Dynamic properties of the coupled Oregonator model with delay*

Xiang Wu, Chunrui Zhang[1]

Department of Mathematics, Northeast Forestry University
Harbin 150040, China
math@nefu.edu.cn

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Abstract. This work explores a coupled Oregonator model. By analyzing the associated characteristic equation, linear stability is investigated and Hopf bifurcations are demonstrated, as well as the stability and direction of the Hopf bifurcation are determined by employing the normal form method and the center manifold reduction. We also discussed the Z_2 equivariant property and the existence of multiple periodic solutions. Numerical simulations are presented to illustrate the results in Section 5.

Keywords: delay, Hopf bifurcation, stability, coupled Oregonator model.

1 Introduction

Delay in dynamical systems is exhibited whenever the system behavior is dependent at least in part on its history. Many technological and biological systems are known to exhibit such behavior, such as coupled laser systems, high-speed milling, population dynamics and gene expression [1–4].

$$\dot{X} = F(X), \tag{1}$$

where $X \in U$, $F \in C^2(U)$, $U \in R^n$ is a compact closure of the open set. When two identical oscillators coupling in the way of linear difference, the equation of motion of the system is

$$\begin{aligned} \dot{X} &= F(X) + K_1(Y - X), \\ \dot{Y} &= F(X) + K_2(X - Y), \end{aligned} \tag{2}$$

where K_1, K_2 are the coupling coefficient matrix (see [5–8]). Chemical diffusion coupling often described in this form [9]. In 1979, Tyson simplified the three-dimensional

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[1]Corresponding author.

oscillator Oregonator to 2-D:

$$\begin{aligned}\varepsilon\frac{\mathrm{d}x}{\mathrm{d}t} &= x(1-x) - hz\frac{x-u}{x+u},\\ \frac{\mathrm{d}z}{\mathrm{d}t} &= x - z,\end{aligned} \tag{3}$$

here $x = [\mathrm{HBrO_2}]$, $z = \mathrm{Ce\ (IV)}$. In 1999, Tianshou Zhou and Chunsuo Zhang proposed a coupled Oregonator model [10]

$$\begin{aligned}\varepsilon\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1(1-x_1) - hz_1\frac{x_1-u}{x_1+u} + D(x_2 - x_1),\\ \frac{\mathrm{d}z_1}{\mathrm{d}t} &= x_1 - z_1,\\ \varepsilon\frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_2(1-x_2) - hz_2\frac{x_2-u}{x_2+u} + D(x_1 - x_2),\\ \frac{\mathrm{d}z_2}{\mathrm{d}t} &= x_2 - z_2.\end{aligned}$$

When electric current is applied, the catalyst $\mathrm{Ce\ (IV)}$ is perturbed and other species are not affected (see [11]). Consequently, in modeling, the perturbation term is introduced only in equation $\mathrm{d}z/\mathrm{d}t = x - z$, and we rewrite this equation as the following form:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = x - z + kz(t-\tau). \tag{4}$$

Then the purpose of this paper is to consider coupled Oregonator model with a delay

$$\begin{aligned}\varepsilon\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1(1-x_1) - hz_1\frac{x_1-u}{x_1+u} + D(x_2 - x_1),\\ \frac{\mathrm{d}z_1}{\mathrm{d}t} &= x_1 - z_1 + kz_1(t-\tau),\\ \varepsilon\frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_2(1-x_2) - hz_2\frac{x_2-u}{x_2+u} + D(x_1 - x_2),\\ \frac{\mathrm{d}z_2}{\mathrm{d}t} &= x_2 - z_2 + kz_2(t-\tau),\end{aligned} \tag{5}$$

where $\varepsilon = 4\times 10^{-2}$, $\delta = 4\times 10^{-4}$, $u = 8\times 10^{-4}$, $h \in (0,1)$ is an adjustable parameter. The remainder of this paper organized as follows. In the next section, we shall consider the stability and the local Hopf bifurcation. Base on the symmetric bifurcation theorem of Golubitsky [12], we also discussed the Z_2 equivariant property and the existence of multiple periodic solutions in Section 3. In Section 4, based on the normal form method and the center manifold reduction introduced by Hassard et al. [13], we derive the formulae determining the direction, stability and the period of the bifurcating periodic solution at the critical value of τ, a conclusion is drawn in this section. To verify the theoretic analysis, numerical simulations are given in Section 5.

2 Stability and local Hopf bifurcations

Through out the paper, we assume that $k < 1$ (resulting in equilibrium point $x_0/z_0 = 1 - k > 0$).

Definition 1. (See [1].) Suppose $S(x_1^s, z_1^s, x_2^s, z_2^s)$ is the uniform steady state of system (5). If $x_1^s = x_2^s$, $z_1^s = z_2^s$, then S is an uniform steady state.

