A note on the max-sum equivalence of randomly weighted sums of heavy-tailed random variables*

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Abstract. This paper investigates the asymptotic behavior for the tail probability of the randomly weighted sums $\sum_{k=1}^{n}\theta_k X_k$ and their maximum, where the random variables X_k and the random weights θ_k follow a certain dependence structure proposed by Asimit and Badescu [1] and Li et al. [2]. The obtained results can be used to obtain asymptotic formulas for ruin probability in the insurance risk models with discounted factors.

Keywords: long-tailed distribution, randomly weighted sum, max-sum equivalence.

1 Introduction

Let $(X_1,\theta_1),\dots,(X_n,\theta_n)$ be n mutually independent random vectors, where $X_1,\dots,X_n$ are real-valued random variables (r.v.s) with distribution functions (d.f.s) $F_1,\dots,F_n$, respectively, and the random weights $\theta_1,\dots,\theta_n$ are nonnegative and nondegenerate at zero r.v.s with d.f.s $G_1,\dots,G_n$, respectively. For each $k=1,\dots,n$, X_k and θ_k

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can be dependent. For $n \geqslant 1$, denote the randomly weighted sum and its maximum, respectively, by

$$S_n^\theta = \sum_{k=1}^{n} \theta_k X_k \quad \text{and} \quad M_n^\theta = \max_{1\leqslant k\leqslant n} S_k^\theta. \tag{1}$$

Such randomly weighted sums and their maximums are often encountered in actuarial and financial situations. For instance, in a discrete-time risk model proposed by Nyrhinen in [3] and in [4], the real-valued r.v. X_k ($k = 1, \dots, n$) can be interpreted as the net loss of an insurance company (i.e. the total claim amount minus the total premium income) during period k, and the random weight θ_k ($k = 1, \dots, n$) can be regarded as the stochastic discount factor from time k to time 0. In this situation, the sum S_n^θ is the present value of all net losses from time 0 to time n and the maximum M_n^θ is the maximal discounted net loss of an insurance company during the first n periods.

In the present paper, we are interested in the asymptotic behavior (as $x \to \infty$) of tail probabilities $\mathbf{P}(S_n^\theta > x)$ and $\mathbf{P}(M_n^\theta > x)$, where the last probability can be understood as the probability of ruin during the first n periods with an initial capital reserve x.

In this paper, we use limit relationships only for x tending to infinity. For two positive functions $u(x)$ and $v(x)$: we write $u(x) \sim v(x)$ if $\lim u(x)/v(x) = 1$ and write $u(x) = o(v(x))$ if $\lim u(x)/v(x) = 0$. In addition, we denote by $x^+ = \max\{x, 0\}$ the positive part of a real number x. For any distribution function V, we denote its tail by $\overline{V}(x) = 1 - V(x)$ for all x. The indicator function of an event A we denote by $\mathbf{1}_A$.

Before discussing the asymptotic properties of probabilities $\mathbf{P}(S_n^\theta > x)$ and $\mathbf{P}(M_n^\theta > x)$ we recall the definitions of some classes of heavy-tailed d.f.s. A d.f. V on $[0, \infty)$ is called subexponential if $\overline{V^{*2}}(x) \sim 2\overline{V}(x)$, where V^{*2} denotes the convolution of V with itself. The class of all subexponential d.f.s, as usually, will be denoted by $\mathscr{S}$. A d.f. V on $[0, \infty)$ is said to belong to the class $\mathscr{L}$ of long-tailed d.f.s if for every positive y, we have $\overline{V}(x + y) \sim \overline{V}(x)$. A d.f. V supported on $[0, \infty)$ belongs to the class $\mathscr{D}$ (has dominatingly varying tail) if $\limsup \overline{F}(xy)/\overline{F}(x) < \infty$ for every fixed $y \in (0, 1)$. If a d.f. V is supported on $\mathbb{R}$, then V belongs to some of classes $\mathscr{S}$, $\mathscr{L}$, $\mathscr{D}$ if the d.f. $V(x)\mathbf{1}_{\{x\geqslant 0\}}$ belongs to the corresponding class. It is known (see, e.g., [5, Chap. 1.4]) that

