Some new fixed point results in non-Archimedean fuzzy metric spaces

Peyman Salimi [a,1], **Calogero Vetro**[b,2], **Pasquale Vetro**[b,2]

[a]Department of Mathematics, Astara Branch, Islamic Azad University, Astara, Iran
salimipeyman@gmail.com

[b]Università degli Studi di Palermo, Dipartimento di Matematica e Informatica
Via Archirafi 34, 90123 Palermo, Italy
cvetro@math.unipa.it; vetro@math.unipa.it

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Abstract. In this paper, we introduce the notions of fuzzy (α, β, φ)-contractive mapping, fuzzy α-ϕ-ψ-contractive mapping and fuzzy α-β-contractive mapping and establish some results of fixed point for this class of mappings in the setting of non-Archimedean fuzzy metric spaces. The results presented in this paper generalize and extend some recent results in fuzzy metric spaces. Also, some examples are given to support the usability of our results.

Keywords: fuzzy metric spaces, non-Archimedean fuzzy metric spaces, fuzzy (α, β, φ)-contractive mappings, fuzzy α-ϕ-ψ-contractive mappings, fuzzy α-β-contractive mappings.

1 Introduction

The concept of fuzzy metric space was introduced in different ways by some authors (see i.e. [1, 2]) and further to this, the fixed point theory in this kind of spaces has been intensively studied (see [3–11]). Here, we underline as the notion of fuzzy metric space, introduced by Kramosil and Michalek [2] was modified by George and Veeramani [12,13] that obtained a Hausdorff topology for this class of fuzzy metric spaces. Recently, Miheţ [14] enlarged the class of fuzzy contractive mappings of Gregori and Sapena [7] and proved a fuzzy Banach contraction result for complete non-Archimedean fuzzy metric spaces, see also Vetro [15]. Now, we briefly describe our reasons for being interested in results of this kind. The applications of fixed point theorems are remarkable in different disciplines of mathematics, engineering and economics in dealing with problems arising in approximation theory, game theory and many others (see [16] and references therein). Consequently, many researchers, following the Banach contraction principle, investigated the existence of weaker contractive conditions or extended previous results

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under relatively weak hypotheses on the metric space. Motivated by Samet et al. [17], we introduce the class of fuzzy (α, β, φ)-contractive mappings, fuzzy α-ϕ-ψ-contractive mappings and fuzzy α-β-contractive mappings. The reader is referred to [18–20] for some discussions and applications on a non-Archimedean metric space and its induced topology. For example, let X be a non-Archimedean metric space, some assumptions on X can allow to extend a group of isometries of X to the group of Mobius transformations on X. Additionally, this result applies when the metric space is a field, that is, the p-adic numbers Q_p, and it is known that many metrics arise from valuations on a ring. Also for this, our results can be of interest in such areas of mathematics as algebra, geometry, group theory, functional analysis and topology. In this paper, we give fixed point results for some new classes of fuzzy contractive mappings. Our results substantially generalize and extend several comparable results in the existing literature, in particular we consider a recent result of Shen et al. [21].

2 Preliminaries

For the sake of completeness, we briefly recall some basic concepts used in the following.

Definition 1. A binary operation $\star : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following assertions:

(T1) $\star$ is commutative and associative;
(T2) $\star$ is continuous;
(T3) $a \star 1 = a$ for all $a \in [0,1]$;
(T4) $a \star b \leqslant c \star d$ whenever $a \leqslant c$ and $b \leqslant d$ for all $a, b, c, d \in [0,1]$.

Definition 2. A fuzzy metric space in the sense of George and Veeramani is an ordered triple $(X, M, \star)$ such that X is a nonempty set, $\star$ a continuous t-norm and M is a fuzzy set on $X \times X \times (0, +\infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$:

(F1) $M(x, y, t) > 0$ for all $t > 0$;
(F2) $M(x, y, t) = 1$ if and only if $x = y$;
(F3) $M(x, y, t) = M(y, x, t)$;
(F4) $M(x, y, t) \star M(y, z, s) \leqslant M(x, z, t + s)$;
(F5) $M(x, y, \cdot) : (0, +\infty) \to (0, 1]$ is continuous.

Then the triple $(X, M, \star)$ is called a fuzzy metric space. If we replace (F4) by

(F6) $M(x, y, t) \star M(y, z, s) \leqslant M(x, z, \max\{t, s\})$,

then the triple $(X, M, \star)$ is called a non-Archimedean fuzzy metric space. Since, (F6) implies (F4) then each non-Archimedean fuzzy metric space is a fuzzy metric space.

Definition 3. Let $(X, M, \star)$ be a fuzzy metric space (or a non-Archimedean fuzzy metric space). Then

(i) a sequence $\{x_n\}$ converges to $x \in X$, if and only if $\lim_{n\to+\infty} M(x_n, x, t) = 1$ for all $t > 0$;

(ii) a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if for all $\epsilon \in (0,1)$ and $t > 0$, there exists n_0 such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $m, n \geqslant n_0$;

(iii) the fuzzy metric space (or the non-Archimedean fuzzy metric space) is called complete if every Cauchy sequence converges to some $x \in X$.

