

Estimation of parameters of finite population L -statistics

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Abstract. We consider the estimation of important parameters of a linear combination of order statistics (L -statistic) in a finite population, emphasizing the influence of auxiliary information on the estimation accuracy. Assuming that values of an auxiliary variable are available for all population units, we construct calibrated estimators for the variance of L -statistics and for the parameters, which define one-term Edgeworth expansions of distributions of L -statistics. The gain of the new estimators is demonstrated by the simulation study.

Keywords: finite population, L -statistic, variance, Edgeworth expansion, bootstrap, calibration.

1 Introduction

Consider a study variable x with the real values $\mathcal{X} = \{x_1, \dots, x_N\}$ in the finite population $\mathcal{U} = \{1, \dots, N\}$. Assume that an auxiliary (completely known) real variable z is available with the values $\mathcal{Z} = \{z_1, \dots, z_N\}$. Let $\mathbb{X} = \{X_1, \dots, X_n\}$ be the measurements of the simple random sample units $\{1, \dots, n\}$, $n < N$, drawn without replacement from the population. Let $\mathbb{Z} = \{Z_1, \dots, Z_n\}$ be the corresponding measurements of the variable z . The L -statistic

$$L = L_n(\mathbb{X}) = \frac{1}{n} \sum_{j=1}^n c_j X_{j:n} \quad (1)$$

is a linear combination of the order statistics $X_{1:n} \leq \dots \leq X_{n:n}$ with the real coefficients

$$c_j = J\left(\frac{j}{n+1}\right), \quad J: (0, 1) \rightarrow \mathbb{R},$$

called weights. Usually, it is convenient to use the function $J(\cdot)$ for the definition of the weights. Here are two examples of L -statistic (1).

Example 1. The trimmed mean is defined as follows: for any fixed numbers $0 < t_1 < t_2 < 1$,

$$M_{t_1;t_2} = ([t_2n] - [t_1n])^{-1} \sum_{j=[t_1n]+1}^{[t_2n]} X_{j:n}, \quad (2)$$

where $[\cdot]$ represents the greatest integer function. This statistic is close (asymptotically equivalent) to the statistic (1), with the weight function $J(u) = (t_2 - t_1)^{-1} \mathbb{I}\{t_1 < u < t_2\}$. Here $\mathbb{I}\{\cdot\}$ is the indicator function. Note that the marginal case, where $t_1 = 0$ and $t_2 = 1$, represents the usual sample mean. In this case $J \equiv 1$. The trimmed means are applied in a robust estimation of the center of population \mathcal{X} .

Example 2. In the case of independent and identically distributed (i.i.d.) observations, the L -statistic, defined by the weight function $J(u) = 6u(1 - u)$, is applied as an efficient estimator of the location parameter for the logistic distribution [1]. Let us denote it by L_{loc} . Therefore, if it is assumed that the population \mathcal{X} is obtained from the logistic distribution, the defined statistic L_{loc} may be useful in the estimation of the center of population \mathcal{X} . Clearly, in many other situations, it may also be meaningful to give smaller weights to extreme observations.

The first object of our interest is an estimation of the variance $\sigma_L^2 = \text{Var } L$. The quality of estimators of σ_L^2 is important, if, e.g., we construct the confidence intervals for a parameter which is some L -functional or, we choose between two or more competing statistics by comparing their variances. To construct estimators for σ_L^2 , we choose the following strategy. Since the L -statistic is a symmetric function of observations, application of Hoeffding's decomposition to symmetric statistics [2] yields

$$L = \mathbf{E}L + U_1 + U_2 + \cdots, \quad (3)$$

where

$$U_1 = \sum_{i=1}^n g_1(X_i) \quad \text{and} \quad U_2 = \sum_{1 \leq i < j \leq n} g_2(X_i, X_j)$$

are the linear and quadratic parts of the decomposition, respectively. Next, for many commonly used statistics higher-order terms in (3) are stochastically negligible. For instance, for statistic (1) normalized by the factor $n^{1/2}$, it follows from Theorem 1 in [2] and by the proof of Theorem 1 in [3] that, if $J(\cdot)$ has a bounded second derivative $J''(\cdot)$ on $(0, 1)$ and $\mathbf{E}X_1^2$ is finite, then the variance of higher than the second-order terms in (3) is of the order $O(n_*^{-2})$ where $n_* = \min\{n, N - n\}$. Therefore, we truncate the variance decomposition formula (2.6) in [2]:

$$\sigma_L^2 \approx \frac{n(N-n)}{N-1} \sigma_1^2 + \binom{n}{2} \binom{N-n}{2} \binom{N-2}{2}^{-1} \sigma_2^2, \quad (4)$$

where

$$\sigma_1^2 = \frac{1}{N} \sum_{k=1}^N g_1^2(x_k) \quad \text{and} \quad \sigma_2^2 = \binom{N}{2}^{-1} \sum_{1 \leq k < l \leq N} g_2^2(x_k, x_l).$$

Here, in the case of L -statistics, the first- and the second-order influence functions $g_1(\cdot)$ and $g_2(\cdot, \cdot)$ have explicit expressions, see [3]. In particular, denote by

$$\mathcal{H}_{N,n,i}(j) = \binom{i}{j} \binom{N-i}{n-j} / \binom{N}{n}$$

the probability that a hypergeometric random variable with the parameters N , n and i attains the value j . Assume that, without loss of generality, $x_1 \leq \dots \leq x_N$. Denote $\Delta x_i = x_{i+1} - x_i$. For $1 \leq k \leq N$,

$$g_1(k; \mathcal{X}) := g_1(x_k) = -n^{-1} \sum_{i=1}^{N-1} \left(\mathbb{I}\{i \geq k\} - \frac{i}{N} \right) a_i \Delta x_i \tag{5}$$

with the numbers $a_i := a_{N,n,i} = \sum_{p=1}^n c_p \mathcal{H}_{N-2,n-1,i-1}(p-1)$. For $1 \leq k < l \leq N$,

$$g_2(k, l; \mathcal{X}) := g_2(x_k, x_l) = -n^{-1} \sum_{i=1}^{N-1} \phi_{k,l}(i) b_i \Delta x_i, \tag{6}$$

where

$$\phi_{k,l}(i) = \begin{cases} i(i-1)/A & \text{if } 1 \leq i < k, \\ -(i-1)(N-i-1)/A & \text{if } k \leq i < l, \\ (N-i-1)(N-i)/A & \text{if } l \leq i < N, \end{cases} \quad A = (N-1)(N-2),$$

and with $b_i := b_{N,n,i} = \sum_{p=2}^n (c_p - c_{p-1}) \mathcal{H}_{N-4,n-2,i-2}(p-2)$. The main variance estimation idea is simple. To estimate the right-hand side of (4), we estimate all values of the functions $g_1(\cdot)$ and $g_2(\cdot, \cdot)$ given by (5) and (6). The same approach is applied in a construction of bootstrap estimators using information of the sample \mathbb{X} only, see [4]. In the present paper, we incorporate auxiliary information into the estimation process.