$$\mathscr{L} \cap \mathscr{D} \subset \mathscr{S} \subset \mathscr{L}.$$

In the last years, a number of papers considering asymptotic behavior of $\mathbf{P}(S_n^\theta > x)$ and $\mathbf{P}(M_n^\theta > x)$ have been contributed to the case where $X_1, \dots, X_n$ are independent identically distributed (i.i.d.) r.v.s, independent of $\theta_1, \dots, \theta_n$, while there is no independence assumption and distribution identity assumption on $\theta_1, \dots, \theta_n$. For example, Tang and Tsitsiashvili [6] considered the case where $X_1, \dots, X_n$ have common subexponential d.f. and the random weights are two-sided bounded, i.e. $\mathbf{P}(a \leqslant \theta_k \leqslant b) = 1$ for all $k = 1, \dots, n$, and some $0 < a \leqslant b < \infty$. In [6], it was proved that for each $n \geqslant 1$

$$\mathbf{P}(M_n^\theta > x) \sim \mathbf{P}(S_n^\theta > x) \sim \sum_{k=1}^{n} \mathbf{P}(\theta_k X_k > x). \tag{2}$$

Similar results can be found in [7–10], among others. In particular, Chen et al. [9] obtained general result by considering nonidentically distributed r.v.s X_k having long-tailed d.f.s. Theorem 2.1 of [9] states that

$$\mathbf{P}(M_n^\theta > x) \sim \mathbf{P}(S_n^\theta > x) \sim \mathbf{P}\left(\sum_{k=1}^n \theta_k X_k^+ > x\right) \tag{3}$$

if the following conditions are satisfied: r.v.s $X_1, \ldots, X_n$ are independent; F_k is long-tailed for each $k = 1, \ldots, n$; $\theta_1, \ldots, \theta_n$ are such that $\mathbf{P}(a \leqslant \theta_k \leqslant b) = 1$ for $k = 1, \ldots, n$ and some $0 < a \leqslant b < \infty$; the sequences $\{X_1, \ldots X_n\}$, $\{\theta_1, \ldots, \theta_n\}$ are mutually independent. In addition, Theorem 2.2 of [9] shows that asymptotic relation (3) still holds for bounded from above random weights, assuming some restriction on the dependence structure of $\{\theta_1, \ldots, \theta_n\}$.

In the present paper, motivated by the results in [9], we study asymptotic behavior of r.v.s in the case of nonidentically distributed r.v.s $X_1, \ldots, X_n$. We also suppose that random vectors $(X_1, \theta_1), \ldots, (X_n, \theta_n)$ are mutually independent, whereas some dependence structure exists between X_k and θ_k for each $k = 1, \ldots, n$. For each pair (X_k, θ_k), we use the dependence structure which was introduced by Asimit and Badescu [1], i.e., for each fixed $k = 1, \ldots, n$, there exists a measurable function $h_k : [0, \infty) \to (0, \infty)$ such that

$$\mathbf{P}(X_k > x \mid \theta_k = t) \sim \overline{F}_k(x) h_k(t) \tag{4}$$

uniformly for $t \geqslant 0$, where the uniformity is understood as

$$\lim_{x\to\infty} \sup_{t\geqslant 0} \left| \frac{\mathbf{P}(X_k > x \mid \theta_k = t)}{\overline{F}_k(x) h_k(t)} - 1 \right| = 0.$$

When t is not a possible value of some θ_k, the conditional probability in (4) is understood as unconditional and therefore $h_k(t) = 1$ for such t.

Some examples of the r.v.s satisfying dependence condition (4) can be found in [1] and [2]. These examples are constructed using the Ali–Mikhail–Haq, the Farlie–Gumbel–Morgenstern and the Frank copulas.