Definition 4. Let $f : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. Then f is an α-admissible mapping if

$$\alpha(x,y) \geqslant 1 \quad \text{implies} \quad \alpha(fx, fy) \geqslant 1, \quad x, y \in X.$$

Definition 5. Let $f : X \to X$, $\beta : X \times (0, +\infty) \to [0, +\infty)$ and $k : (0, +\infty) \to (0, 1)$. Then f is a (k, β)-admissible mapping if

$$\beta(x,t) \leqslant \sqrt{k(t)} \quad \text{implies} \quad \beta(fx,t) \leqslant \sqrt{k(t)}, \quad x \in X,\ t > 0.$$

Denote with Φ the set of all the functions $\varphi : [0,1] \to [0,1]$ with the following properties:

(φ1) φ is decreasing and continuous;

(φ2) $\varphi(\lambda) = 0$ if and only if $\lambda = 1$.

Definition 6. Let $(X, M, \star)$ be a non-Archimedean fuzzy metric space and f be an α-admissible and (k, β)-admissible mapping. If there exists $\varphi \in \Phi$ such that

$$\alpha(x, fx)\alpha(y, fy)\varphi\bigl(M(fx, fy, t)\bigr) \leqslant \beta(x,t)\beta(y,t)\varphi\bigl(M(x,y,t)\bigr) \tag{1}$$

holds for all $x, y \in X$ with $x \neq y$ and all $t > 0$. Then f is called a fuzzy (α, β, φ)-contractive mapping.

3 Main results

The following theorem is our first result on the existence of fixed points for fuzzy (α, β, φ)-contractive mappings.

Theorem 1. *Let $(X, M, \star)$ be a complete non-Archimedean fuzzy metric space, $\alpha : X \times X \to [0, +\infty)$, $\beta : X \times (0, +\infty) \to [0, +\infty)$ and $k : (0, +\infty) \to (0, 1)$. Assume that f is a fuzzy (α, β, φ)-contractive mapping such that the following assertions hold*:

(a) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geqslant 1$ and $\beta(x_0, t) \leqslant \sqrt{k(t)}$ for all $t > 0$;*

(b) *if $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$, and $x_n \to x$ as $n \to +\infty$, then $\alpha(x, fx) \geqslant 1$.*

Then f has a fixed point. Moreover, if $y = fy$ implies $\alpha(y, fy) \geqslant 1$ and for all $x \in X$ and all $t > 0$, $\beta(x,t) < 1$ then f has a unique fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geqslant 1$. Define a sequence $\{x_n\}$ in X by $x_n = f^n x_0 = fx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for f and the result is proved. Hence, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Since f is an α-admissible mapping and $\alpha(x_0, fx_0) = \alpha(x_0, x_1) \geqslant 1$, we deduce that

$\alpha(x_1, x_2) = \alpha(fx_0, fx_1) \geqslant 1$. By continuing this process, we get $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$. Similarly, we deduce that $\beta(x_n, t) \leqslant \sqrt{k(t)}$ for all $n \in \mathbb{N} \cup \{0\}$ and all $t > 0$. Also define $\tau_n(t) = M(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N} \cup \{0\}$ and all $t > 0$. From (1) with $x = x_{n-1}$ and $y = x_n$ we get

$$\begin{aligned} \varphi\bigl(\tau_n(t)\bigr) &= \varphi\bigl(M(x_n, x_{n+1}, t)\bigr) \\ &\leqslant \alpha(x_{n-1}, x_n)\alpha(x_n, x_{n+1})\varphi\bigl(M(x_n, x_{n+1}, t)\bigr) \\ &= \alpha(x_{n-1}, fx_{n-1})\alpha(x_n, fx_n)\varphi\bigl(M(fx_{n-1}, fx_n, t)\bigr) \\ &\leqslant \beta(x_{n-1}, t)\beta(x_n, t)\varphi\bigl(M(x_{n-1}, x_n, t)\bigr) \\ &\leqslant k(t)\varphi\bigl(M(x_{n-1}, x_n, t)\bigr) \\ &< \varphi\bigl(\tau_{n-1}(t)\bigr). \end{aligned} \tag{2}$$

Since φ is decreasing, then $\tau_{n-1}(t) < \tau_n(t)$, that is, the sequence $\{\tau_n(t)\}$ is an increasing sequence for all $t > 0$. Take $\lim_{n\to+\infty} \tau_n(t) = \tau(t)$. We will show that $\tau(t) = 1$ for all $t > 0$. Suppose, to the contrary, that $0 < \tau(t_0) < 1$ for some $t_0 > 0$. Since $\tau_n(t_0) \leqslant \tau(t_0)$ and φ is continuous, by taking the limit as $n \to +\infty$ in (2) with $t = t_0$, we obtain

$$\varphi\bigl(\tau(t_0)\bigr) \leqslant k(t_0)\varphi\bigl(\tau(t_0)\bigr) < \varphi\bigl(\tau(t_0)\bigr),$$

which is a contradiction. Hence, $\tau(t) = 1$ for all $t > 0$. Now, we want show that $\{x_n\}$ is a Cauchy sequence. Assuming it is not true, then there exists $\epsilon \in (0, 1)$ and $t_0 > 0$ such that for all $k \in \mathbb{N}$ there exist $n(k), m(k) \in \mathbb{N}$ with $m(k) > n(k) \geqslant k$ and