Let $S(x_1, z_1, x_2, z_2)$ be an equilibrium point of system (5). Obviously, $S(x_1, z_1, x_2, z_2)$ satisfying equation group

$$\begin{aligned} z_1 &= \frac{x_1}{1-k}, \\ z_2 &= \frac{x_2}{1-k}, \\ x_1(1-x_1) - hz_1\frac{x_1-u}{x_1+u} &+ D(x_2-x_1) = 0, \\ x_2(1-x_2) - hz_2\frac{x_2-u}{x_2+u} &+ D(x_1-x_2) = 0. \end{aligned} \tag{6}$$

Let $x_1 = x_2 = x$, then we have

$$x(1-x) - \frac{h}{1-k}x\frac{x-u}{x+u} = 0. \tag{7}$$

Equation (7) have has three roots $x = 0$, $x = x_+$, $x = x_-$, where

$$x_\pm = \frac{1 - \frac{h}{1-k} - u \pm \sqrt{(1 - \frac{h}{1-k} - u)^2 + 4u(1 + \frac{h}{1-k})}}{2}. \tag{8}$$

So, system (5) has three steady-state solution, $S_-(x_{1-}, z_{1-}, x_{2-}, z_{2-})$, $S_0(0,0,0,0)$, $S_+(x_{1+}, z_{1+}, x_{2+}, z_{2+})$. Obviously, there is an unique uniformly positive steady state. The following is to prove that the uniformly positive steady state is unique. Let $G(x) = x(1-x) - (h/(1-k))x(x-u)/(x+u)$, from (7) and (8) we conclude there is an unique x_+ satisfying $G(x) = 0$, and when $0 < x < x_+$, we have $G(x) > 0$; when $x > x_+$, we have $G(x) < 0$. Further more we obtain $G(x_+) = 0$, $\dot{G}(x_+) < 0$. From (6) we have $x_2 = x_1 - G(x_1)/D$, $D > 0$ and $G(x_1) + G(x_2) = 0$, that is why

$$G\left(x_1 - \frac{G(x_1)}{D}\right) + G(x_1) = 0. \tag{9}$$

The following we will study the function

$$g(x, D) = G\left(x - \frac{G(x)}{D}\right) + G(x) = 0. \tag{10}$$

Clearly $g(x_+, D) = 0$, that means x is a positive real root of Eq. (10). Notice that $0 < x < x_+$, from $\dot{G}(x_+) < 0$ and $G(x) > 0$, we can obtain $x - G(x)/D < x < x_+$ and

$G(x - G(x)/D) > 0$, it follows that $g(x, D) > 0$. On the other hand when $x > x_+$, from $G(x) < 0$, we conclude that $x - G(x)/D > x > x_+$ and $G(x - G(x)/D) < 0$, further more we have $g(x, D) < 0$. Then, the following lemma holds.

Lemma 1. *For any $\varepsilon > 0$, $u > 0$, $h > 0$ and $D > 0$, system* (5) *have an unique uniformly positive steady state $S_+(x_{10}, z_{10}, x_{20}, z_{20})$.*

The following work is expanded around the uniformly positive steady state and we don't consider of other steady state. Let $S_+(x_{10}, z_{10}, x_{20}, z_{20})$ satisfying Eqs. (6), where

$$x_{10} = x_{20} = \frac{1 - \frac{h}{1-k} - u + \sqrt{(1 - \frac{h}{1-k} - u)^2 + 4u(1 + \frac{h}{1-k})}}{2},$$
$$z_{10} = z_{20} = \frac{x_{10}}{1-k}.$$

Let $x_1 = x_1 - x_{10}$, $x_2 = x_2 - x_{20}$, $z_1 = z_1 - z_{10}$, $z_2 = z_2 - z_{20}$. Then we can rewrite (5) as the following equivalent system:

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \frac{1}{\varepsilon}\left((x_1+x_{10})(1-x_1-x_{10}) - h(z_1+z_{10})\frac{(x_1+x_{10})-u}{(x_1+x_{10})+u} + D(x_2-x_1)\right),\\
\frac{\mathrm{d}z_1}{\mathrm{d}t} &= x_1-z_1+kz_1(t-\tau),\\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= \frac{1}{\varepsilon}\left((x_2+x_{20})(1-x_2-x_{20}) - h(z_2+z_{20})\frac{(x_2+x_{20})-u}{(x_2+x_{20})+u}+D(x_1-x_2)\right),\\
\frac{\mathrm{d}z_2}{\mathrm{d}t} &= x_2-z_2+kz_2(t-\tau).
\end{aligned} \tag{11}$$

Set $x_0 = x_{10}$, $z_0 = z_{10}$.

The linearization of system (11) at $(0, 0, 0, 0)$ is

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= a_1x_1 + a_2z_1 + \frac{D}{\varepsilon}(x_2 - x_1),\\
\frac{\mathrm{d}z_1}{\mathrm{d}t} &= x_1 - z_1 + kz_1(t-\tau),\\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= a_1x_2 + a_2z_2 + \frac{D}{\varepsilon}(x_1 - x_2),\\
\frac{\mathrm{d}z_2}{\mathrm{d}t} &= x_2 - z_2 + kz_2(t-\tau),
\end{aligned}$$

where $a_1 = (1/\varepsilon)(-2uhz_0/(u+x_0)^2 + 1 - 2x_0)$, $a_2 = (1/\varepsilon)(uh - hx_0)/(u + x_0)$.

