Multiple cycles and the Bautin bifurcation in the Goodwin model of a class struggle

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Abstract. The aim of this paper is to present the necessary and sufficient conditions for the emergence of a generalized Hopf (i.e., Bautin) bifurcation in the Goodwin's model of a class struggle, and determine the parameter regions where multiple attracting and repelling limit cycles around the steady state may coexist.

Keywords: global indeterminacy, distibutive conflict, business cycles, Goodwin model.

1 Introduction

In the field of economic growth, a huge interest has been devoted to explain the mechanisms through which a nonlinear system looses its stability and starts to oscillate around the steady state. Since the seminal Kalekian analysis, limit cycles represent the theoretical way of characterizing the emergence of persistent oscillations of macro-economic variables, namely the rise of indeterminacy problems (see [1]).

In the quest for explaining the emergence of periodic fluctuations in an economic system the powerful tool of the Andronov–Hopf bifurcation theorem has been employed to derive the set of necessary and sufficient conditions for the rise of limit cycles (see, for example, [2]). Unfortunately, this theorem is not able to tell the full story of the global behavior of the dynamical system, which can exhibit a more complicated picture in the large, when some degenerate conditions occur (see, for example, [3–5]). That is, depending on the parameters configuration, we may obtain both (i) uniqueness of stable limit cycles (i.e., Hopf bifurcation), or (ii) multiple limit cycles of opposite stability (i.e., Bautin bifurcation), that eventually collide and disappear.

An interesting problem to be investigated occurs, in fact, when the first Lyapunov coefficient vanishes, thus letting cycles of different orientation to coexist. In this case, a second order genericity condition, the second Lyapunov coefficient, must be computed, whose control leaves the door open to unexpected outcomes, which is commonly known as a Bautin bifurcation.

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The Bautin bifurcation (also known as a generalized Hopf bifurcation) has found very interesting applications especially in psychology and physics, but has been quite neglected in the economic analysis, mostly because of the mathematical computation behind it.

Basically, whenever a system undergoes a Bautin bifurcation, the structure of the emerging cycles is quite complex, given that, families of stable and unstable limit cycles (i.e., one inside the other) coexist, collide and eventually disappear via a saddle-node bifurcation. The utility of the Bautin bifurcation is that it allows to control for both the stability of the equilibrium point and the orientation of the limit cycle emerging around it, and is particularly important for the understanding of the global behavior of the system being studied, which displays a very interesting bifurcation diagram. In particular, it represents a co-dimension 2 singularity, that is it requires two different independent parameters to be varied for the bifurcation to occur.

Since we believe this might be of particular interest in explaining the periodic fluctuations of an economy, we deeply investigate the global properties of this odd bifurcation, and apply it to the Goodwin model of a class struggle between workers and capitalists, which has been considered in the literature as the classic framework to formalize the Marxian theory of distributive conflict, through the Lotka–Volterra predator-prey model, borrowed from mathematical ecology. We finally show that, the emergence of endogenous cycles in an economic activity is characterized by equilibrium trajectories that converge to the inner steady state, for small parameter shocks, but exhibit persistent fluctuations, and move towards the outer stable limit cycle, for larger disturbances.

A vast literature exists trying to examine the possibility of the cyclical growth in the Goodwinian economy. The Marxian inspiration behind the original Goodwin model [6] is that lowering the wage rate enhances output growth and employment, which consequently increases claims for new (rising) wage renewals. This decreases labour productivity, and eventually depresses economic growth, by pushing workers back in the area of unemployment, in a perpetual and unavoidable oscillating pattern.