$$M(x_{m(k)}, x_{n(k)}, t_0) \leqslant 1 - \epsilon. \tag{3}$$

Assume that $m(k)$ is the least integer exceeding $n(k)$ satisfying the above inequality. Equivalently,

$$M(x_{m(k)-1}, x_{n(k)}, t_0) > 1 - \epsilon \tag{4}$$

and so, for all k, we get

$$\begin{aligned} 1 - \epsilon &\geqslant M(x_{m(k)}, x_{n(k)}, t_0) \\ &\geqslant M(x_{m(k)-1}, x_{m(k)}, t_0) \star M(x_{m(k)-1}, x_{n(k)}, t_0) \\ &> \tau_{m(k)}(t_0) \star (1 - \epsilon). \end{aligned} \tag{5}$$

By taking limit as $n \to +\infty$ in (5), we deduce that

$$\lim_{n\to+\infty} M\bigl(x_{m(k)}, x_{n(k)}, t_0\bigr) = 1 - \epsilon.$$

From

$$\begin{aligned} &M(x_{m(k)+1}, x_{n(k)+1}, t_0) \\ &\quad\geqslant M(x_{m(k)+1}, x_{m(k)}, t_0) \star M(x_{m(k)}, x_{n(k)}, t_0) \star M(x_{n(k)}, x_{n(k)+1}, t_0) \end{aligned}$$

and

$$\begin{aligned} &M(x_{m(k)}, x_{n(k)}, t_0) \\ &\quad\geqslant M(x_{m(k)+1}, x_{m(k)}, t_0) \star M(x_{m(k)+1}, x_{n(k)+1}, t_0) \star M(x_{n(k)}, x_{n(k)+1}, t_0) \end{aligned}$$

we get

$$\lim_{n\to+\infty} M(x_{m(k)+1}, x_{n(k)+1}, t_0) = 1 - \epsilon.$$

Now, by (1) with $x = x_{m(k)}$ and $y = x_{n(k)}$, we have

$$\begin{aligned}
&\varphi\big(M(x_{m(k)+1}, x_{n(k)+1}, t_0)\big)\\
&\quad\leqslant \alpha(x_{m(k)}, x_{m(k)+1})\alpha(x_{n(k)}, x_{n(k)+1})\varphi\big(M(x_{m(k)+1}, x_{n(k)+1}, t_0)\big)\\
&\quad= \alpha(x_{m(k)}, fx_{m(k)})\alpha(x_{n(k)}, fx_{n(k)})\varphi\big(M(fx_{m(k)}, fx_{n(k)}, t_0)\big)\\
&\quad\leqslant \beta(x_{m(k)}, t_0)\beta(x_{n(k)}, t_0)\varphi\big(M(x_{m(k)}, x_{n(k)}, t_0)\big)\\
&\quad\leqslant k(t_0)\varphi\big(M(x_{m(k)}, x_{n(k)}, t_0)\big).
\end{aligned}$$

Using the continuity of the function φ, by taking the limit as $k \to +\infty$ in the above inequality, we get

$$\varphi(1-\epsilon) \leqslant k(t_0)\varphi(1-\epsilon).$$

Now, if $\varphi(1-\epsilon) = 0$ then by $(\varphi 2)$ we have $\epsilon = 0$, which is a contradiction. Otherwise, we assume that $\varphi(1-\epsilon) > 0$. Then $1 \leqslant k(t_0)$, which is a contradiction, since $0 < k(t_0) < 1$. Thus $\{x_n\}$ is a Cauchy sequence. The completeness of $(X, M, \star)$ ensures that the sequence $\{x_n\}$ converges to some $z \in X$, that is, for all $t > 0$,

$$\lim_{n\to+\infty} M(x_n, z, t) = 1.$$

Since, $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, by (F2), we get $0 < \tau_n(t) = M(x_n, x_{n+1}, t) < 1$ for all $t > 0$. Hence, there exists a subsequence $\{x_{n(r)}\}$ of $\{x_n\}$ such that $x_{n(r)} \neq z$ for all $n \in \mathbb{N}$. From (1) with $x = x_{n(r)}$ and $y = z$, we have

$$\begin{aligned}
\varphi\big(M(x_{n(r)+1}, fz, t)\big) &\leqslant \alpha(x_{n(r)}, x_{n(r)+1})\alpha(z, fz)\varphi\big((M(x_{n(r)+1}, fz, t)\big)\\
&= \alpha(x_{n(r)}, fx_{n(r)})\alpha(z, fz)\varphi\big((M(fx_{n(r)}, fz, t)\big)\\
&\leqslant \beta(x_{n(r)}, t)\beta(z, t)\varphi\big(M(x_{n(r)}, z, t)\big)\\
&\leqslant k(t)\varphi\big(M(x_{n(r)}, z, t)\big).
\end{aligned}$$

Taking the limit as $n \to +\infty$ in the above inequality, we have

$$\lim_{n\to+\infty} \varphi\big(M(x_{n(r)+1}, fz, t)\big) \leqslant k(t)\varphi(1) = 0$$

and hence $\lim_{n\to+\infty} M(x_{n(r)+1}, fz, t) = 1$ for all $t > 0$. From

$$M(x_{n(r)+1}, fz, t) \star M(x_{n(r)+1}, z, t) \leqslant M(z, fz, t),$$

by taking the limit as $n \to +\infty$, we obtain

$$\begin{aligned}
1 &= 1 \star 1\\
&= \Big(\lim_{n\to+\infty} M(x_{n(r)+1}, fz, t)\Big) \star \Big(\lim_{n\to+\infty} M(x_{n(r)+1}, z, t)\Big)\\
&\leqslant M(z, fz, t).
\end{aligned}$$

Hence, $M(z, fz, t) = 1$ and so $z = fz$.

