left[(1-s)^{\alpha-1}t^{\alpha-1} - (t-s)^{\alpha-1}\right] + \frac{\lambda}{\alpha}\eta^\alpha s(1-s)^{\alpha-1}t^{\alpha-1} \\
&= \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)\left[(1-s)^{\alpha-2}t^{\alpha-2}(1-s)t - (t-s)^{\alpha-2}(t-s)\right] \\
&\quad + \frac{\lambda}{\alpha}\eta^\alpha s(1-s)^{\alpha-1}t^{\alpha-1} \\
&\geqslant \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(1-s)^{\alpha-2}t^{\alpha-2}\left[(1-s)t - (t-s)\right] + \frac{\lambda}{\alpha}\eta^\alpha s(1-s)^{\alpha-1}t^{\alpha-1} \\
&\geqslant \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(1-s)^{\alpha-2}t^{\alpha-2}s(1-t) + \frac{\lambda}{\alpha}\eta^\alpha(1-\eta)^i s(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\geqslant \frac{\lambda}{\alpha}\eta^\alpha(1-\eta)^i s(1-s)^{\alpha-1-i}t^{\alpha-1}.
\end{aligned}
$$

$$
\begin{aligned}
&\left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)\Gamma(\alpha)G_1(t,s) \\
&\quad = \varDelta_1 t^{\alpha-1}(1-s)^{\alpha-1-i} - \frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1} - \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(t-s)^{\alpha-1} \\
&\quad \leqslant \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)\left[(1-s)^{\alpha-1-i}t^{\alpha-1} - (t-s)^{\alpha-1}\right] \\
&\qquad + \frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha_1-1-i}t^{\alpha-1} - \frac{\lambda}{\alpha}(\eta-s)^\alpha t^{\alpha-1} \\
&\quad \leqslant \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)\left[(1-s)^{\alpha-1-i}t^{\alpha-1} - (t-s)^{\alpha-1}(1-s)^{\alpha-1-i}\right] \\
&\qquad + \frac{\lambda}{\alpha}\eta^\alpha t^{\alpha-1}\left[(1-s)^{\alpha-1-i} - \left(1-\frac{s}{\eta}\right)^\alpha(1-s)^{\alpha-1-i}\right] \\
&\quad = \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(1-s)^{\alpha-1-i}t^{\alpha-1}\left[1 - \left(1-\frac{s}{t}\right)^{\alpha-1}\right] \\
&\qquad + \frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha-1-i}t^{\alpha-1}\left[1 - \left(1-\frac{s}{\eta}\right)^\alpha\right] \\
&\quad \leqslant \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(1-s)^{\alpha-1-i}t^{\alpha-1}\left[1 - \left(1-\frac{s}{t}\right)^n\right] \\
&\qquad + \frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha-1-i}t^{\alpha-1}\left[1 - \left(1-\frac{s}{\eta}\right)^n\right] \\
&\quad = \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(1-s)^{\alpha-1-i}t^{\alpha-1}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[1-\left(1-\frac{s}{t}\right)\right]\left[1+\left(1-\frac{s}{t}\right)+\left(1-\frac{s}{t}\right)^2+\cdots+\left(1-\frac{s}{t}\right)^{n-1}\right] \\
&+\frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha-1-i}t^{\alpha-1}\left[1-\left(1-\frac{s}{\eta}\right)\right] \\
&\times \left[1+\left(1-\frac{s}{\eta}\right)+\left(1-\frac{s}{\eta}\right)^2+\cdots+\left(1-\frac{s}{\eta}\right)^{n-1}\right] \\
\leqslant{}& n\left(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha\right)s(1-s)^{\alpha-1-i}t^{\alpha-2}+n\frac{\lambda}{\alpha}\eta^{\alpha-1}s(1-s)^{\alpha-1-i}t^{\alpha-1} \\
\leqslant{}& n\left(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha+\frac{\lambda}{\alpha}\eta^{\alpha-1}\right)s(1-s)^{\alpha-1-i}.
\end{aligned}
$$

For $\eta \leqslant s \leqslant t$,

$$
\begin{aligned}
&\left(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha\right)\Gamma(\alpha)G_1(t,s) \\
&\quad= \Delta_1 t^{\alpha-1}(1-s)^{\alpha-1-i}-\left(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha\right)(t-s)^{\alpha-1} \\
&\quad= \left(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha\right)\left[(1-s)^{\alpha-1-i}t^{\alpha-1}-(t-s)^{\alpha-1}\right]+\frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\quad\geqslant \left(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha\right)\left[(1-s)^{\alpha-1}t^{\alpha-1}-(t-s)^{\alpha-1}\right]+\frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\quad= \left(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha\right)\left[(1-s)^{\alpha-2}t^{\alpha-2}(1-s)t-(t-s)^{\alpha-2}(t-s)\right] \\
&\qquad+\frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\quad\geqslant \left(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha\right)(1-s)^{\alpha-2}t^{\alpha-2}\left[(1-s)t-(t-s)\right]+\frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\quad= \left(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha\right)(1-s)^{\alpha-2}t^{\alpha-2}s(1-t)+\frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\quad\geqslant \frac{\lambda}{\alpha}\eta^\alpha(1-\eta)^i s(1-s)^{\alpha-1-i}t^{\alpha-1}.
\end{aligned}
$$

$$
\begin{aligned}
&\left(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha\right)\Gamma(\alpha)G_1(t,s) \\
&\quad= \Delta_1 t^{\alpha-1}(1-s)^{\alpha-1-i}-\left(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha\right)(t-s)^{\alpha-1} \\
&\quad= \left(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha\right)\left[(1-s)^{\alpha-1-i}t^{\alpha-1}-(t-s)^{\alpha-1}\right]+\frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha-1-i}t^{\alpha-1}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \left(\varDelta - \frac{\lambda}{\alpha}\eta^{\alpha}\right)\left[(1-s)^{\alpha-1-i}t^{\alpha-1} - (t-s)^{\alpha-1}(1-s)^{\alpha-1-i}\right] \\
&\quad + \frac{\lambda}{\alpha}\eta^{\alpha}(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&= \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^{\alpha}\right)(1-s)^{\alpha-1-i}t^{\alpha-1}\left[1 - \left(1-\frac{s}{t}\right)^{\alpha-1}\right] + \frac{\lambda}{\alpha}\eta^{\alpha}(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\leqslant \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^{\alpha}\right)(1-s)^{\alpha-1-i}t^{\alpha-1}\left[1 - \left(1-\frac{s}{t}\right)^{n}\right] + \frac{\lambda}{\alpha}\eta^{\alpha}(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&= \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^{\alpha}\right)(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\quad \times \left[1 - \left(1-\frac{s}{t}\right)\right]\left[1 + \left(1-\frac{s}{t}\right) + \left(1-\frac{s}{t}\right)^{2} + \cdots + \left(1-\frac{s}{t}\right)^{n-1}\right] \\
&\quad + \frac{\lambda}{\alpha}\eta^{\alpha}(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\leqslant n\left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^{\alpha}\right)s(1-s)^{\alpha-1-i}t^{\alpha-2} + n\frac{\lambda}{\alpha}\eta^{\alpha-1}s(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\leqslant n\left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^{\alpha} + \frac{\lambda}{\alpha}\eta^{\alpha-1}\right)s(1-s)^{\alpha-1-i}.
\end{aligned}
$$