Due to its conservative nature, different generalizations of the original framework have been proposed to explore the connections between monetary and financial forces and real factors in the determination of long-lasting fluctuations (see [7–9]). Interestingly, conditions for stability versus cyclical fluctuations are proved via the Hopf bifurcation theorem, if a dynamic view of technical change is adopted, thus assuming endogenous technical change as a means for never-ending conflict cycles (see [10, 11]).[1]

More recently, Matsumoto [17] has characterized the global dynamics of a time delay version of the Goodwin nonlinear accelerator model, to show the possible emergence, and coexistence, of multiple limit cycles, that is a stable limit cycle coexists with an unstable one that encloses the stable stationary point, through the Hopf bifurcation theorem and the Poincaré–Bendixon theorem. Results are particularly interesting for they show that a stable region characterized by an unstable limit cycle is surrounded by a stable one, that is, the dynamic motion returns to the stationary point for small disturbances but becomes unstable, and exhibits persistent fluctuations, for large disturbances. Further complex analysis has been made on the same framework in [18], by showing a large

[1] Interesting applications of the Hopf bifurcation theorem can be also found in [12–16].

dynamic sensitivity of the model for a particular set of the bifurcation parameters, and the additional possible emergence of different chaotic attractors.

The rest of the paper is organized as follows. In Section 2, we present the model, and derive the steady state conditions. Additionally, we study the local dynamics, and define the conditions for the Bautin singularity to emerge. In section 3, we outline a practical application to provide an economic interpretation of our results. A final section draws some concluding remarks, and suggestions for future research.

2 A Bautin bifurcation in the Goodwin model

Consider the following generalized version of the Goodwin model [6] of a class struggle (see [19]).[2]

$$\begin{aligned} \dot{v} &= \left[\frac{1-u}{\sigma} - (\phi + n)\right] v, \\ \dot{u} &= \left[\psi(v,u) - \phi\right] u \end{aligned} \qquad (\mathcal{S})$$

with the set of parameters $\Omega = (\sigma, \phi, n, \rho, \gamma)$[3], and where $\psi(v,u) = \dot{w}/w$ is the growth rate of the real wage rate, which represents an extended Phillips curve relationship, depending both on the rate of employment, v, and the labor income share, u (see [21]).[4] The formal structure of $(\mathcal{S})$ clearly resembles a Lotka–Volterra system, where the wage rate (w), through the labor income share (u) is the predator, and the employment rate (v) is the prey. Following standard textbooks, we can adopt an additive functional form

$$\psi(u,v) = f(v) + g(u), \qquad (1)$$

where $f(v) = \rho v - \gamma$ with $(\rho, \gamma) \in \mathbb{R}^+$, is the standard Phillips curve relation of the original Goodwin's model [6], and $g(u) = (1-u)^\eta > 0$ for all u, given $\eta > 0$, is the explicit additional term we introduce in the model, with $g_u = -\eta(1-u)^{\eta-1} < 0$, saying that workers claim higher wages when they experience a disadvantage in the income distribution.[5]

[2]In a successive Goodwin's model [20], a more elegant version of the class struggle is presented, by characterizing the conditions for the possible emergence of multiple cycles. This is done by assuming a growing labor productivity, that moves the equilibrium solution from cycle to cycle. Notwithstanding, the wage growth rate still depends on the employment rate solely, (i.e., $\dot{w}/w = f(v)$), which is, on the contrary, more generally defined in the model presented herein, as a key determinant for raising multiple cycles.

[3]Generally, σ is the capital output ratio, ϕ is the growth rate of labor productivity, n is the growth rate of labor supply.

[4]See also [22] for a critique to a linearized Phillips curve, as a rise in firm's profits is not neutral, but pushes unions towards higher wage claims, which mathematically writes $\partial\psi/\partial(1-u) > 0$.

[5]As clearly pointed out in [14], a generalized Phillips curve, $F(v,u)$, can be represented as a nonlinear function, both depending on v and u, that is also to say on v and its growth rate $\hat{v} \equiv \dot{v}/v = u$. Therefore, we can rewrite $F(u,v) = f(v,\hat{v})$, such that $F_v = f_v + f_{\hat{v}}(\partial\hat{v}/\partial v) > 0$ and $F_u = f_{\hat{v}}(\partial\hat{v}/\partial u) < 0$.