Note that Yang et al. [11] obtained relation (2) in the case of dependence (4), when $X_1, \ldots, X_n$ are i.i.d. real-valued r.v.s with common distribution $F \in \mathscr{S}$, and $\theta_1, \ldots, \theta_n$ are bounded from above, i.e. $\mathbf{P}(0 \leqslant \theta_k \leqslant b) = 1$ for all $k = 1, \ldots, n$ and some positive constant b. In this paper, we consider a more general case where $F_1, \ldots, F_n$ can be different and $\theta_1, \ldots, \theta_n$ can be unbounded. We establish relation (3) as in [9] under dependence relation (4) and the assumption that $F_1, \ldots, F_n$ are in $\mathscr{L}$. In the case when $F_1, \ldots, F_n$ belong to the class $\mathscr{L} \cap \mathscr{D}$, we obtain relation (2).

The following statement is the main result of the paper. We remark only that in this main assertion, we suppose $\theta_1, \ldots, \theta_n$ to be strictly positive.

Theorem 1. *Suppose that $(X_1, \theta_1), \ldots, (X_n, \theta_n)$ are mutually independent random vectors, where $X_1, \ldots, X_n$ are real-valued r.v.s with d.f.s $F_1, \ldots,$ F_n, respectively, and $\theta_1, \ldots, \theta_n$ are positive r.v.s with d.f.s $G_1, \ldots, G_n$, respectively. Assume that, for each* *fixed* $k = 1, \dots, n$*, the pair* (X_k, θ_k) *satisfies condition* (4)*. If, for each* $k = 1, \dots, n$*,* $F_k \in \mathscr{L}$ *(respectively,* $F_k \in \mathscr{L} \cap \mathscr{D}$*) and* $\overline{G}_k(x) = o(\overline{F}_k(c_k x))$ *for some positive* c_k*, then relation* (3) (*respectively,* (2)) *holds.*

In the insurance context, researchers are often interested in asymptotic behavior of ruin probability $\mathbf{P}(M_n^\theta > x)$. According to relation (3), in order to obtain asymptotics for this probability, it suffices to find asymptotics of the tail $\mathbf{P}(\sum_{k=1}^n \theta_k X_k^+ > x)$. Theorem 1 states that relation (3) holds in the case $F_k \in \mathscr{L}$, $k = 1, \dots, n$, and dependence structure (4). If, in addition, $F_k \in \mathscr{L} \cap \mathscr{D}$, $k = 1, \dots, n$, then due to relation (2) we can obtain asymptotic formula of ruin probability from the asymptotics of discounted net losses $\mathbf{P}(\theta_k X_k > x)$, $k = 1, \dots, n$. In both cases, the required asymptotics depend on d.f.s F_k, G_k, $k = 1, \dots, n$, and on functions h_k, $k = 1, \dots, n$, given in (4).

2 Proof of Theorem 1

The following lemmas will be used in the proof of Theorem 1. The first lemma is due to Lemma 2.1 in [11].

Lemma 1. *Let* ξ *be a real-valued r.v. with distribution* F_ξ*, and let* η *be a nonnegative and nondegenerate at zero r.v. with distribution* F_η*. Assume that there exists a measurable function* $h : [0, \infty) \to (0, \infty)$ *such that*

$$\mathbf{P}(\xi > x \mid \eta = t) \sim \overline{F}_\xi(x) h(t) \tag{5}$$

uniformly for all $t \in [0, \infty)$*. If* $F_\xi \in \mathscr{L}$ *and* $\overline{F}_\eta(x) = o(\overline{F}_\xi(cx))$ *for some* $c > 0$*, then the d.f.* $F_{\xi\eta}$ *of the product* $\xi\eta$ *belongs to* $\mathscr{L}$*.*

The second lemma shows that similar statement holds for the class of d.f.s with dominatingly varying tails.