The second object of our survey is approximations to the distribution function of the standardized L -statistic

$$F(x) = \mathbf{P} \{L - \mathbf{E}L \leq x\sigma_L\} \tag{7}$$

and the distribution function of the Studentized statistic

$$F_S(x) = \mathbf{P} \{L - \mathbf{E}L \leq x\hat{\sigma}_{LJ}\}. \tag{8}$$

Here

$$\hat{\sigma}_{LJ}^2 = \left(1 - \frac{n}{N}\right) \frac{n-1}{n} \sum_{k=1}^n (L_{(k)} - \bar{L})^2, \quad \bar{L} = \frac{1}{n} \sum_{k=1}^n L_{(k)} \tag{9}$$

is the jackknife estimator of σ_L^2 based on \mathbb{X} . Here $L_{(k)} = L_{n-1}(\mathbb{X} \setminus \{X_k\})$, $1 \leq k \leq n$ are L -statistics with the weights $c'_j = J(j/n)$, $1 \leq j \leq n-1$. A very usual approximation to distributions (7) and (8) is the standard normal distribution function $\Phi(x)$. Theorem 1 of [5] states that if $J(\cdot)$ is bounded and satisfies the Hölder condition

of order $\delta > 1/2$ on $(0, 1)$, and $\mathbf{E} X_1^2$ is finite and a condition of the Lindeberg-type is satisfied for all n_* , then distribution (7) converges to $\Phi(x)$ as $n_* \rightarrow \infty$. The same conditions imply the convergence of (8). It follows from the same theorem and from Proposition 3 in [2]. In the case of a trimmed mean, the requirements for $J(\cdot)$ are replaced by a little stronger than the Lindeberg-type smoothness condition, see Theorem 2 in [5]. However, if, e.g., the sample size is not large enough, the normal approximation can be inaccurate. One of the methods to correct $\Phi(x)$ is Edgeworth expansions. Compared to the theoretical error $O(n_*^{-1/2})$ of the normal approximation, the error of the one-term Edgeworth approximation, under an appropriate conditions, is of the order $o(n_*^{-1/2})$ or, under more restrictive conditions, $O(n_*^{-1})$. The corresponding one-term Edgeworth expansions for distributions (7) and (8) are, see [2, 6],

$$G(x) = \Phi(x) - \frac{(q-p)\alpha + 3\kappa}{6\tau} \Phi'(x)(x^2 - 1) \quad (10)$$

and

$$G_S(x) = \Phi(x) + \frac{(q-p + (q+1)x^2)\alpha + 3(x^2 + 1)\kappa}{6\tau} \Phi'(x). \quad (11)$$

Here $\Phi'(x)$ denotes the derivative of $\Phi(x)$ and $\tau^2 = Npq$, $p = n/N$, $q = 1 - p$, and

$$\begin{aligned} \alpha &= \sigma_1^{-3} \frac{1}{N} \sum_{k=1}^N g_1^3(x_k), \\ \kappa &= \sigma_1^{-3} \tau^2 \binom{N}{2}^{-1} \sum_{1 \leq k < l \leq N} g_2(x_k, x_l) g_1(x_k) g_1(x_l). \end{aligned} \quad (12)$$

If $J''(\cdot)$ is bounded on $(0, 1)$ and $\mathbf{E}|X_1|^{3+\delta}$ is finite for some $\delta > 0$, and a nonlattice condition is satisfied, then the error of approximation (10) is of the order $o(n_*^{-1/2})$ as $n_* \rightarrow \infty$. In the case of (8), by Theorem 2 of [6], in order to ensure the rate $o(n_*^{-1/2})$, the finite $\mathbf{E}|X_1|^{6+\delta}$ and the additional condition $q\tau \rightarrow \infty$ are needed as $n_* \rightarrow \infty$. The case of trimmed means is still not studied for samples without replacement, but one can expect that conditions sufficient for the validity of the short Edgeworth expansions should be similar to that appearing in the i.i.d. case, see [7].

Note that the parameters α and κ completely define the one-term Edgeworth expansions and depend on the functions $g_1(\cdot)$ and $g_2(\cdot, \cdot)$ only. Thus, here the problem is the same as in the case of the variance estimation.

There is a number of variance estimation methods. One of more popular methods, which also appears in our numerical comparisons, is the jackknife estimator (9). It is well studied in the case of symmetric statistics, see [8]. In addition to the mentioned papers [2–4, 6] on the Edgeworth expansions in the cases of finite population symmetric statistics and L -statistics, we note that similar problems are historically important in the classical case of i.i.d. observations, see [7, 9–13].

In Section 2, we apply the calibration technique, see [14], to estimate the functions $g_1(\cdot)$ and $g_2(\cdot, \cdot)$. It is a specific method applied in finite population problems. As a basis

for the construction of the calibrated estimators, we use two different types of estimators of (5) and (6). It is bootstrap estimators derived in [4] and also certain new estimators. The calibrated estimators obtained yield new results in estimating the variance σ_L^2 and the one-term Edgeworth expansions $G(x)$ and $G_S(x)$. In Section 3, to demonstrate the power of calibrated estimators, we present a simulation study for L -statistics of Examples 1 and 2. We compare the accuracies of estimators of the parameters σ_L^2 , α , and κ , and show how the estimators of α and κ affect the efficiency of empirical one-term Edgeworth expansions. We also separately simulate and discuss the estimates of the functions $g_1(\cdot)$ and $g_2(\cdot, \cdot)$. In Section 4, we summarize the main findings of our work.

2 Estimation

We present four different estimators of (5) and (6). Assume that, without loss of generality, $x_1 \leq \dots \leq x_N$. Let $z_{j_1} \leq \dots \leq z_{j_N}$ be the corresponding ordered sequence, which is necessary for the calculation of $g_1(k; \mathcal{Z})$ and $g_2(k, l; \mathcal{Z})$ below. Here j_1, \dots, j_N is some permutation of $1, \dots, N$. Let $Z_{1:n} \leq \dots \leq Z_{n:n}$ be the order statistics of \mathbb{Z} . Denote $\Delta Z_{j:n} = Z_{j+1:n} - Z_{j:n}$.