Given the non-trivial steady state values

$$v^* = \frac{\phi + \gamma - \sigma^\eta (\phi + n)^\eta}{\rho},$$
$$u^* = 1 - \sigma(\phi + n),$$

the Jacobian matrix associated to ($\mathcal{S}$), at (v^*, u^*), becomes

$$J^* = \begin{bmatrix} 0 & [\sigma(\phi+n)]^\eta - \phi - \gamma/(\sigma\rho) \\ \rho[1-\sigma(\phi+n)] & -\eta[\sigma(\phi+n)]^{\eta-1}[1-\sigma(\phi+n)] \end{bmatrix} \tag{2}$$

with the corresponding characteristic equation

$$\det\bigl(\lambda \mathbf{I} - J^*\bigr) = \lambda^2 - \tau\lambda + \Delta, \tag{3}$$

where

$$\tau = -\eta\bigl[\sigma(\phi+n)\bigr]^{\eta-1}\bigl[1-\sigma(\phi+n)\bigr], \tag{4a}$$
$$\Delta = -\bigl[1-\sigma(\phi+n)\bigr]\frac{[\sigma(\phi+n)]^\eta - \phi - \gamma}{\sigma} \tag{4b}$$

represent the trace (τ) and the determinant (Δ) associated to the Jacobian matrix, J^*, respectively.[6]

A pair of complex conjugate eigenvalues emerges only if

$$\lambda_{1,2} = \mu(\eta) \pm \mathrm{i}\omega(\eta), \tag{5a}$$

where

$$\mu(\eta) = \frac{\eta\sigma^{\eta-1}(\phi+n)^{\eta-1}[\sigma(\phi+n)-1]}{2}, \tag{5b}$$
$$\omega(\eta) = 2\sqrt{\frac{[\sigma(\phi+n)-1][\sigma^\eta(\phi+n)^\eta - \phi - \gamma] - \sigma\mu^2(\eta)}{\sigma}} \tag{5c}$$

with η as the Hopf-bifurcation parameter.

As clearly depicted in Fig. 1, limit cycles do emerge at $\eta = 0$, and Eqs. (5b) and (5c) reduce to

$$\mu(0) = 0,$$
$$\omega(0) = \omega_0 = 2\sqrt{\frac{[1-\sigma(\phi+n)][\phi+\gamma-1]}{\sigma}}$$

where $\omega_0 > 0$ by assuming $\phi > 1 - \gamma$.[7]

In the task to justifying the emergence of multiple limit cycles, in what follows we characterize the necessary and sufficient conditions for the Bautin singularity to occur, that is an unstable limit cycle coexists with (and encloses) the stable one. We do this by means of the following.

[6] In the original Goodwin model [6] the capitalist economy is permanently oscillating, that is $\tau = 0$.

[7] Following [22], we assume $\alpha = 0.001$, $\beta = 0.001$, and set $\gamma = 2.54$, $\rho = 1$, and $\sigma = 2$.

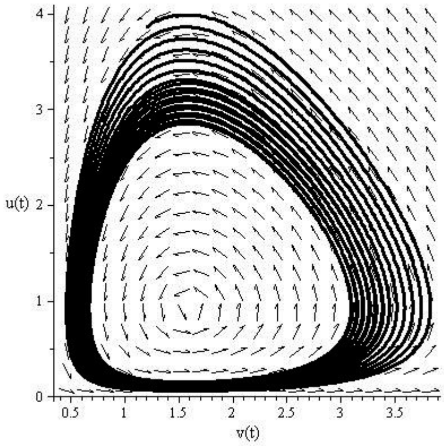


Fig. 1. The Goodwin limit cycle.

Theorem 1 (Bautin bifurcation). *Assume the system*

$$\dot{x} = f(x, \eta), \quad x \in \mathbb{R}^2,\ \eta \in \mathbb{R}^2,$$

with f smooth, and associated complex conjugate eigenvalues $\lambda_{1,2} = \mu(\eta) \pm \mathrm{i}\omega(\eta)$ with $\omega(0) = \omega_0 > 0$. A Bautin bifurcation holds if, for $\eta = 0$, we verify that

$$\text{(i)}\ \mu(0) = 0, \qquad \text{(ii)}\ \ell_1 = 0, \qquad \text{(iii)}\ \ell_2 \neq 0,$$

where ℓ_1 and ℓ_2 are the first and second Lyapunov coefficients, respectively.