Moreover, its corresponding characteristic equation is

$$\begin{vmatrix}
\lambda - (a_1 - \frac{D}{\varepsilon}) & -a_2 & -\frac{D}{\varepsilon} & 0\\
-1 & \lambda + 1 - k\mathrm{e}^{-\lambda\tau} & 0 & 0\\
-\frac{D}{\varepsilon} & 0 & \lambda - (a_1 - \frac{D}{\varepsilon}) & -a_2\\
0 & 0 & -1 & \lambda + 1 - k\mathrm{e}^{-\lambda\tau}
\end{vmatrix}$$
$$= \begin{vmatrix}
\lambda - a_1 & -a_2\\
-1 & \lambda + 1 - k\mathrm{e}^{-\lambda\tau}
\end{vmatrix}
\begin{vmatrix}
\lambda - (a_1 - \frac{2D}{\varepsilon}) & -a_2\\
-1 & \lambda + 1 - k\mathrm{e}^{-\lambda\tau}
\end{vmatrix} = \Delta_1\Delta_2 = 0, \tag{12}$$

where $\Delta_1 = \lambda^2 - b_1\lambda - b_2 - k\lambda e^{-\lambda\tau} + ka_1 e^{-\lambda\tau}$, $\Delta_2 = \lambda^2 - b_1\lambda - b_2 - k\lambda e^{-\lambda\tau} + ka_1 e^{-\lambda\tau} + (2D/\varepsilon)(\lambda + 1 - ke^{-\lambda\tau})$, $b_1 = a_1 - 1$, $b_2 = a_1 + a_2$.

In this section, we will study the distribution of roots of Eq. (12). We first introduce the following important result, which was been proved by Ruan and Wei using Rouche theorem [14]. For $\tau = 0$, the two roots of $\Delta_1 = 0$ have negative real parts if and only if $k + b_1 < 0$, $ka_1 - b_2 > 0$. Because of $D > 0$, we obtain if $k + b_1 < 0$, then $k + b_1 - 2D/\varepsilon < 0$ and if $ka_1 - b_2 > 0$, then $ka_1 - b_2 - k(2D/\varepsilon) + 2D/\varepsilon > 0$. Thus, the two roots of $\Delta_2 = 0$ have negative real parts. We impose the following condition:

(A1) $ka_1 > b_2, k < -b_1$.

Lemma 2. *Let* $\tau = 0$. *Then if* (A1) *is satisfied, all the roots of* (12) *have negative real parts, hence* $(x_{10}, z_{10}, x_{20}, z_{20})$ *is asymptotically stable.*

Next, we mainly focus on the case of $\tau > 0$.

Case 1. If $\lambda = iw_1$ $(w_1 > 0)$ is a purely imaginary root of $\Delta_1 = 0$ for $\tau > 0$, then we have

$$-\omega_1^2 - b_1 i\omega_1 - b_2 - ki\omega_1 e^{-i\omega_1\tau} + ka_1 e^{-i\omega_1\tau} = 0.$$

Separating the real and imaginary parts, we obtain

$$\begin{aligned} &-\omega_1^2 - b_2 - k\omega_1 \sin\omega_1\tau + ka_1\cos\omega_1\tau = 0,\\ &-b_1\omega_1 - k\omega_1\cos\omega_1\tau - ka_1\sin\omega_1\tau = 0, \end{aligned} \tag{13}$$

which implies

$$w_1^4 + (2b_2 + b_1^2 - k^2)w_1^2 + b_2^2 - k^2a_1^2 = 0. \tag{14}$$

Let $z = w_1^2$ and denote

$$u_1 = 2b_2 + b_1^2 - k^2, r_1 = b_2^2 - k^2a_1^2.$$

Then, (15) becomes

$$m^2 + u_1 m + r_1 = 0. \tag{15}$$

In order to seek a positive solution for Eq. (14), we impose the following condition:

(B1) $r_1 < 0$.

Clearly, under the condition (B1), (12) has a unique positive root $m = (1/2) \times (-u_1 + \sqrt{u_1^2 - 4r_1})$.

(B2) $r_1 > 0, u_1 > 0$.

Under the condition (B2), (12) has no positive root.

(B3) $r_1 > 0, u_1 < 0$.

Under the condition (B3), if there are real positive roots, then $|k|$ is very large, h infinitely close to one, does not match with the actual situation.

Summarizing the above discussions, we obtain the following.

Lemma 3. *For the polynomial equation* (15)*, we have the following result*:

(i) *If* $r_1 < 0$*, then equation* $\Delta_1 = 0$ *has a unique positive root* $m = (1/2) \times (-u_1 + \sqrt{u_1^2 - 4r_1})$.