For $t \leqslant s \leqslant \eta$,

$$
\begin{aligned}
&\left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^{\alpha}\right)\Gamma(\alpha)G_1(t,s) \\
&\quad = \varDelta_1 t^{\alpha-1}(1-s)^{\alpha-1-i} - \frac{\lambda}{\alpha}(\eta-s)^{\alpha}t^{\alpha-1} \\
&\quad \geqslant \varDelta_1 t^{\alpha-1}(1-s)^{\alpha-1} - \frac{\lambda}{\alpha}\eta^{\alpha}\left(1-\frac{s}{\eta}\right)^{\alpha}t^{\alpha-1} \\
&\quad \geqslant \varDelta_1 t^{\alpha-1}(1-s)^{\alpha-1} - \frac{\lambda}{\alpha}\eta^{\alpha}(1-s)^{\alpha}t^{\alpha-1} \\
&\quad = \left[\varDelta_1 - \frac{\lambda}{\alpha}\eta^{\alpha}(1-s)\right](1-s)^{\alpha-1}t^{\alpha-1} \\
&\quad = \left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^{\alpha}\right)(1-s)^{\alpha-1}t^{\alpha-1} + \frac{\lambda}{\alpha}\eta^{\alpha}s(1-s)^{\alpha-1}t^{\alpha-1} \\
&\quad \geqslant \frac{\lambda}{\alpha}\eta^{\alpha}(1-\eta)^{i}s(1-s)^{\alpha-1-i}t^{\alpha-1}.
\end{aligned}
$$

$$
\begin{aligned}
&\left(\varDelta_1 - \frac{\lambda}{\alpha}\eta^{\alpha}\right)\Gamma(\alpha)G_1(t,s) \\
&\quad = \varDelta_1 t^{\alpha-1}(1-s)^{\alpha-1-i} - \frac{\lambda}{\alpha}(\eta-s)^{\alpha}t^{\alpha-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\quad + \frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha-1-i}t^{\alpha-1} - \frac{\lambda}{\alpha}\eta^\alpha\left(1-\frac{s}{\eta}\right)^\alpha t^{\alpha-1} \\
&\leqslant \left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\quad + \frac{\lambda}{\alpha}\eta^\alpha t^{\alpha-1}\left[(1-s)^{\alpha-1-i} - \left(1-\frac{s}{\eta}\right)^\alpha(1-s)^{\alpha-1-i}\right] \\
&= \left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(1-s)^{\alpha-1-i}t^{\alpha-1} + \frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha-1-i}t^{\alpha-1}\left[1-\left(1-\frac{s}{\eta}\right)^\alpha\right] \\
&\leqslant \left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(1-s)^{\alpha-1-i}t^{\alpha-1} + \frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha-1-i}t^{\alpha-1}\left[1-\left(1-\frac{s}{\eta}\right)^n\right] \\
&= \left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(1-s)^{\alpha-1-i}t^{\alpha-1} + \frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha-1-i}t^{\alpha-1}\left[1-\left(1-\frac{s}{\eta}\right)\right] \\
&\quad \times \left[1+\left(1-\frac{s}{\eta}\right)+\left(1-\frac{s}{\eta}\right)^2+\cdots+\left(1-\frac{s}{\eta}\right)^{n-1}\right] \\
&\leqslant n\left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)s(1-s)^{\alpha-1-i}t^{\alpha-2} + n\frac{\lambda}{\alpha}\eta^{\alpha-1}s(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\leqslant n\left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1}\right)s(1-s)^{\alpha-1-i}.
\end{aligned}
$$

For $t \leqslant s$, $\eta \leqslant s$,

$$
\begin{aligned}
&\left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)\Gamma(\alpha)G_1(t,s) \\
&\quad = \Delta_1 t^{\alpha-1}(1-s)^{\alpha-1-i} \\
&\quad = \left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(1-s)^{\alpha-1-i}t^{\alpha-1} + \frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\quad \geqslant \frac{\lambda}{\alpha}\eta^\alpha(1-\eta)^i s(1-s)^{\alpha-1-i}t^{\alpha-1}.
\end{aligned}
$$

$$
\begin{aligned}
&\left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)\Gamma(\alpha)G(t,s) \\
&\quad = \Delta_1 t^{\alpha-1}(1-s)^{\alpha-1-i} \\
&\quad = \left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)(1-s)^{\alpha-1-i}t^{\alpha-1} + \frac{\lambda}{\alpha}\eta^\alpha(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\quad \leqslant \left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha\right)s(1-s)^{\alpha-1-i}t^{\alpha-2} + \frac{\lambda}{\alpha}\eta^{\alpha-1}s(1-s)^{\alpha-1-i}t^{\alpha-1} \\
&\quad \leqslant n\left(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1}\right)s(1-s)^{\alpha-1-i}.
\end{aligned}
$$

From above, (a1)–(a3) are complete. The proof is complete. □

Similarly, the general solution of

$$\begin{aligned}&D_{0+}^{\beta}v(t)+y(t)=0,\\&v(0)=v'(0)=\cdots=v^{(n-2)}(0)=0,\\&v^{(j)}(1)=b\int_0^c v(s)\,\mathrm{d}s,\end{aligned}$$

where $0<t<1$, $n-1<\beta\leqslant n$, $0<c\leqslant 1$, $j\in N$, $0\leqslant j\leqslant n-2$ and j is a fixed constant, $\Delta_2-bc^{\beta}/\beta>0$,

$$\Delta_2=\begin{cases}1, & j=0,\\(\beta-1)(\beta-2)\cdots(\beta-j), & j\geqslant 1,\end{cases}$$

$j\in N$ and j is a fixed constant, is

$$v(t)=\int_0^1 G_2(t,s)y(s)\,\mathrm{d}s,$$

where $G_2(t,s)$ can be obtained from $G_1(t,s)$ by replacing α, λ, η, i with β, b, c, j, correspondingly, and satisfy properties (a1)–(a3) with α, λ, η, i replaced by β, b, c, j in case of $G_2(t,s)$, correspondingly.

Lemma 5. (*See* [13].) *Let E be a Banach space and $P\in E$ be a cone. Assume Ω_1 and Ω_2 be two bounded open subsets in E such that $\theta\in\Omega_1$ and $\overline{\Omega}_1\subset\Omega_2$. Let operator $A:(\overline{\Omega}_2\backslash\Omega_1)\cap P\to P$ be completely continuous. Suppose that one of the two conditions*:

(1) $\|Au\|\leqslant\|u\|$ *for all $u\in P\cap\partial\Omega_1$, $\|Au\|\geqslant\|u\|$ for all $u\in P\cap\partial\Omega_2$ and*

(2) $\|Au\|\geqslant\|u\|$ *for all $u\in P\cap\partial\Omega_1$, $\|Au\|\leqslant\|u\|$ for all $u\in P\cap\partial\Omega_2$*

is satisfied. Then A has a fixed point in $(\Omega_2\setminus\overline{\Omega}_1)\cap P$.