Proof. See [23]. □

The application of Theorem 1 needs to put system $(\mathcal{S})$ in a more convenient normal form to work with. That is, via the transformation matrix $\mathbf{T} = \left[\begin{smallmatrix} v^*/\sigma & 0 \\ 0 & \omega_0 \end{smallmatrix}\right]$, and the following change of variables $(v, u)^{\mathrm{T}} = \mathbf{T}(w_1, w_2)^{\mathrm{T}}$, we derive:

$$\dot{z} = F(z) + G(z)x, \tag{$\mathcal{M}$}$$

where $z = (w_1, w_2)^{\mathrm{T}} \in \mathbb{R}^2$ is the vector of state variables[8], $F(z), G(z) : \mathbb{R}^2 \to \mathbb{R}^2$ are smooth functions, $G(z)x$ is the control input with $x = x(z, \mu, \beta) = \beta_1\mu_1 + (\beta_2 + \mu_2) \times (w_1^2 + w_2^2) + \beta_3(w_1^2 + w_2^2)^2$, and the set of artificial parameters $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$, and $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$.[9]

[8] It is easy to note that w_1 becomes an auxiliary variable for the rate of employment, v, and w_2 is a simple linear transformation of the original labor income share, u. Cyclical oscillation of the auxiliary variables (w_1, w_2) are thus topologically equivalent to the economic fluctuations in the vector field of the original variables (v, u).

[9] In the last years, increasing attention has been devoted to analyze systems displaying complex dynamics, aimed at finding the appropriate method to control and stabilize the dynamical behavior of a system around bifurcation points (see [24]). The so-called "bifurcation control" is made through the manipulation of the aforementioned artificial parameters, which can be interpreted as the measure of both fiscal and monetary policy actions made by a government to stabilize an economy subject to unwanted cyclical fluctuations.

Assume that $F(0) = 0$, $G(0) = 0$, and $J = \mathrm{d}F(0) = \left(\begin{smallmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{smallmatrix}\right)$ has a pair of pure imaginary eigenvalues $\lambda_{1,2} = \pm \mathrm{i}\omega_0$ with $\omega_0 \geqslant 0$. A candidate couple of associated eigenvectors is therefore given by $q = (1, -\mathrm{i})^{\mathrm{T}}$, and $p = (1/2, -\mathrm{i}/2)^{\mathrm{T}}$, such that

$$\langle q, p \rangle = 1,$$

where $\langle \cdot, \cdot \rangle$ refers to the scalar product in $\mathbb{R}^2$: $\langle q, p \rangle = q_1 p_1 + q_2 p_2$.

Let also:

$$M = \mathrm{d}G(0) = \begin{pmatrix} \partial G_1/\partial w_1 & \partial G_1/\partial w_2 \\ \partial G_2/\partial w_1 & \partial G_2/\partial w_2 \end{pmatrix} \tag{6}$$

with $\mathrm{tr}(M) = \mathrm{tr}(\mathrm{d}G(0)) \neq 0$. Then there exist $\beta_1, \beta_2 \in \mathbb{R}$ such that system ($\mathcal{M}$) undergoes a Hopf bifurcation at $\mu_1 = 0$, provided that $\beta_2 \neq \hat{\beta}_2$. In this case, it is possible to control the stability and the direction of the limit cycles emerging near the origin by selecting positive or negative β_1 and $\beta_2 \gtrless \hat{\beta}_2$, where $\hat{\beta}_2$ is a solution of $\ell_1 = 0$. In addition, the system undergoes a Bautin bifurcation at $\mu_1 = \mu_2 = 0$, for $\beta_3 \neq \hat{\beta}_3$, and it is therefore possible to control the stability and the direction of the limit cycle emerging near the origin by selecting positive or negative β_1, $\mathrm{tr}(M)$ and $\beta_3 \gtrless \hat{\beta}_3$ where $\hat{\beta}_3$ is a solution of $\ell_2 = 0$ (see [25]).