2.1 Bootstrap estimators

The construction in [4] is based on one of the finite population bootstrap variants [15]. Write $N = mn + t$, $0 \leq t < n$, where m is integer. Denote $\Delta X_{j:n} = X_{j+1:n} - X_{j:n}$. Denote $u_i(k) = -n^{-1}(\mathbb{I}\{i \geq k\} - i/N)a_i$ and $v_i(k, l) = -n^{-1}\phi_{k,l}(i)b_i$. Then, for $1 \leq k \leq N$,

$$\hat{g}_{1B}(k; \mathbb{X}) = \sum_{j=1}^{n-1} \sum_{i=mj}^{mj+t} u_i(k) \mathcal{H}_{n,t,j}(i - mj) \Delta X_{j:n} \tag{13}$$

and, for $1 \leq k < l \leq N$,

$$\hat{g}_{2B}(k, l; \mathbb{X}) = \sum_{j=1}^{n-1} \sum_{i=mj}^{mj+t} v_i(k, l) \mathcal{H}_{n,t,j}(i - mj) \Delta X_{j:n}. \tag{14}$$

By [4], these estimators give more stable estimates of the parameters α and κ , compared to the universal jackknife estimation method from [16], in the sense that they are less sensitive to a non-smoothness of the weight function $J(\cdot)$.

2.2 Calibrated bootstrap estimators

We construct calibrated estimators on the basis of the bootstrap estimators. Rewrite, for $1 \leq k \leq N$,

$$\hat{g}_{1B}(k; \mathbb{X}) = \sum_{j=1}^{n-1} d_j^{(1B)}(k) \sum_{i=mj}^{mj+t} u_i(k) \mathcal{H}_{n,t,j}(i - mj) \Delta X_{j:n} \tag{15}$$

and, for $1 \leq k < l \leq N$,

$$\hat{g}_{2B}(k, l; \mathbb{X}) = \sum_{j=1}^{n-1} d_j^{(2B)}(k, l) \sum_{i=mj}^{mj+t} v_i(k, l) \mathcal{H}_{n,t,j}(i - mj) \Delta X_{j:n}, \quad (16)$$

where $d_j^{(1B)}(k) = 1$, $d_j^{(2B)}(k, l) = 1$, $j = 1, \dots, n-1$.

The weights $d_j^{(1B)}(k)$ and $d_j^{(2B)}(k, l)$ can be modified using auxiliary variables and calibration ideas from [14, 17] to obtain estimators with a smaller variance. We define here the calibrated estimators of $g_1(k; \mathcal{X})$, $1 \leq k \leq N$, and $g_2(k, l; \mathcal{X})$, $1 \leq k < l \leq N$, of the following shape:

$$\hat{g}_{1Bw}(k; \mathbb{X}, \mathcal{Z}) = \sum_{j=1}^{n-1} w_j^{(1B)}(k) \sum_{i=mj}^{mj+t} u_i(k) \mathcal{H}_{n,t,j}(i - mj) \Delta X_{j:n} \quad (17)$$

and

$$\hat{g}_{2Bw}(k, l; \mathbb{X}, \mathcal{Z}) = \sum_{j=1}^{n-1} w_j^{(2B)}(k, l) \sum_{i=mj}^{mj+t} v_i(k, l) \mathcal{H}_{n,t,j}(i - mj) \Delta X_{j:n}, \quad (18)$$

where the new (calibrated) weights $w_j^{(1B)}(k)$ and $w_j^{(2B)}(k, l)$

- minimize the distance measure

$$D(\mathbf{w}) = \sum_{j=1}^{n-1} (w_j - 1)^2; \quad (19)$$

here $w_j = w_j^{(1B)}(k)$ for $\hat{g}_{1Bw}(k; \mathbb{X}, \mathcal{Z})$, and $w_j = w_j^{(2B)}(k, l)$ for $\hat{g}_{2Bw}(k, l; \mathbb{X}, \mathcal{Z})$;

- satisfy the calibration equations

$$\sum_{j=1}^{n-1} w_j^{(1B)}(k) \sum_{i=mj}^{mj+t} u_i(k) \mathcal{H}_{n,t,j}(i - mj) \Delta Z_{j:n} = g_1(k; \mathcal{Z}), \quad (20)$$

and

$$\sum_{j=1}^{n-1} w_j^{(2B)}(k, l) \sum_{i=mj}^{mj+t} v_i(k, l) \mathcal{H}_{n,t,j}(i - mj) \Delta Z_{j:n} = g_2(k, l; \mathcal{Z}), \quad (21)$$

respectively.

Calibration equations (20) and (21) are treated as the requirements to use the new weights in order to obtain the exact estimates of the known values $g_1(k; \mathcal{Z})$ and $g_2(k, l; \mathcal{Z})$. Thus, in the case of quite a high correlation between the study and auxiliary variables, it is

natural to expect that the estimates of $g_1(k; \mathcal{X})$ and $g_2(k, l; \mathcal{X})$ will be more accurate when the calibrated weights $w_j^{(1B)}(k)$ and $w_j^{(2B)}(k, l)$ are applied in (17) and (18), respectively.

The weights $w_j^{(1B)}(k)$ and $w_j^{(2B)}(k, l)$ of estimators (17) and (18) are given by the following proposition that is actually a corollary which follows from the derivation of weights of a calibrated estimator of the finite population total in [14].

Proposition 1. *The weights $w_j^{(1B)}(k)$, $1 \leq k \leq N$, and $w_j^{(2B)}(k, l)$, $1 \leq k < l \leq N$, $j = 1, \dots, n - 1$, of estimators (17) and (18), which minimize the distance measure (19) and satisfy the corresponding equations (20) and (21), are given by*

$$w_j^{(1B)}(k) = 1 + (g_1(k; \mathcal{Z}) - \hat{g}_{1B}(k; \mathbb{Z})) \left(\sum_{t=1}^{n-1} r_t^2(k) \right)^{-1} r_j(k)$$

and

$$w_j^{(2B)}(k, l) = 1 + (g_2(k, l; \mathcal{Z}) - \hat{g}_{2B}(k, l; \mathbb{Z})) \left(\sum_{t=1}^{n-1} q_t^2(k, l) \right)^{-1} q_j(k, l).$$

Here

$$r_j(k) = \sum_{i=m_j}^{m_j+t} u_i(k) \mathcal{H}_{n,t,j}(i - m_j) \Delta Z_{j:n},$$

$$q_j(k, l) = \sum_{i=m_j}^{m_j+t} v_i(k, l) \mathcal{H}_{n,t,j}(i - m_j) \Delta Z_{j:n}.$$

Proof. Let us take the distance measure (19) and calibration equation (20), and define the Lagrange function

$$\Lambda = \sum_{j=1}^{n-1} (w_j^{(1B)}(k) - 1)^2 - \lambda \left(\sum_{j=1}^{n-1} w_j^{(1B)}(k) r_j(k) - g_1(k; \mathcal{Z}) \right).$$

The derivatives $\partial \Lambda / \partial w_j^{(1B)}(k)$ are equal to zero in case

$$w_j^{(1B)}(k) = 1 + \frac{1}{2} \lambda r_j(k). \tag{22}$$

Then, summing (22) multiplied by $r_j(k)$, respectively, and taking into account calibration equation (20), we can find

$$\lambda = 2(g_1(k; \mathcal{Z}) - \hat{g}_{1B}(k; \mathbb{Z})) \left(\sum_{t=1}^{n-1} r_t^2(k) \right)^{-1}.$$

Substituting this expression into (22), we get an equation for $w_j^{(1B)}(k)$.