Lemma 6. (*See* [14].) *Let E be a Banach space and $\Omega\subset E$ be closed and convex. Assume U is a relatively open subset of Ω with $\theta\in U$, and let operator $A:\overline{U}\to\Omega$ be a continuous compact map. Then either*

(1) *A has a fixed point in U or*

(2) *there exists $u\in\partial U$ and $\varphi\in(0,1)$ with $u=\varphi Au$.*

3 Main results and proof

Let $E=C[0,1]$ be the Banach space with the maximum norm $\|u\|=\max_{t\in[0,1]}|u(t)|$. Thus $(E\times E,\|\cdot\|)$ is a Banach space with the norm defined by $\|(u,v)\|=\max\{\|u\|,\|v\|\}$ for all $(u,v)\in E\times E$. We define the cone $P\subset E\times E$ by

$$P=\{(u,v)\in E\times E\mid u(t)\geqslant 0,\ v(t)\geqslant 0,\ 0\leqslant t\leqslant 1\}.$$

Lemma 7. *Let* $n-1<\alpha,\beta\leqslant n$. *Let* $F:(0,1]\times[0,+\infty)\to[0,+\infty)$ *be continuous and satisfy* $\lim_{t\to 0+}F(t,\cdot)=+\infty$. *Assume that there exists* $0<\sigma<1$ *such that* $t^\sigma F(t)$ *is continuous on* $[0,1]$. *Then* $u(t)=\int_0^1 G_1(t,s)F(s)\,\mathrm{d}s$ *is continuous on* $[0,1]$.

Proof. From the continuity of $t^\sigma F(t)$ and $u(t)=\int_0^1 G_1(t,s)t^{-\sigma}t^\sigma F(s)\,\mathrm{d}s$, we know that $u(0)=0$. If $u(t)\to u(t_0)$ when $t\to t_0$ for any $t_0\in[0,1]$, then the proof is complete. In the following we separate the process into three cases.

Case 1 ($t_0=0$, $t\in(0,1]$). Owing to the continuity of $t^\sigma F(t)$, there exists an $M>0$ such that $|t^\sigma F(t)|\leqslant M$ for all $t\in[0,1]$, then

$$
\begin{aligned}
&\bigl|u(t)-u(0)\bigr| \\
&= \Biggl|-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\,\mathrm{d}s \\
&\quad + \frac{1}{\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha}\int_0^1 \frac{\Delta_1(1-s)^{\alpha-1-i}t^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\,\mathrm{d}s \\
&\quad - \frac{1}{\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha}\int_0^\eta \frac{\frac{\lambda}{\alpha}(\eta-s)^{\alpha}t^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\,\mathrm{d}s\Biggr| \\
&\leqslant \Biggl|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\,\mathrm{d}s\Biggr| \\
&\quad + \Biggl|\frac{1}{\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha}\int_0^1 \frac{\Delta_1(1-s)^{\alpha-1-i}t^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\,\mathrm{d}s \\
&\quad - \frac{1}{\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha}\int_0^\eta \frac{\frac{\lambda}{\alpha}(\eta-s)^{\alpha}t^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\,\mathrm{d}s\Biggr| \\
&\leqslant \Biggl|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\,\mathrm{d}s\Biggr| \\
&\quad + \frac{\Delta_1+\frac{\lambda}{\alpha}}{\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha}\Biggl|\int_0^1 \frac{(1-s)^{\alpha-1-i}t^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}s^\sigma F(s)\,\mathrm{d}s\Biggr| \\
&\leqslant M\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}\,\mathrm{d}s + M\frac{\Delta_1+\frac{\lambda}{\alpha}}{\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha}\int_0^1 \frac{(1-s)^{\alpha-1-i}t^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}\,\mathrm{d}s \\
&= \frac{Mt^{\alpha-\sigma}}{\Gamma(\alpha)}B(1-\sigma,\alpha) + \frac{(\Delta_1+\frac{\lambda}{\alpha})Mt^{\alpha-1}}{(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}B(1-\sigma,\alpha-i) \\
&\leqslant \frac{(\Delta_1+\frac{\lambda}{\alpha})M\Gamma(1-\sigma)}{(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(1+\alpha-i-\sigma)}\bigl(t^{\alpha-\sigma}+t^{\alpha-1}\bigr)\to 0 \quad (t\to 0).
\end{aligned}
$$

$B(\cdot)$ mentioned in the above functions represents the Beta function.

Case 2 $(t_0 \in (0,1), t \in (t_0, 1])$.

$$
\begin{aligned}
&\left|u(t) - u(t_0)\right| \\
&\quad = \Bigg| -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\,\mathrm{d}s \\
&\qquad + \frac{1}{\Delta_1 - \frac{\lambda}{\alpha}\eta^{\alpha}} \int_0^1 \frac{\Delta_1 (1-s)^{\alpha-1-i} t^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\,\mathrm{d}s \\
&\qquad - \frac{1}{\Delta_1 - \frac{\lambda}{\alpha}\eta^{\alpha}} \int_0^\eta \frac{\frac{\lambda}{\alpha}(\eta-s)^{\alpha} t^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\,\mathrm{d}s + \int_0^{t_0} \frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\,\mathrm{d}s \\
&\qquad - \frac{1}{\Delta_1 - \frac{\lambda}{\alpha}\eta^{\alpha}} \int_0^1 \frac{\Delta_1 (1-s)^{\alpha-1-i} t_0^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\,\mathrm{d}s \\
&\qquad + \frac{1}{\Delta_1 - \frac{\lambda}{\alpha}\eta^{\alpha}} \int_0^\eta \frac{\frac{\lambda}{\alpha}(\eta-s)^{\alpha} t_0^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\,\mathrm{d}s \Bigg| \\
&\quad = \Bigg| -\int_0^{t_0} \frac{(t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\,\mathrm{d}s \\
&\qquad + \int_0^1 \frac{\Delta_1 (1-s)^{\alpha-1-i} (t^{\alpha-1} - t_0^{\alpha-1})}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^{\alpha})\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\,\mathrm{d}s \\
&\qquad - \int_0^\eta \frac{\frac{\lambda}{\alpha}(\eta-s)^{\alpha} (t^{\alpha-1} - t_0^{\alpha-1})}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^{\alpha})\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\,\mathrm{d}s - \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\,\mathrm{d}s \Bigg| \\
&\quad \leqslant \Bigg| \int_0^{t_0} \frac{(t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\,\mathrm{d}s \Bigg| \\
&\qquad + \frac{\Delta_1 + \frac{\lambda}{\alpha}}{\Delta_1 - \frac{\lambda}{\alpha}\eta^{\alpha}} \int_0^1 \frac{(1-s)^{\alpha-1-i} (t^{\alpha-1} - t_0^{\alpha-1})}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\,\mathrm{d}s \Bigg| \\
&\qquad + \Bigg| \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s)\,\mathrm{d}s \Bigg| \\
&\quad \leqslant M \int_0^{t_0} \frac{(t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma}\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{M(\Delta_1+\frac{\lambda}{\alpha})(t^{\alpha-1}-t_0^{\alpha-1})}{\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha}\int_0^1\frac{(1-s)^{\alpha-1-i}}{\Gamma(\alpha)}s^{-\sigma}\,\mathrm{d}s+M\int_{t_0}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}\,\mathrm{d}s\\
&=\frac{M(t^{\alpha-\sigma}-t_0^{\alpha-\sigma})}{\Gamma(\alpha)}B(1-\sigma,\alpha)+\frac{(\Delta_1+\frac{\lambda}{\alpha})M(t^{\alpha-1}-t_0^{\alpha-1})}{(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}B(1-\sigma,\alpha-i)\\
&\leqslant\frac{(\Delta_1+\frac{\lambda}{\alpha})M\Gamma(1-\sigma)}{(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(1+\alpha-i-\sigma)}\bigl(t^{\alpha-\sigma}-t_0^{\alpha-\sigma}+t^{\alpha-1}-t_0^{\alpha-1}\bigr)\to 0\quad(t\to t_0).
\end{aligned}
$$

Case 3 ($t_0\in(0,1]$, $t\in[0,t_0)$). Similarly to the proof of Case 2, so we omit it. The proof is complete. □

From Lemma 3 we can write the system of BVPs (1) as an equivalent system of integral equations

$$
\begin{aligned}
u(t)&=\int_0^1 G_1(t,s)f\bigl(s,v(s)\bigr)\,\mathrm{d}s,\quad 0\leqslant t\leqslant 1,\\
v(t)&=\int_0^1 G_2(t,s)g\bigl(s,u(s)\bigr)\,\mathrm{d}s,\quad 0\leqslant t\leqslant 1,
\end{aligned}
$$

which can be proved in the same way as Lemma 3.3 in [6]. For convenience, the proof is omitted.