Taylor expanding system ($\mathcal{M}$) around $z = 0$ we obtain

$$\dot{z} = Jz + \mathcal{F}(z), \tag{$\mathcal{Q}$}$$

whose elements, in correspondence of the Hopf bifurcation parameter $\mu_1 = 0$, reduce to:

$$\mathcal{F}(z) = \frac{1}{2} B(z, z) + \frac{1}{3!} C(z, z, z), \tag{7}$$

where $B(z, z) = \mathrm{d}^2 F(0)(z, z)$, and $C(z, z, z) = \mathrm{d}^3 F(0)(z, z, z) + 6(\beta_2 + \mu_2) \times (w_1^2 + w_2^2) M z$.

Before applying Theorem 1, we assume hereafter $G(z) = sz$, with $s \neq 0$ being a parameter which measures the size of the adopted policy law (see [24]).[10] Following simple algebra, this implies that $G(0) = 0$, and $\mathrm{tr}(M) = 2s \neq 0$.

The first Lyapunov coefficient, at the bifurcation point ($\mu_1 = 0$), can be easily computed by means of the following general formula (see [23]:

$$\ell_1(0) = \frac{1}{2\omega^2} \mathrm{Re}(\mathrm{i} g_{20} g_{11} + \omega g_{21}),$$

where $g_{20} = \langle p, B(q, q) \rangle$, $g_{11} = \langle p, B(q, \bar{q}) \rangle$, $g_{21} = \langle p, C(q, q, \bar{q}) \rangle$. It follows consequently:

$$\ell_1 = \frac{1}{2\omega_0^2} + \frac{4s}{\omega_0}(\beta_2 + \mu_2)$$

[10] To simplify the analysis, we use a first order Taylor expansion of the $G(z)$ vector field. We are thus able to capture the direct feedbacks of any policy intervention on the assumed control law term, $G(z)x$, when the leading bifurcation parameters are manipulated to stabilize the emerging cyclical economic process.

such that ℓ_1 vanishes at the bifurcation point $\mu_2 = 0$, in correspondence of the critical *leading parameter* value

$$\hat{\beta}_2 = -\frac{1}{8s\omega_0}.$$

Another genericity condition for the Bautin bifurcation to occur is that the second Lyapunov coefficient be different from zero. We do not provide the general formula, for which we invite the reader to refer to [23], and give instead the general output

$$\ell_2 = \frac{16s\beta_3}{\omega_0},$$

which vanishes at the critical value $\hat{\beta}_3 = 0$.

3 Practical application

The global dynamic behavior of system ($\mathcal{Q}$) for different values of the unfolding parameters (μ_1, μ_2) can be summarized in the following bifurcation diagram (see Fig. 2).

Starting from region 1, and moving counterclockwise, we notice that the system exhibits a single stable equilibrium, which hopf bifurcates and gives rise to a stable limit cycle when passing through region 2. A Bautin bifurcation occurs in region 3, as far as we approach the separatrix $T = \{(\mu_1, \mu_2)\colon\ \mu_2^2 + 4\mu_1 = 0,\ \mu_2 > 0\}$ along which the stability direction of the limit cycle is reverted.

Indeed, we notice that as long as the bifurcation parameter μ_2 approaches to zero, the presence of multiple cycles of different orientation emerges, and the Bautin bifurcation finally occurs.

The control law is therefore designed so that the amplitude of the occurring bifurcating solution is reduced, and the dynamic behavior of the system around the bifurcation point is modified. We may interpret the bifurcation and control parameters as the appropriate fiscal and monetary tools, whose manipulation may represent the policy actions made by a government to stabilize an economy subject to unwanted cyclical fluctuations.