Lemma 2. *Let* ξ *be a real-valued r.v. and* η *be a nonnegative and nondegenerate at zero r.v., such that relation* (5) *holds. If* $F_\xi \in \mathscr{D}$ *and* $\overline{F}_\eta(x) = o(\overline{F}_\xi(x))$*, then* $F_{\xi\eta} \in \mathscr{D}$*.*

Proof. It suffices to prove that

$$\liminf \frac{\overline{F}_{\xi\eta}(2x)}{\overline{F}_{\xi\eta}(x)} > 0. \tag{6}$$

According to (5) and definition of the class $\mathscr{D}$, there exist $c_1 > 0$ and $D \geqslant 2$ such that

$$\frac{1}{2}\,\overline{F}_\xi(z)h(t) \leqslant \mathbf{P}(\xi > z \mid \eta = t) \leqslant \frac{3}{2}\,\overline{F}_\xi(z)h(t) \quad \text{and} \quad \overline{F}_\xi(2z) \geqslant c_1 \overline{F}_\xi(z)$$

for all $z \geqslant D/2$ and $t \geqslant 0$.

For x sufficiently large, the bounds above imply that

$$
\begin{aligned}
\overline{F}_{\xi\eta}(2x) &= \int\limits_{(0,\infty)} \mathbf{P}\biggl(\xi > \frac{2x}{y} \biggm| \eta = y\biggr)\,\mathrm{d}F_\eta(y) \\
&\geqslant \frac{1}{2} \int\limits_{(0,2x/D]} \mathbf{P}\biggl(\xi > \frac{2x}{y}\biggr) h(y)\,\mathrm{d}F_\eta(y) \\
&\geqslant \frac{c_1}{2} \int\limits_{(0,2x/D]} \mathbf{P}\biggl(\xi > \frac{x}{y}\biggr) h(y)\,\mathrm{d}F_\eta(y) \\
&\geqslant \frac{c_1}{3} \int\limits_{(0,2x/D]} \mathbf{P}\biggl(\xi > \frac{x}{y} \biggm| \eta = y\biggr)\,\mathrm{d}F_\eta(y) \\
&= \frac{c_1}{3} \Biggl(\overline{F}_{\xi\eta}(x) - \int\limits_{(2x/D,\infty)} \mathbf{P}\biggl(\xi > \frac{x}{y} \biggm| \eta = y\biggr)\,\mathrm{d}F_\eta(y)\Biggr) \\
&\geqslant \frac{c_1}{3} \biggl(\overline{F}_{\xi\eta}(x) - \overline{F}_\eta\biggl(\frac{2x}{D}\biggr)\biggr).
\end{aligned}
$$

Therefore,

$$
\liminf \frac{\overline{F}_{\xi\eta}(2x)}{\overline{F}_{\xi\eta}(x)} \geqslant \frac{c_1}{3}\biggl(1 - \limsup \frac{\overline{F}_\eta(2x/D)}{\overline{F}_{\xi\eta}(x)}\biggr).
$$

Hence, (6) will follow if we show that

$$
\limsup \frac{\overline{F}_\eta(2x/D)}{\overline{F}_{\xi\eta}(x)} = 0. \tag{7}
$$

The last relation can be proved in the same manner as relation (2.8) in [11]. Namely, if η is bounded (and nondegenerate at zero according to conditions of the lemma), then there exists $c_2 > 0$ such that $\mathbf{E}\bigl(h(\eta)\mathbf{1}_{\{\eta \geqslant c_2\}}\bigr)$ is positive and thus by (5)

$$
\begin{aligned}
\limsup \frac{\overline{F}_\eta(2x/D)}{\overline{F}_{\xi\eta}(x)} &= \limsup \frac{\overline{F}_\eta(2x/D)}{\int_{(0,\infty)} \mathbf{P}(\xi > x/y \mid \eta = y)\mathrm{d}F_\eta(y)} \\
&\leqslant \limsup \frac{\overline{F}_\eta(2x/D)}{\int_{[c_2,\infty)} \mathbf{P}(\xi > x/c_2 \mid \eta = y)\mathrm{d}F_\eta(y)} \\
&= \limsup \frac{\overline{F}_\eta(2x/D)}{\overline{F}_\xi(x/c_2)\int_{[c_2,\infty)} h(y)\mathrm{d}F_\eta(y)} = 0.
\end{aligned}
$$