We define $A:E\times E\to E\times E$ to be an operator, i.e.,

$$
\begin{aligned}
A(u,v)(t)&=\left(\int_0^1 G_1(t,s)s^{-\sigma_1}s^{\sigma_1}f\bigl(s,v(s)\bigr)\,\mathrm{d}s,\int_0^1 G_2(t,s)s^{-\sigma_2}s^{\sigma_2}g\bigl(s,u(s)\bigr)\,\mathrm{d}s\right)\\
&=:\bigl(A_1v(t),A_2u(t)\bigr).
\end{aligned}
$$

Lemma 8. *Let $n-1<\alpha,\beta\leqslant n$. Let $f,g:(0,1]\times[0,+\infty)\to[0,+\infty)$ be continuous and satisfy $\lim_{t\to0+}f(t,\cdot)=+\infty$, $\lim_{t\to0+}g(t,\cdot)=+\infty$. Assume that there exists $0<\sigma_1,\sigma_2<1$ such that $t^{\sigma_1}f(t,y),t^{\sigma_2}g(t,y)$ are continuous on $[0,1]\times[0,\infty)$. Then the operator $A:P\to P$ is completely continuous.*

Proof. For any $(u,v)\in P$, we have that

$$u,v\in P_1=\bigl\{y\in E\bigm| y(t)\geqslant 0,\ 0\leqslant t\leqslant 1\bigr\}.$$

Since

$$A_1v(t)=\int_0^1 G_1(t,s)s^{-\sigma_1}s^{\sigma_1}f\bigl(s,v(s)\bigr)\,\mathrm{d}s,$$

we get that $A_1 : P_1 \to P_1$ by Lemma 7 and the nonnegativity of f. Set $v_0 \in P_1$ and $\|v_0\| = c_0$. If $v \in P_1$ and $\|v - v_0\| < 1$, then $\|v\| < 1 + c_0 := c$. By the continuity of $t^{\sigma_1} f(t, y)$, we get that $t^{\sigma_1} f(t, y)$ is uniformly continuous on $[0, 1] \times [0, c]$, namely, for all $\varepsilon > 0$, exists $\delta > 0$ ($\delta < 1$), when $|y_1 - y_2| < \delta$, we have $|t^{\sigma_1} f(t, y_1) - t^{\sigma_1} f(t, y_2)| < \varepsilon$, for all $t \in [0, 1]$, $y_1, y_2 \in [0, c]$. Obviously, if $\|v - v_0\| < \delta$, then $v_0(t), v(t) \in [0, c]$ and $|v(t) - v_0(t)| < \delta$ for all $t \in [0, 1]$. Hence, we have

$$\left|t^{\sigma_1} f\bigl(t, v(t)\bigr) - t^{\sigma_1} f\bigl(t, v_0(t)\bigr)\right| < \varepsilon \tag{4}$$

for all $t \in [0, 1]$, $v \in P_1$, $\|v - v_0\| < \delta$. It follows from (4), we can get

$$\begin{aligned}
&\|A_1 v - A_1 v_0\| \\
&\quad = \max_{t\in[0,1]} \left|A_1 v(t) - A_1 v_0(t)\right| \\
&\quad \leqslant \max_{t\in[0,1]} \int_0^1 G_1(t, s) s^{-\sigma_1} \left|s^{\sigma_1} f\bigl(s, v(s)\bigr) - s^{\sigma_1} f\bigl(s, v_0(s)\bigr)\right| \mathrm{d}s \\
&\quad < \varepsilon \int_0^1 G_1(t, s) s^{-\sigma_1} \, \mathrm{d}s \leqslant \varepsilon \int_0^1 \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} s(1 - s)^{\alpha-1-i} s^{-\sigma_1} \, \mathrm{d}s \\
&\quad = \varepsilon \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1-i} s^{1-\sigma_1} \, \mathrm{d}s \\
&\quad = \varepsilon \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} B(2 - \sigma_1, \alpha - i) \\
&\quad = \varepsilon \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} \frac{\Gamma(2 - \sigma_1)\Gamma(\alpha - i)}{\Gamma(2 + \alpha - i - \sigma_1)}.
\end{aligned}$$

Owing to the arbitrariness of v_0, we know that $A_1 : P_1 \to P_1$ is continuous. Similarly, we can get that $A_2 : P_2 \to P_2$ is continuous. So, we proved $A : P \to P$ is continuous.

Let $M \subset P$ be bounded. That is to say there exists a constant $l > 0$ such that $\|(u, v)\| \leqslant l$ for all $(u, v) \in M$. Since $t^{\sigma_1} f(t, y)$, $t^{\sigma_2} g(t, y)$ are continuous on $[0, 1] \times [0, +\infty)$, let $L = \max_{t\in[0,1],(u,v)\in M}\{t^{\sigma_1} f(t, v(t)), t^{\sigma_2} g(t, u(t))\} + 1$. Then, for each $(u, v) \in M$, we have

$$\begin{aligned}
\left|A_1 v(t)\right| &\leqslant \int_0^1 G_1(t, s) s^{-\sigma_1} \left|s^{\sigma_1} f\bigl(s, v(s)\bigr)\right| \mathrm{d}s \\
&\leqslant L \int_0^1 \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} s(1 - s)^{\alpha-1-i} s^{-\sigma_1} \, \mathrm{d}s \\
&= L \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} \frac{\Gamma(2 - \sigma_1)\Gamma(\alpha - i)}{\Gamma(2 + \alpha - i - \sigma_1)}.
\end{aligned}$$

Hence, we have

$$\|A_1 v\| = \max_{t\in[0,1]} |A_1 v(t)| \leqslant L \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} \frac{\Gamma(2-\sigma_1)\Gamma(\alpha-i)}{\Gamma(2+\alpha-i-\sigma_1)}.$$

Similarly, we have

$$\|A_2 u\| = \max_{t\in[0,1]} |A_2 u(t)| \leqslant L \frac{n(\Delta_2 - \frac{b}{\beta}c^\beta + \frac{b}{\beta}c^{\beta-1})}{(\Delta_2 - \frac{b}{\beta}c^\beta)\Gamma(\beta)} \frac{\Gamma(2-\sigma_2)\Gamma(\beta-j)}{\Gamma(2+\beta-j-\sigma_2)}.$$

Thus,

$$\begin{aligned}\|A(u,v)\| &= \max_{t\in[0,1]} \bigl\{|A_1 v|, |A_2 u|\bigr\} \\ &\leqslant \max\biggl\{\frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} \frac{\Gamma(2-\sigma_1)\Gamma(\alpha-i)}{\Gamma(2+\alpha-i-\sigma_1)}, \\ &\qquad \frac{n(\Delta_2 - \frac{b}{\beta}c^\beta + \frac{b}{\beta}c^{\beta-1})}{(\Delta_2 - \frac{b}{\beta}c^\beta)\Gamma(\beta)} \frac{\Gamma(2-\sigma_2)\Gamma(\beta-j)}{\Gamma(2+\beta-j-\sigma_2)}\biggr\} L.\end{aligned}$$

Therefore, $A(M)$ is bounded.