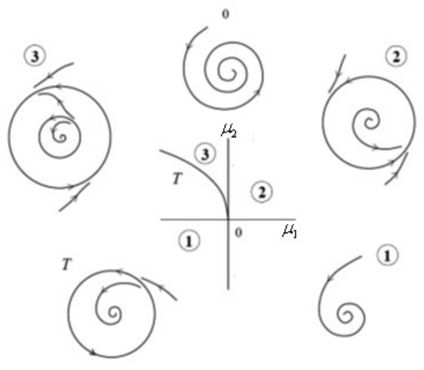
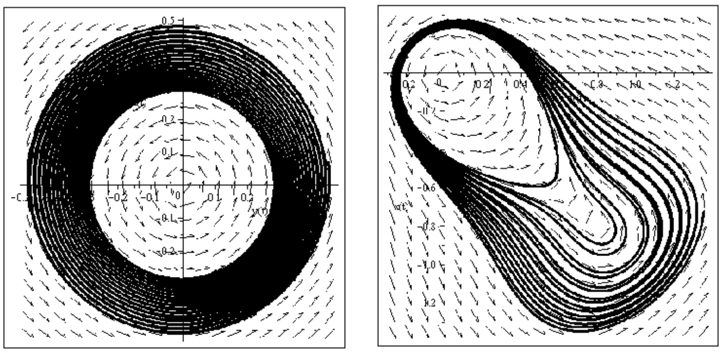


Fig. 2. The Bautin bifurcation diagram.

To clarify our statement, we concentrate our attention on the case with $s = -1$, and provide an example to show the behavior of a system undergoing the Bautin bifurcation, by choosing different values for β_1, β_2, β_3, μ_1, μ_2, that allow us to locate the economy in the region 3 of the Bautin bifurcation diagram.[11]

If we want to study the possible effects of a cut in public subsidies, let assume, for example, μ_1 as the rate of subsidies for housing, and μ_2 as the inflation rate. In this case, $\beta_1 > 0$ can measure an increase in the demand for housing due to the lowering cost of mortgages, while $\beta_2 < 0$ might represent a decrease in the real interest rate, with $\beta_3 > 0$ signalling an increase in the existing money supply.

Fig. 3 shows, in the auxiliary variables (w_1, w_2) space, how the economy might give rise to a cycling behavior, as long as a policy to reduce the amount of public subsidies from the budget to housing development is conducted (i.e., $\mu_1 \to 0$), along with the adoption of a different inflation targeting. Interestingly, we may notice that if the policymaker acts to reduce and control inflation ($\mu_2 = 0.5$) the waves of business cycle can be stabilized. On the contrary, an inflationary rate out of control ($\mu_2 = 2$) allows for the emergence of multiple limit cycles of different orientation (the Bautin bifurcation) along which the economy is trapped in an unavoidable oscillating pattern.[12]



μ_2=0.5 μ_2=2

Fig. 3. The $w_1 - w_2$ oscillating space.

[11]The case with $s = 1$ can be reduced to the one analyzed via a convenient time-rescaling and variable transformation.

[12]It is worth noting that Fig. 3 assures the possibility of negative values for the auxiliary variables (w_1, w_2). The following counterintuitive example may serve to provide an economic interpretation of this finding. As recently shown by the IMF predictions, it can be estimated that the US unemployment rate will hit 0.0% in December of 2021, and finally turn negative, around -0.1%, in January 2022. This is justified by the fact that increasing the country's various job replacement programs may act as a dropping device of the labor force out of the employment pool for good, which is where most of those who are "unemployed" will end up, thus both flattening the number of those unemployed, and finally soaring the number of those actually employed. Moreover, as shown in [26] using data from the US National Income and Product Accounts (NIPA), in the past 30 years, a decline in labor income share has been evident, and mainly due to the rapid grow of top executive salaries, joint with the relative ease with which current employees can be replaced by the reserve army of unemployed job seekers, which reduces the workforce bargaining power in negotiating wages. Real wages thus inivitably fall, with a rise in productivity, and a subsequent labor income share to fall decisively.

4 Concluding remarks

We analyzed a generalized version of the standard Goodwin model [6] of a class struggle to show the possibility of emerging multiple limit cycles of different orientation. To this end, we developed the whole steps of necessary and sufficient conditions to verify the emergence of a Bautin bifurcation, and validate the theory by means of some numerical examples through the change of the appropriate control parameters. Besides its complicate mathematical structure, this technique is able to provide very interesting results, and a promising application both in business cycles or financial and monetary problems.

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