The proof for the case of a calibrated bootstrap estimator of the function $g_2(\cdot, \cdot)$ is similar. \square

2.3 Horvitz–Thompson type estimators

To construct more typical calibrated estimators, following [14], we introduce simple estimators which are similar to the usual Horvitz–Thompson estimators in the sense that, in expressions (5) and (6), population characteristics are replaced by their empirical analogs. Let $1 \leq i_1 < \dots < i_n \leq N$ denote the positions of the order statistics $X_{1:n} \leq \dots \leq X_{n:n}$ in the ordered set \mathcal{X} . Since these indexes are unknown, we introduce their estimators $1 \leq \hat{i}_1 < \dots < \hat{i}_n \leq N$, which are the positions of the order statistics $Z_{1:n} \leq \dots \leq Z_{n:n}$ in the auxiliary ordered set \mathcal{Z} . Then, for $1 \leq k \leq N$, define

$$\hat{g}_{1HT}(k; \mathbb{X}, \mathcal{Z}) = -n^{-1} \sum_{j=1}^{n-1} \left(\mathbb{I}\{\hat{i}_j \geq k\} - \frac{\hat{i}_j}{N} \right) a_{\hat{i}_j} \Delta X_{j:n} \quad (23)$$

and, for $1 \leq k < l \leq N$,

$$\hat{g}_{2HT}(k, l; \mathbb{X}, \mathcal{Z}) = -n^{-1} \sum_{j=1}^{n-1} \phi_{k,l}(\hat{i}_j) b_{\hat{i}_j} \Delta X_{j:n}. \quad (24)$$

2.4 Calibrated estimators

The following estimators are based on the Horvitz–Thompson type estimators (23) and (24) which are modified analogously as the bootstrap estimators (13) and (14), i.e., we consider the calibrated estimators of (5) and (6) of the form: for $1 \leq k \leq N$,

$$\hat{g}_{1w}(k; \mathbb{X}, \mathcal{Z}) = -n^{-1} \sum_{j=1}^{n-1} w_j^{(1)}(k) \left(\mathbb{I}\{\hat{i}_j \geq k\} - \frac{\hat{i}_j}{N} \right) a_{\hat{i}_j} \Delta X_{j:n}, \quad (25)$$

and, for $1 \leq k < l \leq N$,

$$\hat{g}_{2w}(k, l; \mathbb{X}, \mathcal{Z}) = -n^{-1} \sum_{j=1}^{n-1} w_j^{(2)}(k, l) \phi_{k,l}(\hat{i}_j) b_{\hat{i}_j} \Delta X_{j:n}. \quad (26)$$

Here the calibrated weights $w_j^{(1)}(k)$ and $w_j^{(2)}(k, l)$

- minimize the distance measure (19), where $w_j = w_j^{(1)}(k)$ for $\hat{g}_{1w}(k; \mathbb{X}, \mathcal{Z})$, and $w_j = w_j^{(2)}(k, l)$ for $\hat{g}_{2w}(k, l; \mathbb{X}, \mathcal{Z})$;
- satisfy the calibration equations

$$-n^{-1} \sum_{j=1}^{n-1} w_j^{(1)}(k) \left(\mathbb{I}\{\hat{i}_j \geq k\} - \frac{\hat{i}_j}{N} \right) a_{\hat{i}_j} \Delta Z_{j:n} = g_1(k; \mathcal{Z}), \quad (27)$$

and

$$-n^{-1} \sum_{j=1}^{n-1} w_j^{(2)}(k, l) \phi_{k,l}(\hat{i}_j) b_{\hat{i}_j} \Delta Z_{j:n} = g_2(k, l; \mathcal{Z}), \quad (28)$$

respectively.

Proposition 2. *The weights $w_j^{(1)}(k)$, $1 \leq k \leq N$, and $w_j^{(2)}(k, l)$, $1 \leq k < l \leq N$, $j = 1, \dots, n - 1$, of estimators (25) and (26) which minimize the distance measure (19) and satisfy corresponding equations (27) and (28) are given by*

$$w_j^{(1)}(k) = 1 + (g_1(k; \mathcal{Z}) - \hat{g}_{1HT}(k; \mathbb{Z}, \mathcal{Z})) \left(\sum_{t=1}^{n-1} p_{i_t}^2(k) \right)^{-1} p_{i_j}(k)$$

and

$$w_j^{(2)}(k, l) = 1 + (g_2(k, l; \mathcal{Z}) - \hat{g}_{2HT}(k, l; \mathbb{Z}, \mathcal{Z})) \left(\sum_{t=1}^{n-1} s_{i_t}^2(k, l) \right)^{-1} s_{i_j}(k, l).$$

Here

$$p_{i_j}(k) = -n^{-1} \left(\mathbb{I}\{\hat{i}_j \geq k\} - \frac{\hat{i}_j}{N} \right) a_{i_j} \Delta Z_{j:n},$$

$$s_{i_j}(k, l) = -n^{-1} \phi_{k,l}(\hat{i}_j) b_{i_j} \Delta Z_{j:n}.$$

The proof is similar to that of Proposition 1.

2.5 Estimators of variance and parameters of Edgeworth expansions

Using the bootstrap estimators (13) and (14) of all the values of functions $g_1(\cdot)$ and $g_2(\cdot, \cdot)$, we define bootstrap estimators of the variance of L -statistic and parameters (12):

$$\hat{\sigma}_{LB}^2 = \frac{n(N-n)}{N-1} \hat{\sigma}_{1B}^2 + \binom{n}{2} \binom{N-n}{2} \binom{N-2}{2}^{-1} \hat{\sigma}_{2B}^2,$$

$$\hat{\alpha}_B = \hat{\sigma}_{1B}^{-3} \frac{1}{N} \sum_{k=1}^N \hat{g}_{1B}^3(k; \mathbb{X}),$$

$$\hat{\kappa}_B = \hat{\sigma}_{1B}^{-3} T^2 \binom{N}{2}^{-1} \sum_{1 \leq k < l \leq N} \hat{g}_{2B}(k, l; \mathbb{X}) \hat{g}_{1B}(k; \mathbb{X}) \hat{g}_{1B}(l; \mathbb{X}),$$

where

$$\hat{\sigma}_{1B}^2 = \frac{1}{N} \sum_{k=1}^N \hat{g}_{1B}^2(k; \mathbb{X}) \quad \text{and} \quad \hat{\sigma}_{2B}^2 = \binom{N}{2}^{-1} \sum_{1 \leq k < l \leq N} \hat{g}_{2B}^2(k, l; \mathbb{X}).$$

Similarly, by substituting estimators (17), (18), (23), (24), (25), and (26) into (4) and (12), we define the calibrated bootstrap $(\hat{\sigma}_{LBw}^2, \hat{\alpha}_{Bw}, \hat{\kappa}_{Bw})$, Horvitz–Thompson type $(\hat{\sigma}_{LHT}^2, \hat{\alpha}_{HT}, \hat{\kappa}_{HT})$ and calibrated estimators $(\hat{\sigma}_{Lw}^2, \hat{\alpha}_w, \hat{\kappa}_w)$ of the parameters $(\sigma_L^2, \alpha, \kappa)$.