(ii) *If* $r_1 > 0$*, then equation* $\Delta_1 = 0$ *has no positive root.*

Suppose that Eq. (15) has positive roots. Without loss of generality, we assume that it has a positive root defined by m. Then, Eq. (14) has a positive root ω_1, moreover ω_1 must satisfies the following equation:

$$\left(\frac{w_1^2 + a_1 b_2}{k(w_1^2 + a_1^2)}\right)^2 + \left(\frac{w_1^3 + w_1(a_1^2 + a_2)}{kw_1^2 + ka_1^2}\right)^2 = 1.$$

By (13), we have

$$\cos(w_1\tau) = \frac{w_1^2 + a_1 b_2}{k(w^2 + a_1^2)}, \qquad \sin(w_1\tau) = -\frac{w_1^3 + w_1(a_1^2 + a_2)}{kw_1^2 + ka_1^2}.$$

Thus, denote

$$\alpha_1 = -\frac{w_1^3 + w_1(a_1^2 + a_2)}{kw_1^2 + ka_1^2}, \qquad \beta_1 = \frac{w_1^2 + a_1 b_2}{k(w^2 + a_1^2)},$$

$$\tau_{1j} = \begin{cases} \frac{1}{w_1}(\arccos\beta_1 + 2j\pi), & \alpha_1 \geqslant 0, \\ \frac{1}{w_1}(2\pi - \arccos\beta_1 + 2j\pi), & \alpha_1 < 0, \end{cases}$$

where $j = 0, 1, 2, \ldots$, then $\pm \mathrm{i}w_1$ is a pair of purely imaginary roots of (12) with $\tau = \tau_{1j}$.

If (B1) hold, we have $k < 0$, then

$$\tau_{1j} = \frac{1}{w_1}(\arccos b_1 + 2j\pi), \quad j \in \{1, 2, \ldots\}.$$

Case 2. If $\lambda = \mathrm{i}w_2$ ($w_2 > 0$) is a purely imaginary root of $\Delta_2 = 0$ for $\tau > 0$, then we have

$$-\left(b_1 - \frac{2D}{\varepsilon}\right)\mathrm{i}\omega_2 - \left(b_2 - \frac{2D}{\varepsilon}\right) - k\mathrm{i}\omega_2 \mathrm{e}^{-\mathrm{i}\omega_2\tau} + k\left(a_1 - \frac{2D}{\varepsilon}\right)\mathrm{e}^{-\mathrm{i}\omega_2\tau} = 0.$$

Separating the real and imaginary parts, we obtain

$$\begin{aligned} &-\omega_2^2 - \left(b_2 - \frac{2D}{\varepsilon}\right) - k\omega_2 \sin\omega_2\tau + k\left(a_1 - \frac{2D}{\varepsilon}\right)\cos\omega_2\tau = 0, \\ &-\left(b_1 - \frac{2D}{\varepsilon}\right)\omega_2 - k\omega_2\cos\omega_2\tau - k\left(a_1 - \frac{2D}{\varepsilon}\right)\sin\omega_2\tau = 0, \end{aligned} \tag{16}$$

which implies

$$w_2^4 + \left(2\left(b_2 - \frac{2D}{\varepsilon}\right) + \left(b_1 - \frac{2D}{\varepsilon}\right)^2 - k^2\right)w_2^2 + \left(b_2 - \frac{2D}{\varepsilon}\right)^2 - k^2\left(a_1 - \frac{2D}{\varepsilon}\right)^2 = 0. \tag{17}$$

Denote

$$u_2 = 2\left(b_2 - \frac{2D}{\varepsilon}\right) + \left(b_1 - \frac{2D}{\varepsilon}\right)^2 - k^2,$$
$$r_2 = \left(b_2 - \frac{2D}{\varepsilon}\right)^2 - k^2\left(a_1 - \frac{2D}{\varepsilon}\right)^2.$$

Then, Eq. (17) becomes

$$m^2 + u_2 m + r_2 = 0. \tag{18}$$

In order to seek a positive solution for (18), we impose the following condition:

(C1) $r_2 < 0$.
Clearly, under the condition (C1), (18) has a unique positive root $m = (1/2) \times (-u_2 + \sqrt{u_2^2 - 4r_2})$.

(C2) $r_2 > 0$, $u_2 > 0$.
Under the condition (C2), (18) has no positive root.

(C3) $r_2 > 0$, $u_2 < 0$.
Under the condition (C3), if $u_2^2 - 4r_2 > 0$, then (18) has a pair of roots

$$m_{1,2} = \frac{-u_2 \pm \sqrt{u_2^2 - 4r_2}}{2}.$$

Summarizing the above discussions, we obtain the following.

Lemma 4. *For the polynomial equation* (18)*, we have the following result*:

(i) *If* $r_2 < 0$*, then equation* $\Delta_2 = 0$ *has a unique positive root* $m = (1/2) \times (-u_2 + \sqrt{u_2^2 - 4r_2})$.