If η is unbounded, then $\overline{F}_\eta(x) > 0$ for all x, and assumption (5) together with condition $\overline{F_\eta}(x) = o(\overline{F_\xi}(x))$ of the lemma imply

$$\limsup \frac{\overline{F}_\eta(x)}{\overline{F}_{\xi\eta}(xD/2)} \leqslant \limsup \frac{\overline{F}_\eta(x)}{\int_{[D/2,\infty)} \mathbf{P}(\xi > x \mid \eta = y)\mathrm{d}F_\eta(y)} = \frac{1}{\mathbf{E}(h(\eta)\mathbf{1}_{\{\eta \geqslant D/2\}})} \limsup \frac{\overline{F}_\eta(x)}{\overline{F}_\xi(x)} = 0$$

for every fixed positive D. Hence, the estimate (7) holds in both cases and the lemma is proved. $\square$

The following statement is due to [12] and shows that the class $\mathscr{L} \cap \mathscr{D}$ is closed under convolution of different d.f.s and has the max-sum equivalence property.

Lemma 3. (*See* [12, *Thm.* 2.1].) *If d.f.s* $V_1 \in \mathscr{L} \cap \mathscr{D}$, $V_2 \in \mathscr{L} \cap \mathscr{D}$, *then* $V_1 * V_2 \in \mathscr{L} \cap \mathscr{D}$ *and* $\overline{V_1 * V_2}(x) \sim \overline{V}_1(x) + \overline{V}_2(x)$.

The next lemma follows from Theorem 2.1 in [9].

Lemma 4. *Assume that* $Y_1, Y_2, \ldots$ *are independent real-valued r.v.s such that d.f. of* Y_k *is long-tailed for each* $k = 1, 2, \ldots$. *Then, for each* $n = 1, 2, \ldots$, *it holds*

$$\mathbf{P}\left(\sum_{k=1}^n Y_k > x\right) \sim \mathbf{P}\left(\sum_{k=1}^n Y_k^+ > x\right). \tag{8}$$

Proof of Theorem 1. First, consider the case where $F_k \in \mathscr{L}$ for all $k = 1, \ldots, n$. Since $S_n^\theta \leqslant M_n^\theta \leqslant \sum_{k=1}^n \theta_k X_k^+$, it suffices to prove

$$\mathbf{P}\left(\sum_{k=1}^n \theta_k X_k > x\right) \sim \mathbf{P}\left(\sum_{k=1}^n \theta_k X_k^+ > x\right). \tag{9}$$

Relation (9) follows from Lemma 4, noting that $(\theta_k X_k)^+ = \theta_k X_k^+$ and that d.f. of $\theta_k X_k$ belongs to $\mathcal{L}$ by Lemma 1 for each $k = 1, \ldots, n$.

In the case $F_k \in \mathcal{L} \cap \mathcal{D}$, the result follows immediately from the obtained asymptotic relations and Lemmas 1–3. Indeed, by Lemma 1 and Lemma 2, for each k, r.v. $\theta_k X_k^+$ belongs to $\mathscr{L} \cap \mathscr{D}$. Since vectors $(X_1, \theta_1), \ldots, (X_n, \theta_n)$ are independent, Lemma 3 implies that

$$\mathbf{P}\left(\sum_{k=1}^n \theta_k X_k^+ > x\right) \sim \sum_{k=1}^n \mathbf{P}\big(\theta_k X_k^+ > x\big),$$

where $\mathbf{P}(\theta_k X_k^+ > x) = \mathbf{P}(\theta_k X_k > x)$ for $x \geqslant 0$. This and obtained asymptotic relation (3) proves (2) and, hence, the theorem. $\square$

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