Next, we prove that A is equicontinuous. Let, for all $\varepsilon > 0$,

$$\delta = \min\biggl\{\frac{\varepsilon(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(1+\alpha-i-\sigma_1)}{2^{n+1}(\Delta_1 + \frac{\lambda}{\alpha})L\Gamma(1-\sigma_1)}, \frac{\varepsilon(\Delta_2 - \frac{b}{\beta}c^\beta)\Gamma(1+\beta-j-\sigma_2)}{2^{n+1}(\Delta_2 + \frac{b}{\beta})L\Gamma(1-\sigma_2)}\biggr\}.$$

Then, for any $(u,v) \in M, t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $0 < t_2 - t_1 < \delta$, we have

$$\begin{aligned}&\bigl|A_1 v(t_2) - A_1 v(t_1)\bigr| \\ &\quad = \left|\int_0^1 G_1(t_2,s) f\bigl(s,v(s)\bigr)\,\mathrm{d}s - \int_0^1 G_1(t_1,s) f\bigl(s,v(s)\bigr)\,\mathrm{d}s\right| \\ &\quad = -\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1} s^{\sigma_1} f\bigl(s,v(s)\bigr)\,\mathrm{d}s \\ &\qquad + \int_0^1 \frac{\Delta_1(1-s)^{\alpha-1-i} t_2^{\alpha-1}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} s^{-\sigma_1} s^{\sigma_1} f\bigl(s,v(s)\bigr)\,\mathrm{d}s \\ &\qquad - \int_0^\eta \frac{\frac{\lambda}{\alpha}(\eta-s)^\alpha t_2^{\alpha-1}}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} s^{-\sigma_1} s^{\sigma_1} f\bigl(s,v(s)\bigr)\,\mathrm{d}s \\ &\qquad + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma_1} s^{\sigma_1} f\bigl(s,v(s)\bigr)\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&-\int_0^1 \frac{\varDelta_1(1-s)^{\alpha-1-i}t_1^{\alpha-1}}{(\varDelta_1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}f\bigl(s,v(s)\bigr)\,\mathrm{d}s \\
&+\int_0^\eta \frac{\frac{\lambda}{\alpha}(\eta-s)^{\alpha}t_1^{\alpha-1}}{(\varDelta_1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}f\bigl(s,v(s)\bigr)\,\mathrm{d}s\biggr| \\
={}&\biggl|-\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}f\bigl(s,v(s)\bigr)\,\mathrm{d}s \\
&+\int_0^1 \frac{\varDelta_1(1-s)^{\alpha-1-i}(t_2^{\alpha-1}-t_1^{\alpha-1})}{(\varDelta_1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}f\bigl(s,v(s)\bigr)\,\mathrm{d}s \\
&-\int_0^\eta \frac{\frac{\lambda}{\alpha}(\eta-s)^{\alpha}(t_2^{\alpha-1}-t_1^{\alpha-1})}{(\varDelta_1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}f\bigl(s,v(s)\bigr)\,\mathrm{d}s \\
&-\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}f\bigl(s,v(s)\bigr)\,\mathrm{d}s\biggr| \\
\leqslant{}&\biggl|\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}f\bigl(s,v(s)\bigr)\,\mathrm{d}s\biggr| \\
&+\biggl|\int_0^1 \frac{(\varDelta_1+\frac{\lambda}{\alpha})(1-s)^{\alpha-1-i}(t_2^{\alpha-1}-t_1^{\alpha-1})}{(\varDelta_1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}f\bigl(s,v(s)\bigr)\,\mathrm{d}s\biggr| \\
&+\biggl|\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}s^{\sigma_1}f\bigl(s,v(s)\bigr)\,\mathrm{d}s\biggr| \\
\leqslant{}& L\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}\,\mathrm{d}s \\
&+\frac{L(\varDelta_1+\frac{\lambda}{\alpha})(t_2^{\alpha-1}-t_1^{\alpha-1})}{\varDelta_1-\frac{\lambda}{\alpha}\eta^\alpha}\int_0^1 \frac{(1-s)^{\alpha-1-i}}{\Gamma(\alpha)}s^{-\sigma_1}\,\mathrm{d}s \\
&+L\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma_1}\,\mathrm{d}s \\
={}&\frac{L(t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1})}{\Gamma(\alpha)}B(1-\sigma_1,\alpha)+\frac{(\varDelta_1+\frac{\lambda}{\alpha})L(t_2^{\alpha-1}-t_1^{\alpha-1})}{(\varDelta_1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)}B(1-\sigma_1,\alpha-i) \\
\leqslant{}&\frac{(\varDelta_1+\frac{\lambda}{\alpha})L\Gamma(1-\sigma_1)}{(\varDelta_1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(1+\alpha-i-\sigma_1)}\bigl(t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1}+t_2^{\alpha-1}-t_1^{\alpha-1}\bigr).
\end{aligned}
$$

Similarly,

$$\begin{aligned}&\left|A_2u(t_2)-A_2u(t_1)\right|\\&\quad\leqslant\frac{(\Delta_2+\frac{b}{\beta})L\Gamma(1-\sigma_2)}{(\Delta_2-\frac{b}{\beta}c^\beta)\Gamma(1+\beta-j-\sigma_2)}\left(t_2^{\beta-\sigma_2}-t_1^{\beta-\sigma_2}+t_2^{\beta-1}-t_1^{\beta-1}\right).\end{aligned}$$

Hence, we figure on $t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1}, t_2^{\alpha-1}-t_1^{\alpha-1}, t_2^{\beta-\sigma_2}-t_1^{\beta-\sigma_2}, t_2^{\beta-1}-t_1^{\beta-1}$ in the following three cases:

Case 1. If $0\leqslant t_1<\delta$, $0\leqslant t_2<2\delta$, then

$$t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1}\leqslant t_2^{\alpha-\sigma_1}\leqslant(2\delta)^{\alpha-\sigma_1}<2^n\delta,$$
$$t_2^{\alpha-1}-t_1^{\alpha-1}\leqslant t_2^{\alpha-1}\leqslant(2\delta)^{\alpha-1}<2^n\delta.$$

Case 2. If $0\leqslant t_1<t_2\leqslant\delta$, then

$$t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1}\leqslant t_2^{\alpha-\sigma_1}\leqslant\delta^{\alpha-\sigma_1}<2^n\delta,$$
$$t_2^{\alpha-1}-t_1^{\alpha-1}\leqslant t_2^{\alpha-1}\leqslant\delta^{\alpha-1}<2^n\delta.$$

Case 3. If $\delta\leqslant t_1<t_2\leqslant 1$, then

$$\begin{aligned}&t_2^{\alpha-\sigma_1}-t_1^{\alpha-\sigma_1}\\&\quad=(\alpha-\sigma_1)\int_{t_1}^{t_2}x^{\alpha-\sigma_1-1}\,\mathrm{d}x\leqslant(\alpha-\sigma_1)(t_2-t_1)t_2^{\alpha-\sigma_1-1}\leqslant(\alpha-\sigma_1)\delta<2^n\delta,\\&t_2^{\alpha-1}-t_1^{\alpha-1}\\&\quad=(\alpha-1)\int_{t_1}^{t_2}x^{\alpha-2}\,\mathrm{d}x\leqslant(\alpha-1)(t_2-t_1)t_2^{\alpha-2}\leqslant(\alpha-1)\delta<2^n\delta.\end{aligned}$$

Hence,

$$\left|A_1v(t_2)-A_1v(t_1)\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Similarly,

$$\left|A_2u(t_2)-A_2u(t_1)\right|<\varepsilon.$$

Therefore, $A(M)$ is equicontinuous, and by Arzelà–Ascoli's theorem, we obtain that $A(M)$ is a relatively compact set, then we prove operator $A: P\to P$ is completely continuous. □

Now we give the following three results of this paper.