(ii) *If* $r_2 > 0$*,* $u_2 < 0$*, then equation* $\Delta_2 = 0$ *has a pair of roots* $m_{1,2} = (1/2) \times (-u_2 \pm \sqrt{u_2^2 - 4r_2})$.

Suppose that Eq. (18) has some positive roots. Without loss of generality, we assume that it has a positive root defined by m. Then, (14) has a positive root ω_2, moreover ω_2 must satisfies the following equations:

$$\left(\frac{w_2^2 + (a_1 - \frac{2D}{\varepsilon})(b_2 - \frac{2D}{\varepsilon})}{k(w^2 + (a_1 - \frac{2D}{\varepsilon})^2)}\right)^2 + \left(\frac{w_2^3 + w_2((a_1 - \frac{2D}{\varepsilon})^2 + a_2)}{kw_2^2 + k(a_1 - \frac{2D}{\varepsilon})^2}\right)^2 = 1.$$

By (16), we have

$$\cos(w_2\tau) = \frac{w_2^2 + (a_1 - \frac{2D}{\varepsilon})(b_2 - \frac{2D}{\varepsilon})}{k(w_2^2 + (a_1 - \frac{2D}{\varepsilon})^2)},$$
$$\sin(w_2\tau) = -\frac{w_2^3 + w_2((a_1 - \frac{2D}{\varepsilon})^2 + a_2)}{kw_2^2 + k(a_1 - \frac{2D}{\varepsilon})^2}.$$

Thus, denoting

$$\alpha_2 = -\frac{w_2^3 + w_2((a_1 - \frac{2D}{\varepsilon})^2 + a_2)}{kw_2^2 + k(a_1 - \frac{2D}{\varepsilon})^2},$$
$$\beta_2 = \frac{w_2^2 + (a_1 - \frac{2D}{\varepsilon})(b_2 - \frac{2D}{\varepsilon})}{k(w_2^2 + (a_1 - \frac{2D}{\varepsilon})^2)},$$
$$\tau_{2j} = \begin{cases} \frac{1}{w_2}(\arccos\beta_2 + 2j\pi), & \alpha_2 \geqslant 0, \\ \frac{1}{w_2}(2\pi - \arccos\beta_2 + 2j\pi), & \alpha_2 < 0, \end{cases}$$

where $j = 0, 1, 2, \dots$, then $\pm \mathrm{i}w_2$ is a pair of purely imaginary roots of (12) with $\tau = \tau_{2j}$.

If (C1) hold, we have

$$\tau_{2j} = \frac{1}{w_2}(\arccos\beta_2 + 2j\pi), \quad j \in \{1, 2, \dots\}.$$

As the same, for condition (C3) and $u_2^2 - 4r_2 > 0$ is satisfied, then (17) has at least two roots $w_2^{(1)}$ and $w_2^{(2)}$. So, we have

$$\tau_{2j}^{(1)} = \begin{cases} \frac{1}{w_2^{(1)}}(\arccos\beta_2^{(1)} + 2j\pi), & \alpha_2^{(1)} \geqslant 0, \\ \frac{1}{w_2^{(1)}}(2\pi - \arccos\beta_2^{(1)} + 2j\pi), & \alpha_2^{(1)} < 0, \end{cases}$$

where

$$\alpha_2^{(1)} = -\frac{(w_2^{(1)})^3 + w_2^{(1)}((a_1 - \frac{2D}{\varepsilon})^2 + a_2)}{k(w_2^{(1)})^2 + k(a_1 - \frac{2D}{\varepsilon})^2},$$
$$\beta_2^{(1)} = \frac{(w_2^{(1)})^2 + (a_1 - \frac{2D}{\varepsilon})(b_2 - \frac{2D}{\varepsilon})}{k((w_2^{(1)})^2 + (a_1 - \frac{2D}{\varepsilon})^2)}.$$

And

$$\tau_{2j}^{(2)} = \begin{cases} \frac{1}{w_2^{(2)}}(\arccos\beta_2^{(2)} + 2j\pi), & \alpha_2^{(2)} \geqslant 0, \\ \frac{1}{w_2^{(2)}}(2\pi - \arccos\beta_2^{(2)} + 2j\pi), & \alpha_2^{(2)} < 0, \end{cases}$$

where

$$\alpha_2^{(2)} = -\frac{(w_2^{(2)})^3 + w_2^{(2)}((a_1 - \frac{2D}{\varepsilon})^2 + a_2)}{k(w_2^{(2)})^2 + k(a_1 - \frac{2D}{\varepsilon})^2},$$
$$\beta_2^{(2)} = \frac{(w_2^{(2)})^2 + (a_1 - \frac{2D}{\varepsilon})(b_2 - \frac{2D}{\varepsilon})}{k((w_2^{(2)})^2 + (a_1 - \frac{2D}{\varepsilon})^2)}$$

Because the condition (C3) and $u_2^2 - 4r_2 > 0$ is satisfied, $(\mathrm{d}(\operatorname{Re}\lambda(\tau))/\mathrm{d}\tau)^{-1}_{\tau=\tau_{2j}^{(1)},\tau_{2j}^{(2)}} > 0$

(we will proof of it in later). So, $\tau_{2j} = \min\{\tau_{2j}^{(1)}, \tau_{2j}^{(2)}\}$.