Replacing the true parameters α and κ in (10) and (11) by their estimators defined above, we obtain the corresponding empirical Edgeworth expansions for which we use

the following notation: $G_B(x)$, $G_{Bw}(x)$, $G_{HT}(x)$, $G_w(x)$ and $G_{SB}(x)$, $G_{SBw}(x)$, $G_{SHT}(x)$, $G_{Sw}(x)$ where, for example, $G_B(x)$ and $G_{SB}(x)$ are defined as follows:

$$G_B(x) = \Phi(x) - \frac{(q-p)\hat{\alpha}_B + 3\hat{\kappa}_B}{6\tau} \Phi'(x)(x^2 - 1)$$

and

$$G_{SB}(x) = \Phi(x) + \frac{(q-p + (q+1)x^2)\hat{\alpha}_B + 3(x^2+1)\hat{\kappa}_B}{6\tau} \Phi'(x).$$

3 Simulation study

The simulation study is performed to observe the efficiency of the estimators of variance of L -statistic and to check how the empirical Edgeworth expansions improve the standard normal approximation to distributions (7) and (8).

Case 1. First, we consider an artificial population $\mathcal{U}_{\mathcal{F}}$ of size $N = 120$. The auxiliary variable z , that is defined on $\mathcal{U}_{\mathcal{F}}$, is simulated from the Fisher distribution $\mathcal{F}(5, 4)$. The values of the study variable x are generated according to the formula: $x_k = 2 + z_k + 0.7\sqrt{z_k}\varepsilon_k$, $\varepsilon_k \sim \mathcal{N}(0, 1)$. The variables x and z are strongly correlated with the correlation coefficient $\rho(x, z) = 0.92$. The L -statistic of interest is L_{loc} (see Example 2).

Case 2. The second population of size $N = 150$ is also artificial. Let us denote it by \mathcal{U}_E . In this case, the study variable x is simulated from the exponential distribution $\mathcal{E}(0.001)$ with the expectation equal to 1000. The values of the auxiliary variable z are generated according to the formula: $z_k = x_k + 1100\xi_k$, $\xi_k \sim \mathcal{U}(0, 1)$. As in the previous case, the correlation between the variables x and z is high ($\rho(x, z) = 0.94$). The trimmed mean $M_{0.25;0.75}$ (see Example 1) is chosen for analysis, as an example of L -statistic.

$R = 1000$ simple random samples of fixed size ($n = 40$ for Case 1, and $n = 50$ for Case 2) are drawn and for each of them the estimators of $g_1(k; \mathcal{X})$, $1 \leq k \leq N$, and $g_2(k, l; \mathcal{X})$, $1 \leq k < l \leq N$, are computed. Consequently, the estimates of variance of L_{loc} , $M_{0.25;0.75}$ and parameters (12), required for the construction of the empirical Edgeworth expansions, are also produced.

In the next subsections, we analyze the quality of these estimators by estimating some of the following quality measures: expectation (**E**), bias (**Bias**), standard error (**S**), and mean square error (**MSE**). For any estimator $\hat{\theta}$ of the finite population parameter θ , these characteristics of accuracy are estimated by the following equations:

$$\begin{aligned} \widehat{\mathbf{E}}(\hat{\theta}) &= \frac{1}{R} \sum_{r=1}^R \hat{\theta}_r, & \widehat{\mathbf{Bias}}(\hat{\theta}) &= \frac{1}{R} \sum_{r=1}^R \hat{\theta}_r - \theta, \\ \widehat{\mathbf{S}}(\hat{\theta}) &= \sqrt{\frac{1}{R} \sum_{r=1}^R (\hat{\theta}_r - \widehat{\mathbf{E}}(\hat{\theta}))^2}, & \widehat{\mathbf{MSE}}(\hat{\theta}) &= \frac{1}{R} \sum_{r=1}^R (\hat{\theta}_r - \theta)^2, \end{aligned}$$

where $\hat{\theta}_r$ is the estimate of θ computed from the r th simulated sample.

3.1 Estimation of functions $g_1(\cdot)$ and $g_2(\cdot, \cdot)$

The quality of the estimators of variance of the L -statistic and parameters (12) depends on the estimators of functions $g_1(\cdot)$ and $g_2(\cdot, \cdot)$. Thus, we should first analyze the statistical properties of these estimators. For the best estimators of $g_1(\cdot)$ and $g_2(\cdot, \cdot)$, we expect the highest accuracy of corresponding estimators of the parameters mentioned.

The following empirical study is performed for Case 1, i.e., for the case of population $\mathcal{U}_{\mathcal{F}}$ and L -statistic L_{loc} . The results for Case 2 are similar. Because of many values of the function $g_1(\cdot)$, we illustrate the performance of estimators of $g_1(k; \mathcal{X})$, $1 \leq k \leq N$, only graphically. Figure 1 shows the estimated expectation for some more important pairs of estimators as well as real values of $g_1(\cdot)$. Thus, one can estimate visually the bias for each value.

The calibrated estimators $\hat{g}_{1w}(k; \mathbb{X}, \mathcal{Z})$, $1 \leq k \leq N$, have the lowest bias on the average. As one can see, both calibration methods reduce the bias, i.e., both calibrated estimators are less biased than the respective simple estimators (13) and (23). The Horvitz–Thompson type estimator (23) is more biased than bootstrap estimator (13). The highest bias of estimators is observed on the right side of the values.

Figure 2 illustrates the estimated mean square error for some pairs of estimators discussed above.

The calibrated bootstrap estimators $\hat{g}_{1Bw}(k; \mathbb{X}, \mathcal{Z})$, $1 \leq k \leq N$, have the lowest mean square error on the average. It is of interest to note that for some values of $g_1(\cdot)$ this measure of accuracy is about 13 times lower than that of bootstrap estimator (13), but there exist a few values of $g_1(\cdot)$ for which the auxiliary information does not help, i.e., the mean square error of the calibrated bootstrap estimator is higher than that of the bootstrap estimator. Such a situation is not observed in the case of the pair $(\hat{g}_{1HT}, \hat{g}_{1w})$: for all values of $g_1(\cdot)$, the mean square error of calibrated estimator (25) is lower than the same characteristics of Horvitz–Thompson type estimator (23). Compared to the corresponding estimators (13) and (23), the most significant gain in reduction of the mean square error of both calibrated estimators is observed on the right side of the values.