Theorem 1. *Let* $n-1<\alpha, \beta\leqslant n$. *Let* $f, g: (0,1]\times[0,+\infty)\to[0,+\infty)$ *be continuous and satisfy* $\lim_{t\to 0+}f(t,\cdot)=+\infty$, $\lim_{t\to 0+}g(t,\cdot)=+\infty$. *Assume that there exists* $0<\sigma_1,\sigma_2<1$ *such that* $t^{\sigma_1}f(t,y)$, $t^{\sigma_2}g(t,y)$ *are continuous on* $[0,1]\times[0,+\infty)$ *and* *there exist* $t_0 \in (0,1)$ *and two positive constants* ρ, ξ *subjecting to* $\rho > \max\{\xi(\alpha-1)/(m_1 t_0^{\alpha-1}), \xi(\beta-1)/(m_2 t_0^{\beta-1})\}$, *where*

$$m_1 = \frac{\frac{\lambda}{\alpha}\eta^\alpha(1-\eta)^i}{\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha} \int\limits_{t_0}^{1} s^{1-\sigma_1}(1-s)^{\alpha-1-i}\,\mathrm{d}s,$$

$$m_2 = \frac{\frac{b}{\beta}c^\beta(1-c)^j}{\Delta_2 - \frac{b}{\beta}c^\beta} \int\limits_{t_0}^{1} s^{1-\sigma_2}(1-s)^{\beta-1-j}\,\mathrm{d}s.$$

Further suppose:

(i) *for all* $(t,y) \in [0,1] \times [0,\xi]$,

$$t^{\sigma_1} f(t,y) \geqslant \frac{\xi\Gamma(\alpha)}{m_1 t_0^{\alpha-1}}, \qquad t^{\sigma_2} g(t,y) \geqslant \frac{\xi\Gamma(\beta)}{m_2 t_0^{\beta-1}};$$

(ii) *for all* $(t,y) \in [0,1] \times [0,\rho]$,

$$t^{\sigma_1} f(t,y) \leqslant \frac{\rho\Gamma(2+\alpha-i-\sigma_1)(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)}{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})\Gamma(2-\sigma_1)},$$

$$t^{\sigma_2} g(t,y) \leqslant \frac{\rho\Gamma(2+\beta-j-\sigma_2)(\Delta_2 - \frac{b}{\beta}c^\beta)}{n(\Delta_2 - \frac{b}{\beta}c^\beta + \frac{b}{\beta}c^{\beta-1})\Gamma(2-\sigma_2)}.$$

Then BVP (1) *has at least one positive solution.*

Theorem 2. *Let* $n-1 < \alpha, \beta \leqslant n$. *Let* $f, g : (0,1] \times [0,+\infty) \to [0,+\infty)$ *be continuous and satisfy* $\lim_{t\to 0+} f(t,\cdot) = +\infty$, $\lim_{t\to 0+} g(t,\cdot) = +\infty$. *Assume that there exists* $0 < \sigma_1, \sigma_2 < 1$ *such that* $t^{\sigma_1} f(t,y), t^{\sigma_2} g(t,y)$ *are continuous on* $[0,1] \times [0,+\infty)$. *Suppose they satisfy the following conditions*:

(iii) *there exist two continuous and nondecreasing functions* $\varphi, \psi : [0,+\infty) \to (0,+\infty)$ *such that*

$$t^{\sigma_1} f(t,y) \leqslant \varphi(y), \quad t^{\sigma_2} g(t,y) \leqslant \psi(y) \quad \forall (t,y) \in [0,1] \times [0,+\infty);$$

(iv) *there exists an* $r > 0$, *yielding*

$$\frac{r}{\max\{\varphi(r), \psi(r)\}} > \max\biggl\{ \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha} \frac{\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-i-\sigma_1)}, \frac{n(\Delta_2 - \frac{b}{\beta}c^\beta + \frac{b}{\beta}c^{\beta-1})}{\Delta_2 - \frac{b}{\beta}c^\beta} \frac{\Gamma(2-\sigma_2)}{\Gamma(2+\beta-j-\sigma_2)} \biggr\}.$$

Then the BVP (1) *has a positive solution.*

Theorem 3. *Let* $n-1<\alpha,\beta\leqslant n$. *Let* $f,g:(0,1]\times[0,+\infty)\to[0,+\infty)$ *be continuous and satisfy* $\lim_{t\to 0+} f(t,\cdot)=+\infty$, $\lim_{t\to 0+} g(t,\cdot)=+\infty$. *Assume that there exists* $0<\sigma_1,\sigma_2<1$ *such that* $t^{\sigma_1}f(t,y)$, $t^{\sigma_2}g(t,y)$ *are continuous on* $[0,1]\times[0,+\infty)$. *Suppose they satisfy the following condition*:

(v) *there exists two positive constants* L_1, L_2 *such that, for all* $(t,\cdot)\in[0,1]\times[0,+\infty)$,

$$\left|t^{\sigma_1}f(t,y)-t^{\sigma_1}f(t,x)\right|\leqslant L_1|y-x|,\ \left|t^{\sigma_2}g(t,y)-t^{\sigma_2}g(t,x)\right|\leqslant L_2|y-x|,$$

where

$$\frac{n(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha+\frac{\lambda}{\alpha}\eta^{\alpha-1})}{\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha}\frac{\Gamma(2-\sigma_1)L_1}{\Gamma(2+\alpha-i-\sigma_1)}<1,$$
$$\frac{n(\Delta_2-\frac{b}{\beta}c^\beta+\frac{b}{\beta}c^{\beta-1})}{\Delta_2-\frac{b}{\beta}c^\beta}\frac{\Gamma(2-\sigma_2)L_2}{\Gamma(2+\beta-j-\sigma_2)}<1.$$

Then the BVP (1) *has a unique solution. In addition, we can get the unique solution by constructing iterative sequence and error estimate of the* n *times iterated.*

Proof of Theorem 1. From the conditions we obtain $\rho>\max\{\xi(\alpha-1)/(m_1t_0^{\alpha-1}),\ \xi(\beta-1)/(m_2t_0^{\beta-1})\}>\xi$. We divide the demonstration into two steps.