Here we consider the whole system (12). Note that when $\tau = 0$, Eq. (12) becomes

$$\left(\lambda^2 - (k+b_1)\lambda + ka_1 - b_2\right)\left(\lambda^2 - (k+b_1)\lambda + ka_1 - b_2 + \frac{2D}{\varepsilon}(\lambda+1-k)\right) = 0. \quad (19)$$

Using Lemmas 1–3, we have the following results.

Lemma 5. *For the exponential polynomial equation* (12)*, we have*:

(i) *If* $\min\{r_1, r_2\} > 0$, $u_2^2 - 4r_2 < 0$ *and the condition* (A1) *is satisfied, then all roots with positive real parts of Eq.* (12) *has the same sum as those of the polynomial equation* (19) *for all* $\tau > 0$.

(ii) *If* (B1) *is satisfied, then all roots with positive real parts of equation* $\Delta_1 = 0$ *has the same sum as those of the polynomial equation* $\lambda^2 - (k + b_1)\lambda + ka_1 - b_2 = 0$ *for* $\tau \in [0, \tau_{10})$.

(iii) *If* (C1) *is satisfied, then all roots with positive real parts of equation* $\Delta_2 = 0$ *has the same sum as those of the polynomial equation* $\lambda^2 - (k + b_1)\lambda + ka_1 - b_2 + (2D/\varepsilon)(\lambda + 1 - k) = 0$ *for* $\tau \in [0, \tau_{20})$.

(iv) *If* (C3) *is satisfied, then all roots with positive real parts of equation* $\Delta_2 = 0$ *has the same sum as those of the polynomial equation* $\lambda^2 - (k + b_1)\lambda + ka_1 - b_2 + (2D/\varepsilon)(\lambda + 1 - k) = 0$ *for* $\tau \in [0, \tau_{20})$.

For convenience, we make some hypotheses as follows:

(P1) (1) $\min\{r_1, r_2\} > 0$;
(2) $ka_1 > b_2$, $k < b_1$;
(3) $u_2 > 0$ or $u_2 < 0$, $u_2^2 - 4r_2 < 0$.

(P2) (1) $r_1 < 0, r_2 > 0$;
(2) $k + b_1 - 2D/\varepsilon < 0, ka_1 - b_2 - k(2D/\varepsilon) + 2D/\varepsilon > 0$;
(3) $u_2 > 0$ or $u_2 < 0, u_2^2 - 4r_2 < 0$ is satisfied.

Then denoting $\tau_j = \tau_{1j}$.

(P3) $r_1 > 0$, $ka_1 > b_2$, $k < b_1$, and one of the following holds,
(1) $r_2 < 0$;
(2) $u_2 < 0$ and $u_2^2 - 4r_2 > 0$.

Then denote $\tau_j = \tau_{2j}$.

(P4) One of the following is satisfied:
(1) $r_1 < 0$ and $r_2 < 0$;
(2) $r_1 < 0$ and $r_2 > 0$, $u_2 < 0$, $u_2^2 - 4r_2 > 0$.

Let $\lambda(\tau) = a(\tau) + \mathrm{i}w(\tau)$ be the root of Eq. (5) near $\tau = \tau_j$ satisfying $a(\tau_j) = 0$, $w(\tau_j) = w_j$. Then, the following transversality condition holds.

Lemma 6. *Suppose that one the the hypothesis* (P2), (P3), (P4) *is satisfied, then* $(\mathrm{d}(\operatorname{Re}\lambda(\tau))/\mathrm{d}\tau)^{-1}_{\tau=\tau_{1j}} > 0$, $(\mathrm{d}(\operatorname{Re}\lambda(\tau))/\mathrm{d}\tau)^{-1}_{\tau=\tau_{2j}} > 0$.

Proof. Substituting $\lambda(\tau)$ into (12) and differentiating the resulting equation in τ, we obtain

$$\begin{aligned}
&2\big(\lambda^2-(k+b_1)\lambda+ka_1-b_2\big)\\
&\quad\times\left(\big(2\lambda-b_1\lambda+\tau k\lambda\mathrm{e}^{-\lambda\tau}-k\mathrm{e}^{-\lambda\tau}-\tau a_1\mathrm{e}^{-\lambda\tau}\big)\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}-\lambda ka_1\mathrm{e}^{-\lambda\tau}+k\lambda^2\mathrm{e}^{-\lambda\tau}\right)\\
&\quad+\frac{2D}{\varepsilon}\big(\lambda+1-k\mathrm{e}^{-\lambda\tau}\big)\\
&\quad\times\left(\big(2\lambda-b_1\lambda+\tau k\lambda\mathrm{e}^{-\lambda\tau}-k\mathrm{e}^{-\lambda\tau}-\tau a_1\mathrm{e}^{-\lambda\tau}\big)\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}-\lambda ka_1\mathrm{e}^{-\lambda\tau}+k\lambda^2\mathrm{e}^{-\lambda\tau}\right)\\
&\quad+\frac{2D}{\varepsilon}\big(\lambda^2-(k+b_1)\lambda+ka_1-b_2\big)\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}+\lambda k\mathrm{e}^{-\lambda\tau}+\tau k\mathrm{e}^{-\lambda\tau}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)=0.
\end{aligned}$$