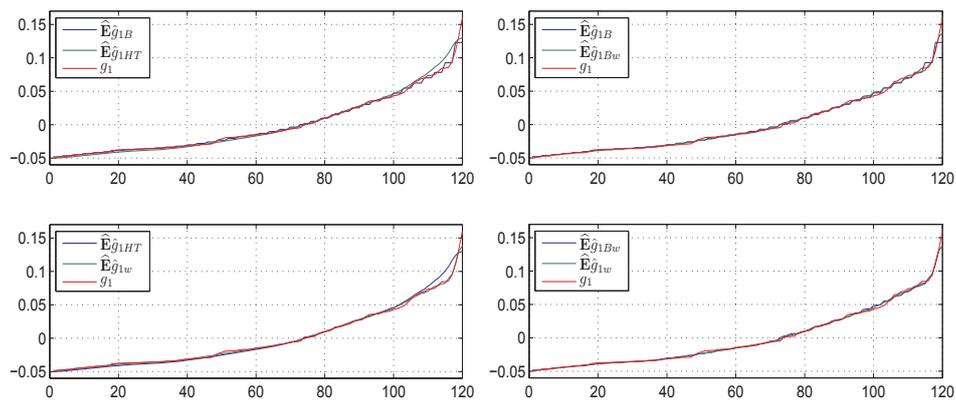


Fig. 1. Estimated expectation of estimators of $g_1(\cdot)$

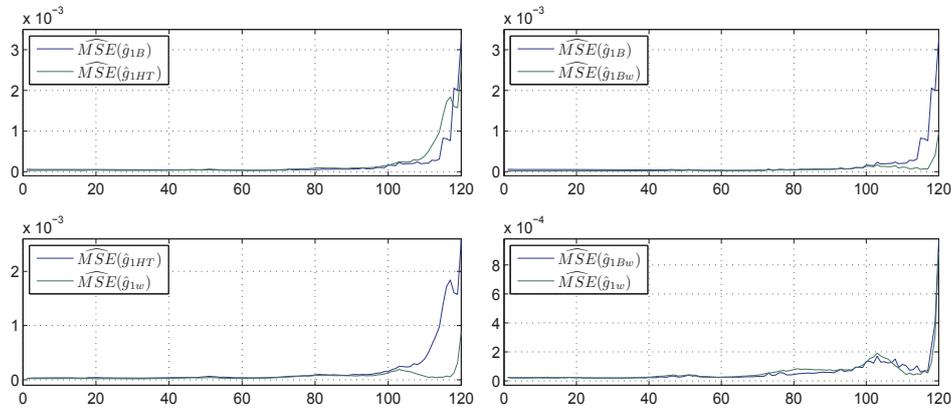


Fig. 2. Estimated mean square error of estimators of $g_1(\cdot)$

We do not present the results for the case of the function $g_2(\cdot, \cdot)$, because the pattern for the expectation and mean square error of the estimators of this function remains very similar as in the case of the function $g_1(\cdot)$.

3.2 Estimation of variance of the L -statistic and parameters α and κ

High quality estimators of the variance σ_L^2 are preferable in solving many practical statistical problems. When constructing empirical Edgeworth expansions, one prefers to use estimators of the parameters α and κ with a smaller variance. Naturally, we consider here empirical statistical properties of the estimators of variance of the L -statistic and parameters α and κ .

Table 1 shows the estimated bias and mean square error of the estimators $\hat{\sigma}_{LJ}^2$, $\hat{\sigma}_{LB}^2$, $\hat{\sigma}_{LBw}^2$, $\hat{\sigma}_{LHT}^2$ and $\hat{\sigma}_{Lw}^2$. In the second column, the true values of variance of the statistics L_{loc} and $M_{0.25;0.75}$ are also given. As expected, the calibrated and calibrated bootstrap estimators outperform all the other ones. They are almost unbiased and have the lowest mean square error. For Case 1, the jackknife estimator $\hat{\sigma}_{LJ}^2$ is a little more efficient than the bootstrap estimator $\hat{\sigma}_{LB}^2$ and the Horvitz–Thompson type estimator $\hat{\sigma}_{LHT}^2$ (see MSE), but this implication cannot be obtained in Case 2, where the bootstrap estimator is much more efficient. The quality of the Horvitz–Thompson type estimator $\hat{\sigma}_{LHT}^2$ is lower as compared to that of the bootstrap estimator $\hat{\sigma}_{LB}^2$, but the calibration of weights of $\hat{g}_{1HT}(k; \mathbb{X}, \mathcal{Z})$ and $\hat{g}_{2HT}(k, l; \mathbb{X}, \mathcal{Z})$ leads to the estimator $\hat{\sigma}_{Lw}^2$ of quite a high quality which is similar to that of $\hat{\sigma}_{LBw}^2$. A more significant gain in reduction of the bias and mean square error (when calibrating the weights of bootstrap and Horvitz–Thompson type estimators of $g_1(\cdot)$ and $g_2(\cdot, \cdot)$) is observed for Case 1, but the results of additional computation show that the relative root mean square error ($=\sqrt{\text{MSE}(\cdot)}/\sigma_L^2$) of all estimators is lower for Case 2.

In Tables 2 and 3, we present the estimated bias and mean square error of the estimators of parameters α and κ as well as their true values. Again, the calibrated estimators are

Table 1. Characteristics of accuracy of the estimators of σ_L^2 .

	$\sigma_L^2 \times 10^3$		$\hat{\sigma}_{LJ}^2$	$\hat{\sigma}_{LB}^2$	$\hat{\sigma}_{LBw}^2$	$\hat{\sigma}_{LHT}^2$	$\hat{\sigma}_{Lw}^2$
Case 1	52.686	Bias $\times 10^3$	-3.046	1.095	-0.245	9.902	0.176
		MSE $\times 10^4$	3.971	5.070	1.107	6.748	1.027
	$\sigma_L^2 \times 10^{-2}$		$\hat{\sigma}_{LJ}^2$	$\hat{\sigma}_{LB}^2$	$\hat{\sigma}_{LBw}^2$	$\hat{\sigma}_{LHT}^2$	$\hat{\sigma}_{Lw}^2$
Case 2	155.993	Bias $\times 10^{-2}$	-5.665	8.810	3.956	-1.900	-0.031
		MSE $\times 10^{-6}$	20.727	7.861	6.538	18.899	8.523

Table 2. Characteristics of accuracy of the estimators of parameters α and κ for Case 1.

α		$\hat{\alpha}_B$	$\hat{\alpha}_{Bw}$	$\hat{\alpha}_{HT}$	$\hat{\alpha}_w$
1.140	Bias	-0.099	-0.094	-0.020	-0.096
	MSE	0.169	0.058	0.161	0.054
κ		$\hat{\kappa}_B$	$\hat{\kappa}_{Bw}$	$\hat{\kappa}_{HT}$	$\hat{\kappa}_w$
0.4184	Bias	0.012	0.010	0.060	0.009
	MSE	0.016	0.005	0.023	0.006

Table 3. Characteristics of accuracy of the estimators of parameters α and κ for Case 2.