Step 1. Let $\Omega_1=\{(u,v)\in P\mid \|u\|<\xi,\ \|v\|<\xi\}$ such that $0\leqslant u(t),v(t)\leqslant\xi$ for any $(u,v)\in P\cap\partial\Omega_1$ and for all $t\in[0,1]$. By condition (i) and Lemma 4, we get

$$\begin{aligned}A_1v(t_0)&=\int_0^1 G_1(t_0,s)s^{-\sigma_1}s^{\sigma_1}f\bigl(s,v(s)\bigr)\,\mathrm{d}s\\&\geqslant\int_{t_0}^1 G_1(t_0,s)s^{-\sigma_1}s^{\sigma_1}f\bigl(s,v(s)\bigr)\,\mathrm{d}s\\&\geqslant\frac{\xi\Gamma(\alpha)}{m_1t_0^{\alpha-1}}\int_{t_0}^1\frac{\frac{\lambda}{\alpha}\eta^\alpha(1-\eta)^i}{\Gamma(\alpha)(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha)}t_0^{\alpha-1}s(1-s)^{\alpha-1-i}s^{-\sigma_1}\,\mathrm{d}s=\xi.\end{aligned}$$

Hence,

$$\|A_1v\|=\max_{t\in[0,1]}\bigl|A_1v(t)\bigr|\geqslant\xi\quad\forall v\in P\cap\partial\Omega_1.$$

Similarly,

$$\|A_2u\|=\max_{t\in[0,1]}\bigl|A_2u(t)\bigr|\geqslant\xi\quad\forall u\in P\cap\partial\Omega_1.$$

Therefore,

$$\bigl\|A(u,v)\bigr\|\geqslant\xi=\bigl\|(u,v)\bigr\|.$$

Step 2. Let $\Omega_2 = \{(u,v) \in P \mid \|u\| < \rho,\ \|v\| < \rho\}$. For any $(u,v) \in P \cap \partial\Omega_2$, $t \in [0,1]$, we have that $0 \leqslant u(t), v(t) \leqslant \rho$. By condition (ii) and Lemma 4, we get

$$\begin{aligned} A_1 v(t) &= \int_0^1 G_1(t,s) s^{-\sigma_1} s^{\sigma_1} f\big(s, v(s)\big)\,\mathrm{d}s \\ &\leqslant \frac{\rho\Gamma(2+\alpha-i-\sigma_1)(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha)}{n(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha+\frac{\lambda}{\alpha}\eta^{\alpha-1})\Gamma(2-\sigma_1)} \int_0^1 \frac{n(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha+\frac{\lambda}{\alpha}\eta^{\alpha-1})}{(\Delta_1-\frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} s(1-s)^{\alpha-1-i} s^{-\sigma_1}\,\mathrm{d}s \\ &\leqslant \rho. \end{aligned}$$

Then we obtain

$$\|A_1 v\| \leqslant \rho \quad \forall (u,v) \in P \cap \partial\Omega_2.$$

Similarly,

$$\|A_2 u\| = \max_{t\in[0,1]} \big|A_2 u(t)\big| \leqslant \rho \quad \forall (u,v) \in P \cap \partial\Omega_2.$$

Therefore, $\|A(u,v)\| \leqslant \rho = \|(u,v)\|$.

Besides, by Lemma 8, operator $A : P \to P$ is completely continuous. Then with Lemma 5, our proof is complete. □

Proof of Theorem 2. Let $U = \{(u,v) \in P \mid \|u\| < r,\ \|v\| < r\}$, so that $U \subset P$. By Lemma 8, we get to know that operator $A : \overline{U} \to P$ is completely continuous. And if there exists $(u,v) \in \partial U$ and $\tilde{\lambda} \in (0,1)$, we have $(u,v) = \tilde{\lambda} A(u,v)$, then by (iii) for $t \in [0,1]$, we obtain

$$\begin{aligned} u(t) &= \tilde{\lambda} A_1 v(t) = \tilde{\lambda} \int_0^1 G_1(t,s) f\big(s, v(s)\big)\,\mathrm{d}s < \int_0^1 G_1(t,s) s^{-\sigma_1} s^{\sigma_1} f\big(s, v(s)\big)\,\mathrm{d}s \\ &\leqslant \int_0^1 G_1(t,s) s^{-\sigma_1} \varphi\big(v(s)\big)\,\mathrm{d}s \leqslant \varphi\big(\|v\|\big) \int_0^1 G_1(t,s) s^{-\sigma_1}\,\mathrm{d}s \\ &\leqslant \varphi\big(\|v\|\big) \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1-i} s^{1-\sigma_1}\,\mathrm{d}s \\ &= \varphi\big(\|v\|\big) \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} B(2-\sigma_1, \alpha-i) \\ &\leqslant \varphi\big(\big\|(u,v)\big\|\big) \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha} \frac{\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-i-\sigma_1)}. \end{aligned}$$

Hence,

$$\|u\| < \varphi\big(\big\|(u,v)\big\|\big) \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha} \frac{\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-i-\sigma_1)},$$

i.e.,

$$\frac{\|u\|}{\varphi(\|(u,v)\|)} < \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha} \frac{\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-i-\sigma_1)}.$$

Similarly,

$$\frac{\|v\|}{\psi(\|(u,v)\|)} < \frac{n(\Delta_2 - \frac{b}{\beta}c^\beta + \frac{b}{\beta}c^{\beta-1})}{\Delta_2 - \frac{b}{\beta}c^\beta} \frac{\Gamma(2-\sigma_2)}{\Gamma(2+\beta-j-\sigma_2)}.$$

Consequently,

$$\frac{\|(u,v)\|}{\max\{\varphi(\|(u,v)\|), \psi(\|(u,v)\|)\}} < \max\biggl\{\frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha} \frac{\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-i-\sigma_1)}, \frac{n(\Delta_2 - \frac{b}{\beta}c^\beta + \frac{b}{\beta}c^{\beta-1})}{\Delta_2 - \frac{b}{\beta}c^\beta} \frac{\Gamma(2-\sigma_2)}{\Gamma(2+\beta-j-\sigma_2)}\biggr\}.$$

Again by (iv) we know $\|(u,v)\| \neq r$ which contradicts that $(u,v) \in \partial U$. Then based on Lemma 6, there is a fixed point $(u,v) \in U$. Therefore the BVP (1) has a positive solution. □

Proof of Theorem 3. We shall use Banach fixed point theorem. From (v), for any $v_1, v_2 \in P_1$, $t \in [0,1]$, we can get that

$$\begin{aligned}
&\bigl|A_1 v_2(t) - A_1 v_1(t)\bigr| \\
&\quad \leqslant \int_0^1 G_1(t,s) s^{-\sigma_1} \bigl| s^{\sigma_1} f\bigl(s, v_2(s)\bigr) - s^{\sigma_1} f\bigl(s, v_1(s)\bigr)\bigr| \,\mathrm{d}s \\
&\quad \leqslant L_1 \bigl|v_2(t) - v_1(t)\bigr| \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1-i} s^{1-\sigma_1} \,\mathrm{d}s \\
&\quad \leqslant L_1 \bigl|v_2(t) - v_1(t)\bigr| \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha)\Gamma(\alpha)} B(2-\sigma_1, \alpha - i) \\
&\quad \leqslant L_1 \bigl|v_2(t) - v_1(t)\bigr| \frac{n(\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha + \frac{\lambda}{\alpha}\eta^{\alpha-1})}{\Delta_1 - \frac{\lambda}{\alpha}\eta^\alpha} \frac{\Gamma(2-\sigma_1)}{\Gamma(2+\alpha-i-\sigma_1)}.
\end{aligned}$$

Similarly, for any $u_1, u_2 \in P_1$, $t \in [0,1]$,

$$\begin{aligned}
&\bigl|A_2 u_2(t) - A_2 u_1(t)\bigr| \\
&\quad \leqslant \frac{n(\Delta_2 - \frac{b}{\beta}c^\beta + \frac{b}{\beta}c^{\beta-1})}{\Delta_2 - \frac{b}{\beta}c^\beta} \frac{\Gamma(2-\sigma_2)}{\Gamma(2+\beta-j-\sigma_2)} L_2 \bigl|u_2(t) - u_1(t)\bigr|.
\end{aligned}$$