Now, we assume that $y = fy$ implies $\alpha(y, fy) \geqslant 1$ and $\beta(x, t) < 1$ for all $x \in X$ and all $t > 0$. We show that z is the unique fixed point of f. Assume that $w \neq z$ is another fixed point of f and $M(z, w, t) < 1$ for all $t > 0$, then we have

$$\begin{aligned}\varphi\big(M(z,w,t)\big) &= \varphi\big(M(fz,fw,t)\big)\\ &\leqslant \alpha(z,fz)\alpha(w,fw)\varphi\big(M(fz,fw,t)\big)\\ &\leqslant \beta(z,t)\beta(w,t)\varphi\big(M(z,w,t)\big)\\ &< \varphi\big(M(z,w,t)\big),\end{aligned}$$

which is a contradiction and hence $M(z, w, t) = 1$ for $t > 0$, that is, $w = z$. □

Definition 7. Let $(X, M, \star)$ be a non-Archimedean fuzzy metric space and $f : X \to X$ be an α-admissible mapping. Also, suppose that $\psi, \phi : [0, 1] \to [0, 1]$ are two continuous functions such that ψ is decreasing, $\psi(t) > \psi(1) - \phi(1)$ and $\phi(t) > 0$ for all $t \in (0, 1)$. We say, f is a fuzzy α-ϕ-ψ-contractive mapping if

$$\alpha(x,fx)\alpha(y,fy)\psi\big(M(fx,fy,t)\big) \leqslant \psi\big(M(x,y,t)\big) - \phi\big(M(x,y,t)\big) \tag{6}$$

holds for all $x, y \in X$ and all $t > 0$.

For this class of mappings we have the following result of existence and uniqueness of fixed point.

Theorem 2. *Let $(X, M, \star)$ be a complete non-Archimedean fuzzy metric space, $\alpha : X \times X \to [0, +\infty)$, $\psi, \phi : [0, 1] \to [0, 1]$ as in Definition 7 and f be a fuzzy α-ϕ-ψ-contractive mapping such that the following assertions hold*:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geqslant 1$;*

(ii) *if $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x, fx) \geqslant 1$.*

Then f has a fixed point. Moreover, if $y = fy$ implies $\alpha(y, fy) \geqslant 1$, then f has a unique fixed point.

Proof. Define a sequence $\{x_n\}$ in X by $x_n = f^n x_0 = fx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for f and the result is proved. Hence, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Then, $0 < M(x_n, x_{n+1}, t) < 1$. Since f is an α-admissible mapping and $\alpha(x_0, fx_0) = \alpha(x_0, x_1) \geqslant 1$, we deduce that $\alpha(x_1, x_2) = \alpha(fx_0, fx_1) \geqslant 1$. By continuing this process, we get $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$. From (6) with $x = x_{n-1}$ and $y = x_n$, we obtain

$$\begin{aligned}\psi(M(x_n,x_{n+1},t)) &\leqslant \alpha(x_{n-1},x_n)\alpha(x_n,x_{n+1})\psi\big(M(fx_{n-1},fx_n,t)\big)\\ &\leqslant \psi(M(x_{n-1},x_n,t)) - \phi\big(M(x_{n-1},x_n,t)\big)\\ &< \psi\big(M(x_{n-1},x_n,t)\big).\end{aligned} \tag{7}$$

Since ψ is decreasing, then $M(x_{n-1}, x_n, t) < M(x_n, x_{n+1}, t)$. It follows that $\{M(x_n, x_{n+1}, t)\}$ is an increasing sequence in $(0, 1]$ and hence there exists $l(t) \in (0, 1]$ such that

$$\lim_{n\to+\infty} M(x_n, x_{n+1}, t) = l(t)$$

for all $t > 0$. Let us prove that $l(t) = 1$ for all $t > 0$. Suppose that there exists $t_0 > 0$ such that $0 < l(t_0) < 1$. By taking the limit as $n \to +\infty$ in (7), we have

$$\psi\bigl(l(t_0)\bigr) \leqslant \psi\bigl(l(t_0)\bigr) - \phi\bigl(l(t_0)\bigr).$$