α		$\hat{\alpha}_B$	$\hat{\alpha}_{Bw}$	$\hat{\alpha}_{HT}$	$\hat{\alpha}_w$
0.335	Bias	-0.010	-0.007	-0.003	0.002
	MSE	0.014	0.008	0.009	0.009
κ		$\hat{\kappa}_B$	$\hat{\kappa}_{Bw}$	$\hat{\kappa}_{HT}$	$\hat{\kappa}_w$
0.471	Bias	0.041	0.014	0.024	0.030
	MSE	0.064	0.038	0.055	0.054

of the highest quality. Especially in Case 1 (see Table 2), we observe the biggest difference between the simple estimators $\hat{\alpha}_B$, $\hat{\alpha}_{HT}$, $\hat{\kappa}_B$, $\hat{\kappa}_{HT}$, and the corresponding calibrated estimators $\hat{\alpha}_{Bw}$, $\hat{\alpha}_w$, $\hat{\kappa}_{Bw}$, $\hat{\kappa}_w$, whose mean square error is lower about 3 times. For the estimator $\hat{\alpha}_w$, this improvement is followed by an increased absolute bias. In Case 2 (see Table 3), we see not so great gain in reduction of MSE. The calibration approach slightly improves $\hat{\alpha}_{HT}$ and $\hat{\kappa}_{HT}$. It is of interest to note that for both cases, all the estimators of κ overestimate this parameter, whereas $\hat{\alpha}_B$, $\hat{\alpha}_{Bw}$ and $\hat{\alpha}_{HT}$ underestimate α .

In opposition to the case of estimation of the variance σ_L^2 , the Horvitz–Thompson type estimator of the parameter α outperforms the bootstrap estimator $\hat{\alpha}_B$. Only in Case 2, the estimator $\hat{\kappa}_{HT}$ is more efficient than $\hat{\kappa}_B$. The calibrated estimators $\hat{\alpha}_{Bw}$ and $\hat{\alpha}_w$ are of a similar quality. The statistical properties of $\hat{\kappa}_{Bw}$ are similar to that of $\hat{\kappa}_w$, only in Case 1. In the exponential population, when the L -statistic is the trimmed mean $M_{0.25;0.75}$, the estimator $\hat{\kappa}_{Bw}$ is more successful as compared to $\hat{\kappa}_w$.

We note that a number of other simulation experiments (with different populations and L -statistics) give results similar to that of Cases 1 and 2.

3.3 Estimation of distributions

In the tables below, we compare the distributions $\tilde{F}(x)$, $\Phi(x)$, $G(x)$, $G_B(x)$, $G_{Bw}(x)$, $G_{HT}(x)$, $G_w(x)$ and $\tilde{F}_S(x)$, $\Phi(x)$, $G_S(x)$, $G_{SB}(x)$, $G_{SBw}(x)$, $G_{SHT}(x)$, $G_{Sw}(x)$ by taking their q -quantiles, $q = 0.01, 0.05, 0.10, 0.90, 0.95, 0.99$. Here $\tilde{F}(x)$ and $\tilde{F}_S(x)$ are Monte–Carlo approximations to the exact distributions $F(x)$ and $F_S(x)$, respectively. The quantiles of empirical Edgeworth expansions are estimators of quantiles of the corresponding true distributions, to which we give the estimated expectation ($\hat{\mathbf{E}}$) and standard error ($\hat{\mathbf{S}}$) (see Tables 4–7), based on the same set of simple random samples mentioned at the beginning of Section 3.

At first, we discuss the simulation results for Case 1 which show (see Table 4) that the one-term Edgeworth expansion $G(x)$ is much more better approximation of $\tilde{F}(x)$ than the normal approximation $\Phi(x)$. The inverses of empirical Edgeworth expansions $G_B(x)$,

Table 4. Approximations to $F^{-1}(q)$ for Case 1.

$q =$	0.01	0.05	0.10	0.90	0.95	0.99
$\tilde{F}^{-1}(q)$	-2.069	-1.549	-1.242	1.321	1.739	2.541
$\Phi^{-1}(q)$	-2.326	-1.645	-1.282	1.282	1.645	2.326
$G^{-1}(q)$	-2.099	-1.564	-1.252	1.319	1.743	2.544
$\hat{\mathbf{E}} G_B^{-1}(q)$	-2.101	-1.565	-1.252	1.320	1.744	2.542
$\hat{\mathbf{E}} G_{Bw}^{-1}(q)$	-2.100	-1.564	-1.252	1.319	1.743	2.543
$\hat{\mathbf{E}} G_{HT}^{-1}(q)$	-2.080	-1.557	-1.249	1.324	1.755	2.562
$\hat{\mathbf{E}} G_w^{-1}(q)$	-2.100	-1.564	-1.252	1.319	1.743	2.543
$\hat{\mathbf{S}} G_B^{-1}(q)$	0.064	0.022	0.008	0.013	0.033	0.059
$\hat{\mathbf{S}} G_{Bw}^{-1}(q)$	0.034	0.012	0.004	0.007	0.017	0.032
$\hat{\mathbf{S}} G_{HT}^{-1}(q)$	0.066	0.023	0.009	0.016	0.037	0.061
$\hat{\mathbf{S}} G_w^{-1}(q)$	0.037	0.013	0.005	0.008	0.019	0.034

Table 5. Approximations to $F_S^{-1}(q)$ for Case 1.

$q =$	0.01	0.05	0.10	0.90	0.95	0.99
$\tilde{F}_S^{-1}(q)$	-3.239	-2.119	-1.594	1.119	1.408	1.936
$\Phi^{-1}(q)$	-2.326	-1.645	-1.282	1.282	1.645	2.326
$G_S^{-1}(q)$	-2.770	-1.972	-1.524	1.088	1.351	1.728
$\hat{\mathbf{E}} G_{SB}^{-1}(q)$	-2.747	-1.957	-1.516	1.095	1.364	1.760
$\hat{\mathbf{E}} G_{SBw}^{-1}(q)$	-2.755	-1.960	-1.515	1.093	1.361	1.749
$\hat{\mathbf{E}} G_{SHT}^{-1}(q)$	-2.775	-1.983	-1.538	1.079	1.341	1.717
$\hat{\mathbf{E}} G_{Sw}^{-1}(q)$	-2.754	-1.959	-1.515	1.093	1.361	1.750
$\hat{\mathbf{S}} G_{SB}^{-1}(q)$	0.098	0.095	0.081	0.053	0.079	0.153
$\hat{\mathbf{S}} G_{SBw}^{-1}(q)$	0.053	0.051	0.043	0.028	0.043	0.083
$\hat{\mathbf{S}} G_{SHT}^{-1}(q)$	0.092	0.095	0.085	0.053	0.079	0.146
$\hat{\mathbf{S}} G_{Sw}^{-1}(q)$	0.054	0.053	0.044	0.029	0.044	0.085

Table 6. Approximations to $F^{-1}(q)$ for Case 2.