So, we have

$$\begin{aligned}
&\max_{t\in[0,1]}\left|A(u_2,v_2)-A(u_1,v_1)\right| \\
&\quad=\left\|A(u_2,v_2)-A(u_1,v_1)\right\|=\left\|(A_1v_2,A_2u_2)-(A_1v_1,A_2u_1)\right\| \\
&\quad=\left\|(A_1v_2-A_1v_1,A_2u_2-A_2u_1)\right\|\leqslant\tilde{L}\left\|(v_2-v_1,u_2-u_1)\right\|,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{L}=\max\biggl\{&\frac{n(\varDelta_1-\frac{\lambda}{\alpha}\eta^\alpha+\frac{\lambda}{\alpha}\eta^{\alpha-1})}{\varDelta_1-\frac{\lambda}{\alpha}\eta^\alpha}\frac{\Gamma(2-\sigma_1)L_1}{\Gamma(2+\alpha-i-\sigma_1)}, \\
&\frac{n(\varDelta_2-\frac{b}{\beta}c^\beta+\frac{b}{\beta}c^{\beta-1})}{\varDelta_2-\frac{b}{\beta}c^\beta}\frac{\Gamma(2-\sigma_2)L_2}{\Gamma(2+\beta-j-\sigma_2)}\biggr\}<1.
\end{aligned}$$

By the contraction mapping principle, the BVP (1) has a unique solution. □

Example 1. For any $n-1<\alpha,\beta\leqslant n$, take $t_0=1/4$, $\xi>0$, and $\rho>\max\{4^{\alpha-1}\times\xi(\alpha-1)/m_1,4^{\beta-1}\xi(\beta-1)/m_2\}$, $\rho>0$. Choose $\sigma_1=\sigma_2=1/2$. Consider the boundary value problem to the singular system of fractional equations

$$\begin{aligned}
&D_{0+}^{\alpha}u(t)+\frac{c_1+v}{\sqrt{t}}=0,\quad 0<t<1, \\
&D_{0+}^{\beta}v(t)+\frac{c_2+u}{\sqrt{t}}=0,\quad 0<t<1, \\
&u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\qquad u(1)=\frac{1}{2}\int_0^{1/2}u(s)\,\mathrm{d}s, \\
&v(0)=v'(0)=\cdots=v^{(n-2)}(0)=0,\qquad v(1)=\frac{1}{2}\int_0^{1/2}v(s)\,\mathrm{d}s,
\end{aligned}\tag{5}$$

where c_1, c_2 are constants satisfying

$$\begin{aligned}
\frac{4^{\alpha-1}\xi\Gamma(\alpha)}{m_1}\leqslant c_1\leqslant\frac{\rho[\Gamma(2+\alpha-\sigma_1)-(\alpha-1+\frac{(\mu n/\alpha)\eta^{\alpha-1}}{(1-\mu\eta^\alpha/\alpha)})\Gamma(2-\sigma_1)]}{[\alpha-1+\frac{(\mu n/\alpha)\eta^{\alpha-1}}{(1-\mu\eta^\alpha/\alpha)}]\Gamma(2-\sigma_1)}, \\
\frac{4^{\beta-1}\xi\Gamma(\beta)}{m_2}\leqslant c_2\leqslant\frac{\rho[\Gamma(2+\beta-\sigma_2)-(\beta-1+\frac{(\mu n/\beta)\eta^{\beta-1}}{(1-\mu\eta^\beta/\beta)})\Gamma(2-\sigma_2)]}{[\beta-1+\frac{(\mu n/\beta)\eta^{\beta-1}}{(1-\mu\eta^\beta/\beta)}]\Gamma(2-\sigma_2)}.
\end{aligned}$$

Denote $f(t,y)=(c_1+y)/\sqrt{t}$, $g(t,y)=(c_2+y)/\sqrt{t}$. Then f, g are continuous on $(0,1]\times[0,+\infty)$ and $\lim_{t\to0+}f(t,\cdot)=+\infty$, $\lim_{t\to0+}g(t,\cdot)=+\infty$. All conditions of Theorem 3 hold. Therefore, BVP (5) has at least one positive solution.

Example 2. Consider the boundary value problem to the following singular system of fractional equations:

$$
\begin{aligned}
&D_{0+}^{9/2}u(t) + \frac{(t-\frac{1}{2})^2 \ln(2+v(t))}{\sqrt{t}} = 0, \quad 0<t<1,\\
&D_{0+}^{13/3}v(t) + \frac{(t-\frac{1}{2})^2 \ln(2+u(t))}{\sqrt{t}} = 0, \quad 0<t<1,\\
&u(0)=u'(0)=u''(0)=u^{(3)}(0)=0, \qquad u(1)=\frac{1}{2}\int_0^{1/2} u(s)\,\mathrm{d}s,\\
&v(0)=v'(0)=v''(0)=v^{(3)}(0)=0, \qquad v(1)=\frac{1}{2}\int_0^{1/2} v(s)\,\mathrm{d}s.
\end{aligned}
\tag{6}
$$

Denote f, g are continuous on $(0,1]\times[0,+\infty)$ and $\lim_{t\to 0+} f(t,\cdot) = +\infty$, $\lim_{t\to 0+} g(t,\cdot) = +\infty$. Choose $\sigma_1=\sigma_2=1/2$ and $\varphi(y)=\psi(y)=\ln(2+y)$, then we have $\sqrt{t}(t-1/2)^2\ln(2+v(t))/\sqrt{t} \leqslant \ln(2+v(t))$ for all $(t,y)\in[0,1]\times[0,+\infty)$. $\varphi,\psi:[0,+\infty)\to(0,+\infty)$ are continuous, nondecreasing functions, so, condition (iii) of Theorem 2 holds. Next, set $r=1$. Then condition (iv) of Theorem 2 holds. Therefore, BVP (6) has at least one positive solution.

Example 3. Consider the boundary value problem to the following singular system of fractional equations:

$$
\begin{aligned}
&D_{0+}^{13/2}u(t) + \frac{(t+\frac{1}{10})^3 (v(t)+1)}{2\sqrt{t}} = 0, \quad 0<t<1,\\
&D_{0+}^{20/3}v(t) + \frac{(t+\frac{1}{3})^2 (u(t)+1)}{15\sqrt[3]{t}} = 0, \quad 0<t<1,\\
&u(0)=u'(0)=\cdots=u^{(5)}(0)=0, \qquad u'(1)=\frac{1}{2}\int_0^{1/3} u(s)\,\mathrm{d}s,\\
&v(0)=v'(0)=\cdots=v^{(5)}(0)=0, \qquad v''(1)=\frac{3}{4}\int_0^{2/3} v(s)\,\mathrm{d}s,
\end{aligned}
\tag{7}
$$

where $f(t,v(t)) = (t+1/10)^3(v(t)+1)/2\sqrt{t}$, $g(t,u(t)) = (t+1/3)^2(u(t)+1)/15\sqrt[3]{t}$, and f, g are continuous on $(0,1]\times[0,+\infty)$ and $\lim_{t\to 0+} f(t,\cdot) = +\infty$, $\lim_{t\to 0+} g(t,\cdot) = +\infty$. Choose $\sigma_1 = 1/2$, $\sigma_2 = 1/3$. Clearly, $|t^{\sigma_1} f(t,v_2(t)) - t^{\sigma_1} f(t,v_1(t))| \leqslant (1331/2000)\|v_2-v_1\|$, $|t^{\sigma_2} g(t,u_2(t)) - t^{\sigma_2} f(t,u_1(t))| \leqslant (16/135)\times \|u_2-u_1\|$ for all $t\in[0,1]$. By a simple calculation, we know $\tilde{L}<1$. All conditions of Theorem 3 hold. Therefore, BVP (7) has a unique solution.

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