Then $\phi(l(t_0)) = 0$, which is a contradiction and so $l(t) = 1$ for all $t > 0$. We will show that $\{x_n\}$ is a Cauchy sequence. Again, assuming it is not true and proceeding as in the proof of Theorem 1, there exist $\epsilon \in (0, 1)$ and $t_0 > 0$ such that for all $k \in \mathbb{N}$ there exist $n(k), m(k) \in \mathbb{N}$ with $m(k) > n(k) \geqslant k$ such that

$$\lim_{n\to+\infty} M(x_{m(k)}, x_{n(k)}, t_0) = 1 - \epsilon$$

and

$$\lim_{n\to+\infty} M(x_{m(k)+1}, x_{n(k)+1}, t_0) = 1 - \epsilon.$$

From (6) with $x = x_{m(k)}$ and $y = x_{n(k)}$, we deduce

$$\begin{aligned}
&\psi\bigl(M(x_{m(k)+1}, x_{n(k)+1}, t_0)\bigr)\\
&\quad= \psi\bigl(M(fx_{m(k)}, fx_{n(k)}, t_0)\bigr)\\
&\quad\leqslant \alpha(x_{m(k)}, fx_{m(k)})\alpha(x_{n(k)}, fx_{n(k)})\psi\bigl(M(fx_{m(k)}, fx_{n(k)}, t_0)\bigr)\\
&\quad\leqslant \psi\bigl(M(x_{m(k)}, x_{n(k)}, t_0)\bigr) - \phi\bigl(M(x_{m(k)}, x_{n(k)}, t_0)\bigr).
\end{aligned}$$

Applying the continuity of the functions ϕ and ψ, by taking the limit as $k \to +\infty$ in the above inequality, we get

$$\psi(1 - \epsilon) \leqslant \psi(1 - \epsilon) - \phi(1 - \epsilon)$$

and so $\phi(1 - \epsilon) = 0$, which is a contradiction. Then $\{x_n\}$ is a Cauchy sequence. Since $(X, M, \star)$ is a complete non-Archimedean fuzzy metric space, then the sequence $\{x_n\}$ converges to some $z \in X$, that is, for all $t > 0$, we have

$$\lim_{n\to+\infty} M(x_n, z, t) = 1.$$

Assume that there exists $t_0 > 0$ such that $0 < M(z, fz, t_0) < 1$. Then by (6) and (ii) we get,

$$\begin{aligned}
\psi\bigl(M(x_{n+1}, fz, t_0)\bigr) &= \psi\bigl(M(fx_n, fz, t_0)\bigr)\\
&\leqslant \alpha(x_n, fx_n)\alpha(z, fz)\psi\bigl(M(fx_n, fz, t_0)\bigr)\\
&\leqslant \psi\bigl(M(x_n, z, t_0)\bigr) - \phi\bigl(M(x_n, z, t_0)\bigr).
\end{aligned}$$

By taking the limit as $n \to +\infty$ in the above inequality, we have

$$\psi\big(M(z, fz, t_0)\big) \leqslant \psi(1) - \phi(1),$$

which is a contradiction. Hence, $M(z, fz, t) = 1$ for all $t > 0$, that is, $z = fz$.

Now, we assume that $y = fy$ implies $\alpha(y, fy) \geqslant 1$. If z, w are two fixed points of f such that there exists $t_0 > 0$ with $0 < M(z, w, t_0) < 1$, using (6), we get

$$\alpha(z, fz)\alpha(w, fw)\psi\big(M(fz, fw, t_0)\big) \leqslant \psi\big(M(z, w, t_0)\big) - \phi\big(M(z, w, t_0)\big).$$

Then

$$\psi\big(M(z, w, t_0)\big) \leqslant \psi\big(M(z, w, t_0)\big) - \phi\big(M(z, w, t_0)\big),$$

which implies $\phi(M(z, w, t_0)) = 0$, that is a contradiction. It follows that $M(z, w, t) = 1$ for all $t > 0$ and so $w = z$. □

Definition 8. Let $(X, M, \star)$ be a non-Archimedean fuzzy metric space, $\alpha : X \times X \to [0, +\infty)$ and $f : X \to X$ be an α-admissible mapping. We say, that f is a fuzzy α-β-contractive mapping if there exists a function $\beta : [0, 1] \to [1, +\infty)$ such that for any sequence $\{t_n\} \subseteq [0, 1]$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 1$ and

$$M(fx, fy, t) \geqslant \alpha(x, fx)\alpha(y, fy)\beta\big(M(x, y, t)\big)M(x, y, t) \tag{8}$$

for all $x, y \in X$ and all $t > 0$.

For the class of fuzzy α-β-contractive mappings we have the following result of existence and uniqueness of the fixed point.

Theorem 3. *Let $(X, M, \star)$ be a complete non-Archimedean fuzzy metric space, $\alpha : X \times X \to [0, +\infty)$, $\beta : [0, 1] \to [1, +\infty)$ as in Definition 8 and f be a fuzzy α-β-contractive self-mapping on X. Also suppose that the following assertions hold*:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geqslant 1$;*

(ii) *if $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x, fx) \geqslant 1$.*

Then f has a fixed point. Moreover, if $y = fy$ implies $\alpha(y, fy) \geqslant 1$ then f has a unique fixed point.