Denote

$$\Delta_1'=\big(2\lambda-b_1\lambda+\tau k\lambda\mathrm{e}^{-\lambda\tau}-k\mathrm{e}^{-\lambda\tau}-\tau a_1\mathrm{e}^{-\lambda\tau}\big)\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}-\lambda ka_1\mathrm{e}^{-\lambda\tau}+k\lambda^2\mathrm{e}^{-\lambda\tau},$$

we have

$$\begin{aligned}
&2\Delta_1\Delta_1'+\frac{2D}{\varepsilon}\big(\lambda+1-k\mathrm{e}^{-\lambda\tau}\big)\Delta_1'+\frac{2D}{\varepsilon}\Delta_1\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\\
&\quad+\frac{2D}{\varepsilon}\left(\lambda k\mathrm{e}^{-\lambda\tau}+\tau k\mathrm{e}^{-\lambda\tau}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\Delta_1=0.
\end{aligned}\tag{20}$$

We first focus on the case $\Delta_1=0$. Under the condition (P2), when $\tau=\tau_{1j}$, we have $\Delta_1=0$. Thus Eq. (20) becomes

$$\begin{aligned}
&\frac{2D}{\varepsilon}\big(\lambda+1-k\mathrm{e}^{-\lambda\tau}\big)\\
&\quad\times\left(\big(2\lambda-b_1\lambda+\tau k\lambda\mathrm{e}^{-\lambda\tau}-k\mathrm{e}^{-\lambda\tau}-\tau a_1\mathrm{e}^{-\lambda\tau}\big)\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}-\lambda ka_1\mathrm{e}^{-\lambda\tau}+k\lambda^2\mathrm{e}^{-\lambda\tau}\right)=0.
\end{aligned}$$

Then

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}=\frac{2\lambda-b_1\lambda+\tau k\lambda\mathrm{e}^{-\lambda\tau}-k\mathrm{e}^{-\lambda\tau}-\tau a_1\mathrm{e}^{-\lambda\tau}}{\lambda ka_1\mathrm{e}^{-\lambda\tau}-k\lambda^2\mathrm{e}^{-\lambda\tau}}.$$

We can easily obtain

$$\begin{aligned}
\left(\frac{\mathrm{d}(\operatorname{Re}\lambda(\tau))}{\mathrm{d}\tau}\right)^{-1}_{\tau=\tau_{1j}}&=\operatorname{Re}\left\{\frac{2\lambda-b_1\lambda+\tau k\lambda\mathrm{e}^{-\lambda\tau}-k\mathrm{e}^{-\lambda\tau}-\tau a_1\mathrm{e}^{-\lambda\tau}}{\lambda ka_1\mathrm{e}^{-\lambda\tau}-k\lambda^2\mathrm{e}^{-\lambda\tau}}\right\}_{\tau=\tau_{1j}}\\
&=\left(\frac{1}{w_1^2+a_1^2}+\frac{(3-a_1)(w_1^2+b_2)+\tau ka_1-\tau a_1}{k(a_1^2+w_0^2)}\right).
\end{aligned}$$

For $a_1 < 0$, we have $k < 0$. In the previous part of this paper we know $|k|$ can't be very large. As mentioned above, it can be obtained that

$$\operatorname{sgn}\left[\left(\frac{1}{w_1^2 + a_1^2} + \frac{(3 - a_1)(w_1^2 + b_2) + \tau k a_1 - \tau a_1}{k(a_1^2 + w_1^2)}\right)\right] > 0.$$

For $a_1 > 0$, we have $0 < k < 1 - 2f$ and $a_1 < 2$. It can be obtained that

$$\operatorname{sgn}\left[\left(\frac{1}{w_1^2 + a_1^2} + \frac{(3 - a_1)(w_1^2 + b_2) + \tau k a_1 - \tau a_1}{k(a_1^2 + w_1^2)}\right)\right] > 0.$$

Then we focus on the case where $\Delta_2 = 0$. As the same, we have

$$\begin{aligned}&\left(\frac{\mathrm{d}(\operatorname{Re}\lambda(\tau))}{\mathrm{d}\tau}\right)^{-1}_{\tau=\tau_{2j}}\\ &\quad= \operatorname{sgn}\left[\left(\frac{k^2 + (3 - (a_1 - \frac{2D}{\varepsilon}))(w_2^2 + (b_2 - \frac{2D}{\varepsilon})) + \tau(a_1 - \frac{2D}{\varepsilon})(k - 1)}{k((a_1 - \frac{2D}{\varepsilon})^2 + w_2^2)}\right)\right].\end{aligned}$$