$q =$	0.01	0.05	0.10	0.90	0.95	0.99
$\tilde{F}^{-1}(q)$	-2.098	-1.571	-1.263	1.316	1.722	2.492
$\Phi^{-1}(q)$	-2.326	-1.645	-1.282	1.282	1.645	2.326
$G^{-1}(q)$	-2.135	-1.576	-1.256	1.312	1.726	2.511
$\widehat{\mathbf{E}} G_B^{-1}(q)$	-2.124	-1.573	-1.255	1.316	1.734	2.520
$\widehat{\mathbf{E}} G_{Bw}^{-1}(q)$	-2.132	-1.576	-1.256	1.314	1.729	2.513
$\widehat{\mathbf{E}} G_{HT}^{-1}(q)$	-2.130	-1.575	-1.256	1.315	1.731	2.515
$\widehat{\mathbf{E}} G_w^{-1}(q)$	-2.127	-1.574	-1.255	1.315	1.732	2.517
$\widehat{\mathbf{S}} G_B^{-1}(q)$	0.089	0.031	0.012	0.019	0.046	0.083
$\widehat{\mathbf{S}} G_{Bw}^{-1}(q)$	0.069	0.024	0.009	0.014	0.035	0.065
$\widehat{\mathbf{S}} G_{HT}^{-1}(q)$	0.084	0.029	0.011	0.017	0.042	0.078
$\widehat{\mathbf{S}} G_w^{-1}(q)$	0.082	0.029	0.011	0.017	0.042	0.077

Table 7. Approximations to $F_S^{-1}(q)$ for Case 2.

$q =$	0.01	0.05	0.10	0.90	0.95	0.99
$\tilde{F}_S^{-1}(q)$	-3.260	-2.091	-1.532	1.203	1.556	2.202
$\Phi^{-1}(q)$	-2.326	-1.645	-1.282	1.282	1.645	2.326
$G_S^{-1}(q)$	-2.621	-1.844	-1.427	1.152	1.455	1.951
$\widehat{\mathbf{E}} G_{SB}^{-1}(q)$	-2.623	-1.852	-1.436	1.146	1.447	1.938
$\widehat{\mathbf{E}} G_{SBw}^{-1}(q)$	-2.618	-1.846	-1.430	1.151	1.453	1.949
$\widehat{\mathbf{E}} G_{SHT}^{-1}(q)$	-2.620	-1.849	-1.433	1.149	1.451	1.944
$\widehat{\mathbf{E}} G_{Sw}^{-1}(q)$	-2.624	-1.851	-1.435	1.147	1.448	1.939
$\widehat{\mathbf{S}} G_{SB}^{-1}(q)$	0.104	0.088	0.071	0.056	0.082	0.161
$\widehat{\mathbf{S}} G_{SBw}^{-1}(q)$	0.080	0.067	0.054	0.043	0.062	0.124
$\widehat{\mathbf{S}} G_{SHT}^{-1}(q)$	0.096	0.081	0.065	0.052	0.075	0.149
$\widehat{\mathbf{S}} G_{Sw}^{-1}(q)$	0.095	0.080	0.064	0.051	0.074	0.146

$G_{Bw}(x)$ and $G_w(x)$ have a similar bias, but the lowest variance belongs to the inverses of calibrated Edgeworth expansions $G_{Bw}(x)$ and $G_w(x)$, which are more efficient than $G_B(x)$, $G_{HT}(x)$ and $\Phi(x)$. The inverse $G_{HT}^{-1}(\cdot)$ is the most biased estimator of $G^{-1}(\cdot)$. It seems that $G_B(x)$ is a slightly better approximation of $\tilde{F}(x)$ than $G_{HT}(x)$.

Table 5 shows that $G_S(x)$ outperforms $\Phi(x)$, but the approximation accuracy is lower than it is in the case of a standardized L -statistic (see Table 4). In the case of the Studentized statistic, one can see that the inverses of calibrated empirical Edgeworth expansions are similarly biased. $G_{SHT}^{-1}(\cdot)$ has the lowest bias on the average, whereas $G_{SB}^{-1}(\cdot)$ is most biased. The inverses of calibrated empirical Edgeworth expansions $G_{SBw}(x)$ and $G_{Sw}(x)$ have a significantly lower variance which guarantees that $G_{SBw}(x)$ and $G_{Sw}(x)$ are more efficient than $G_{SB}(x)$ and $G_{SHT}(x)$.

Tables 6 and 7 present the approximation results of Case 2, i.e., in the case of population \mathcal{U}_E and L -statistic $M_{0.25;0.75}$. Here the Edgeworth expansion $G(x)$ approximates

$\tilde{F}(x)$ similarly as in the previous case (see Table 6), but $G_S(x)$ fails on the right tail (see Table 7). Now, not only $G_{SB}^{-1}(\cdot)$ is the most biased estimator for $G_S^{-1}(\cdot)$, but also $G_B^{-1}(\cdot)$ for $G^{-1}(\cdot)$. The calibrated estimators are differently biased in both standardized and Studentized cases.

Taking a variability into account (compare standard errors), we conclude that approximations $G_{HT}(x)$ and $G_w(x)$, $G_{SHT}(x)$ and $G_{Sw}(x)$ can be of a similar quality for some quantiles. It can be explained by similar statistical properties of $\hat{\alpha}_{HT}$ and $\hat{\alpha}_w$, $\hat{\kappa}_{HT}$ and $\hat{\kappa}_w$ (see Table 3). The calibrated approximations $G_{Bw}(x)$ and $G_{SBw}(x)$ outperform $G_B(x)$ and $G_{SB}(x)$, respectively.

4 Conclusions

This article expands the range of applications of the calibration technique that was first time proposed for estimating the finite population totals in [14]. As it has been expected, the auxiliary information, highly correlated with the corresponding study variable, and calibration methods lead to improved estimates of the values of functions $g_1(\cdot)$ and $g_2(\cdot, \cdot)$ (in some cases, significantly). The estimators of parameters σ_L^2 , α and κ , based on the calibrated estimates of $g_1(\cdot)$ and $g_2(\cdot, \cdot)$, are the best ones among the others considered in this paper. Because of complicated expressions of the estimators, it is difficult to derive conditions using which one can compare them, or to check their consistency. The simulation results show that, in most cases, the bootstrap type estimators are slightly more accurate than the Horvitz–Thompson type estimators, whereas the calibrated bootstrap estimators outperform the calibrated ones.

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