Proof. Define a sequence $\{x_n\}$ in X by $x_n = f^n x_0 = fx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for f and the result is proved. Hence, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Then, $0 < M(x_n, x_{n+1}, t) < 1$. Since f is an α-admissible mapping and $\alpha(x_0, fx_0) = \alpha(x_0, x_1) \geqslant 1$, we deduce that $\alpha(x_1, x_2) = \alpha(fx_0, fx_1) \geqslant 1$. By continuing this process, we get $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$. From (8) we get

$$\begin{aligned} &M(fx_{n-1}, fx_n, t) \\ &\quad \geqslant \alpha(x_{n-1}, fx_{n-1})\alpha(x_n, fx_n)\beta\big(M(x_{n-1}, x_n, t)\big)M(x_{n-1}, x_n, t) \\ &\quad \geqslant \beta\big(M(x_{n-1}, x_n, t)\big)M(x_{n-1}, x_n, t), \end{aligned}$$

and so

$$M(x_n, x_{n+1}, t) \geqslant \beta\big(M(x_{n-1}, x_n, t)\big)M(x_{n-1}, x_n, t) \geqslant M(x_{n-1}, x_n, t). \tag{9}$$

Then, $\{M(x_n, x_{n+1}, t)\}$ is an increasing sequence in $(0,1]$. Thus there exists $l(t) \in (0,1]$ such that $\lim_{n\to+\infty} M(x_n, x_{n+1}, t) = l(t)$ for all $t > 0$. We will prove that $l(t) = 1$ for all $t > 0$. By (9) we deduce

$$\frac{M(x_n, x_{n+1}, t)}{M(x_{n-1}, x_n, t)} \geqslant \beta\big(M(x_{n-1}, x_n, t)\big) \geqslant 1,$$

which implies $\lim_{n\to+\infty} \beta(M(x_{n-1}, x_n, t)) = 1$. Regarding the property of the function β, we conclude that

$$\lim_{n\to+\infty} M(x_n, x_{n+1}, t) = 1.$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Proceeding as in the proof of Theorem 1, there exist $\epsilon \in (0,1)$ and $t_0 > 0$ such that, for all $k \in \mathbb{N}$, there exist $n(k), m(k) \in \mathbb{N}$ with $m(k) > n(k) \geqslant k$ such that

$$\lim_{n\to+\infty} M(x_{m(k)}, x_{n(k)}, t_0) = 1 - \epsilon$$

and

$$\lim_{n\to+\infty} M(x_{m(k)+1}, x_{n(k)+1}, t_0) = 1 - \epsilon.$$

From (8) with $x = x_{m(k)}$ and $y = x_{n(k)}$ we deduce

$$\begin{aligned} &M(fx_{m(k)}, fx_{n(k)}, t_0) \\ &\quad \geqslant \alpha(x_{m(k)}, fx_{m(k)})\alpha(x_{n(k)}, fx_{n(k)})\beta\big(M(x_{m(k)}, x_{n(k)}, t)\big)M(x_{m(k)}, x_{n(k)}, t_0) \\ &\quad \geqslant \beta\big(M(x_{m(k)}, x_{n(k)}, t)\big)M(x_{m(k)}, x_{n(k)}, t_0), \end{aligned}$$

which implies

$$\frac{M(x_{m(k)+1}, x_{n(k)+1}, t_0)}{M(x_{m(k)}, x_{n(k)}, t_0)} \geqslant \beta\big(M(x_{m(k)}, x_{n(k)}, t_0)\big) \geqslant 1.$$

Taking the limit as $k \to +\infty$ in the above inequality we get

$$\lim_{k\to+\infty} \beta\big(M(x_{m(k)}, x_{n(k)}, t_0)\big) = 1,$$

which implies

$$1 - \epsilon = \lim_{k\to+\infty} M(x_{m(k)}, x_{n(k)}, t_0) = 1$$

and so $\epsilon = 0$, which is a contradiction. Then $\{x_n\}$ is a Cauchy sequence. Since $(X, M, \star)$ is a complete space, then the sequence $\{x_n\}$ converges to some $z \in X$ such that, for all $t > 0$, we have

$$\lim_{n\to+\infty} M(x_n, z, t) = 1.$$

By (8) we get

$$\begin{aligned} M(fx_n, fz, t) &\geqslant \alpha(x_n, fx_n)\alpha(z, fz)\beta\big(M(x_n, z, t)\big)M(x_n, z, t) \\ &\geqslant M(x_n, z, t). \end{aligned}$$

Taking the limit as $n \to +\infty$ in the above inequality, we have

$$\lim_{n\to+\infty} M(fx_n, fz, t) = 1$$

for all $t > 0$ and then

$$\begin{aligned} M(z, fz, t) &\geqslant \lim_{n\to+\infty} M(fx_n, z, t) \star \lim_{n\to+\infty} M(fx_n, fz, t) \\ &= 1 \star 1 = 1, \end{aligned}$$

that is, $z = fz$.

Now, we assume that $y = fy$ implies $\alpha(y, fy) \geqslant 1$. We show that z is the unique fixed point of f. Suppose that y, z are two fixed points of f and there exists $t_0 > 0$ such that $0 < M(y, z, t_0) < 1$. Using (8), we deduce

$$M(fz, fy, t_0) \geqslant \alpha(z, fz)\alpha(y, fy)\beta\big(M(z, y, t_0)\big)M(z, y, t_0)$$

and hence

$$1 = \frac{M(z, y, t_0)}{M(z, y, t_0)} \geqslant \beta\big(M(z, y, t_0)\big) \geqslant 1,$$

which implies $M(y, z, t_0) = 1$ that is a contradiction. Therefore, $M(y, z, t) = 1$ for all $t > 0$ and so $y = z$. □

Finally, we briefly discuss a recent result of Shen et al. [21]. Precisely, we consider the following theorem.

Theorem 4. (*See* [21].) *Let $(X, M, \star)$ be a complete fuzzy metric space and f be a self-mapping on X. Assume that $k : (0, +\infty) \to (0, 1)$ is a function and $\varphi \in \Phi$. Also, suppose that*

$$\varphi\big(M(fx, fy, t)\big) \leqslant k(t)\varphi\big(M(x, y, t)\big) \tag{10}$$

holds for all $x, y \in X$ with $x \neq y$ and all $t > 0$. Then f has a unique fixed point.

Shen et al. [21] claimed that if $\{x_n\}$ is not a Cauchy sequence, then there exist $0 < \epsilon < 1$ and two sequences $\{p(n)\}$ and $\{q(n)\}$ such that, for all $t > 0$, we have

$$\begin{aligned} &p(n) > q(n) \geqslant n, \qquad M(x_{p(n)}, x_{q(n)}, t) \leqslant 1 - \epsilon, \\ &M(x_{p(n)-1}, x_{q(n)-1}, t) > 1 - \epsilon, \qquad M(x_{p(n)-1}, x_{q(n)}, t) > 1 - \epsilon. \end{aligned} \tag{11}$$

Here, we note that if (X, d) is a complete metric space, then $(X, M, \star)$ is a complete fuzzy metric space if

$$M(x, y, t) = \frac{t}{t + d(x, y)} \quad \text{for all } x, y \in X \text{ and } t > 0. \tag{12}$$

Now, if $M(x_{p(n)}, x_{q(n)}, t) \leqslant 1 - \epsilon$ holds for all $t > 0$, clearly, we have

$$\lim_{t \to +\infty} M(x_{p(n)}, x_{q(n)}, t) \leqslant 1 - \epsilon,$$

which is a contradiction with respect to (12) that implies

$$\lim_{t \to +\infty} M(x_{p(n)}, x_{q(n)}, t) = 1.$$

Thus, the proof is wrong.

On the other hand, by taking $\alpha(x, y) = 1$ and $\beta(x, t)^2 = k(t)$ in Theorem 1, we deduce the following correct version of Theorem 4.

Theorem 5. *Let $(X, M, \star)$ be a complete non-Archimedean fuzzy metric space and f be a self-mapping on X. Assume that $k : (0, +\infty) \to (0, 1)$ is a function and $\varphi \in \Phi$. Also, suppose that*

$$\varphi\bigl(M(fx, fy, t)\bigr) \leqslant k(t)\varphi\bigl(M(x, y, t)\bigr)$$

holds for all $x, y \in X$ with $x \neq y$ and all $t > 0$. Then f has a unique fixed point.

4 Examples

In this section, we will present some examples to illustrate the usefulness of the proposed theoretical results.

Example 1. Let $X = [0, +\infty)$, $a \star b = \min\{a, b\}$,

$$M(x, y, t) = \begin{cases} 1/(1 + \max\{x, y\}) & \text{if } x \neq y, \\ 1 & \text{if } x = y, \end{cases}$$

for all $t > 0$, $fx = x/(2(x + 2))$, $\beta^2(x, t) = k(t) = 1/2$, $\alpha(x, y) = 1$ for all $x, y \in X$ and $t > 0$. Also define $\varphi(t) = 1 - t$ for all $t \in [0, 1]$.

Proof. Clearly, $(X, M, \star)$ is a non-Archimedean fuzzy metric space. Without loss of generality we assume that $x > y$. Since

$$fx = \frac{x}{2(x + 2)} \leqslant \frac{x}{x + 2}$$

then $xfx + 2fx \leqslant x$. Thus

$$\max\{x, y\} \max\{fx, fy\} + 2 \max\{fx, fy\} \leqslant \max\{x, y\}.$$

Therefore,

$$\begin{aligned} &\max\{x, y\} \max\{fx, fy\} + \max\{fx, fy\} + \max\{x, y\} \\ &\quad \leqslant 2 \max\{x, y\} - \max\{fx, fy\} \end{aligned}$$

and so

$$\begin{aligned}&\big(1+\max\{fx,fy\}\big)\big(1+\max\{x,y\}\big)\\&\quad=1+\max\{x,y\}\max\{fx,fy\}+\max\{fx,fy\}+\max\{x,y\}\\&\quad\leqslant 2\max\{x,y\}-\max\{fx,fy\}+1\\&\quad=2\big(1+\max\{x,y\}\big)-\big(1+\max\{fx,fy\}\big).\end{aligned}$$

Hence, we have

$$1\leqslant\frac{2(1+\max\{x,y\})-(1+\max\{fx,fy\})}{(1+\max\{fx,fy\})(1+\max\{x,y\})}=2M(fx,fy,t)-M(x,y,t),$$

which implies

$$1-M(fx,fy,t)\leqslant\frac{1}{2}\big(1-M(x,y,t)\big),$$

that is,

$$\alpha(x,fx)\alpha(y,fy)\varphi\big(M(fx,fy,t)\big)\leqslant\beta(x,t)\beta(y,t)\varphi\big(M(x,y,t)\big)$$

for all $x,y\in X$ with $x\neq y$ and hence f is a fuzzy (α,β,φ)-contractive mapping. Then all the conditions of Theorem 1 hold and f has a fixed point (here $x=0$ is a fixed point of f). Moreover, for all $x\in X$, we have $\alpha(x,fx)\geqslant 1$ and so the fixed point of f is unique. □

Example 2. Let $(X,M,\star)$ be the non-Archimedean fuzzy metric space and β, φ be the functions considered in Example 1. Also, define

$$fx=\begin{cases}x/(2(x+2)) & \text{if } x\in[0,1],\\(1+3\cos^2(\pi x))/(3+\cos(\pi x)) & \text{if } x\in(1,+\infty),\end{cases}$$

and

$$\alpha(x,y)=\begin{cases}1 & \text{if } x\in[0,1],\\0 & \text{otherwise.}\end{cases}$$

Let $x,y\in[0,1]$ and $x\neq y$, then $fx\neq fy$. Hence

$$M(x,y,t)=\frac{1}{1+\max\{x,y\}}\quad\text{and}\quad M(fx,fy,t)=\frac{1}{1+\max\{fx,fy\}}.$$

Also, $fx=x/(2(x+2))$ and $\alpha(x,y)=1$. By the similar method in the proof of Example 1, we can show that

$$\alpha(x,fx)\alpha(y,fy)\varphi\big(M(fx,fy,t)\big)\leqslant\beta(x,t)\beta(y,t)\varphi\big(M(x,y,t)\big).$$

Otherwise, $\alpha(x,fx)\alpha(y,fy)=0$ and so the condition (1) trivially holds.

Let $x, y \in X$, if $\alpha(x, y) \geqslant 1$ then $x, y \in [0, 1]$. On the other hand, for all $x \in [0, 1]$, we have $fx \leqslant 1$. It follows that $\alpha(fx, fy) \geqslant 1$, that is, f is α-admissible and hence f is a fuzzy (α, β, φ)-contractive mapping. In reason of the above arguments, $\alpha(0, f0) \geqslant 1$.

Now, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\{x_n\} \subset [0, 1]$ and hence $x \in [0, 1]$. This implies that $\alpha(x, fx) \geqslant 1$. Therefore, all the conditions of Theorem 1 hold and f has a unique fixed point.

Example 3. Let $X = [1, +\infty)$, $a \star b = \min\{a, b\}$ and $M(x, y, t) = \min\{x, y\}/\max\{x, y\}$ for all $t > 0$. Define

$$fx = \begin{cases} \pi/3 & \text{if } x \in [1, 3], \\ \sqrt{1 + x^2 + \mathrm{e}^x} & \text{if } x \in (3, +\infty). \end{cases}$$

Also define

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [1, 3], \\ 0 & \text{otherwise,} \end{cases}$$

$\psi(t) = 1 - t/2$ and $\phi(t) = t/2$ for all $t \in [0, 1]$.

Clearly, $(X, M, \star)$ is a non-Archimedean fuzzy metric space, $\psi, \phi : [0, 1] \to [0, 1]$ are continuous, ψ is decreasing, $\psi(t) > \psi(1) - \phi(1)$ and $\phi(t) > 0$ for all $t \in (0, 1)$.

Let, $x, y \in [1, 3]$. Then $\psi(M(fx, fy, t)) = 0$ and hence

$$\alpha(x, fx)\alpha(y, fy)\psi\bigl(M(fx, fy, t)\bigr) = 0 \leqslant \psi\bigl(M(x, y, t)\bigr) - \phi\bigl(M(x, y, t)\bigr).$$

Otherwise, $\alpha(x, fx)\alpha(y, fy) = 0$ and so

$$\alpha(x, fx)\alpha(y, fy)\psi\bigl(M(fx, fy, t)\bigr) = 0 \leqslant \psi\bigl(M(x, y, t)\bigr) - \phi\bigl(M(x, y, t)\bigr).$$

Since f is α-admissible we obtain that f is a fuzzy α-ϕ-ψ-contractive mapping. By the similar proof as in Example 2 the conditions (i) and (ii) of Theorem 2 hold. Then by Theorem 2, f has a unique fixed point.

Example 4. Let $(X, M, \star)$ be the non-Archimedean fuzzy metric space considered in Example 3. Define

$$fx = \begin{cases} 2x & \text{if } x \in [1, 3], \\ 1/\sqrt{1 + x} & \text{if } x \in (3, +\infty). \end{cases}$$

Also define

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [1, 3], \\ 0 & \text{otherwise,} \end{cases}$$

and $\beta(t) = 1$ for all $t \in [0, 1]$.

Let, $x, y \in [1, 3]$ and $x < y$. Then

$$M(fx, fy, t) = \frac{x}{y} \geqslant \frac{x}{y} = \alpha(x, fx)\alpha(y, fy)\beta\bigl(M(x, y, t)\bigr)M(x, y, t).$$

Otherwise, $\alpha(x, fx)\alpha(y, fy) = 0$ and so

$$M(fx, fy, t) \geqslant 0 = \alpha(x, fx)\alpha(y, fy)\beta\big(M(x, y, t)\big)M(x, y, t).$$

By the similar proof as in Example 2 the conditions (i) and (ii) of Theorem 3 hold. Then by Theorem 3, f has a unique fixed point.

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