In the previous part of this paper we know $|k|$ and w_2 can't be very large, ε is very small, it can be obtained that

$$\operatorname{sgn}\left[\left(\frac{k + (3 - (a_1 - \frac{2D}{\varepsilon}))(w_2^2 + (b_2 - \frac{2D}{\varepsilon})) + \tau(a_1 - \frac{2D}{\varepsilon})(k - 1)}{k((a_1 - \frac{2D}{\varepsilon})^2 + w_2^2)}\right)\right] > 0.$$

So under the condition (P3), we have

$$\left(\frac{\mathrm{d}(\operatorname{Re}\lambda(\tau))}{\mathrm{d}\tau}\right)^{-1}_{\tau=\tau_{2j}} > 0.$$

As the same, under the condition (P4), we have

$$\left(\frac{\mathrm{d}(\operatorname{Re}\lambda(\tau))}{\mathrm{d}\tau}\right)^{-1}_{\tau=\tau_{1j}} > 0, \qquad \left(\frac{\mathrm{d}(\operatorname{Re}\lambda(\tau))}{\mathrm{d}\tau}\right)^{-1}_{\tau=\tau_{2j}} > 0.$$

In the case of $r_1 < 0$, $r_2 < 0$, we have $k < 0$. Then we have the following results:

$$\tau_{1j} = \frac{1}{w_1}(\arccos b_1 + 2j\pi), \qquad \tau_{2j} = \frac{1}{w_2}(\arccos b_2 + 2j\pi), \qquad \tau_j = \min\{\tau_{1j}, \tau_{2j}\}.$$

In addition, the second condition of (P4) is not established. Then condition (P4) modified as $r_1 < 0$, $r_2 < 0$. Therefore, the transversality condition holds and Hopf-bifurcation occurs at $\tau = \tau_j$. □

From Lemmas 4 and 5, we have the following.

Theorem 1. (i) *If the condition* (P1) *is satisfied, then the zero solution of system* (5) *is asymptotically stable for all* $\tau > 0$.

(ii) *If one of the hypothesis* (P2), (P3), (P4) *is satisfied, then the zero solution of system* (5) *is asymptotically stable for* $\tau \in (0, \tau_0)$, *and unstable for* $\tau > \tau_0$. *The system* (5) *undergoes a Hopf bifurcation at the zero solution when* $\tau = \tau_j$ $(j = 0, 1, 2 \ldots)$.

3 Existence of multiple periodic solutions

In the following, we consider the symmetric properties of Eq. (5). Using the theories of functional differential equation, we know that the system (5) is Z_2-equivariant with

$$(\rho U)_r = U_{r+1} \pmod 2$$

for any U_r in R^2. It is much interesting to consider the spatio-temporal patterns of bifurcating periodic solutions. For this purpose, we give the concepts of some spatiotemporal symmetric periodic solutions. Assume that the state $(u_1(t), v_1(t), u_2(t), v_2(t))$ can possess two different types of symmetry: spatial and temporal. The oscillators $(u_1(t), v_1(t))$ and $(u_2(t), v_2(t))$ are in-phase if the state taking the form

$$\big(u(t), v(t), u(t), v(t)\big)$$

for all times t. On the other hand, oscillator $(u_1(t), v_1(t))$, is half a period out of phase with (anti-synchronous) oscillator $(u_2(t), v_2(t))$ means the state taking the form

$$\left(u(t), v(t), u\left(t + \frac{T}{2}\right), v\left(t + \frac{T}{2}\right)\right).$$

Now, we explore the possible (spatial) symmetry of the system (5). Consider the action of $Z_2 \times S^1$ on $([-\tau, 0], R^4)$ with

$$(r, \theta)x(t) = rx(t + \theta), \quad (r, \theta) \in Z_2 \times S^1,$$

where S^1 is the temporal. Let $T = 2\pi/\omega_1$ or $T = 2\pi/\omega_2$, and denote P_T the Banach space of all continuous T-periodic function $x(t)$. Denoting SP_T the subspace of P_T consisting of all T-periodic solution of system (5) with $\tau = \tau_{kj}$ $(k = 1, 2)$, then for each subgroup $\Sigma \subset Z_2 \times S^1$,

$$\mathrm{Fix}(\Sigma, SP_T) = \big\{x \in SP_T,\ (r, \theta)x = x \text{ for all } (r, \theta) \in \Sigma\big\}$$

is a subspace.

Theorem 2. *The trivial solution of system* (5) *undergoes a Hopf bifurcation at giving rise to one branch of in-phase (respectively, anti-phase) periodic solutions.*

Proof. Let ω_1 satisfies Eq. (15). The corresponding eigenvectors of $\Delta_1 = 0$ can be chosen as

$$q_1(\theta) = \left(\frac{a_2}{\mathrm{i}w_1 + a_1},\ 1,\ \frac{a_2}{\mathrm{i}w_1 + a_1},\ 1\right)^{\mathrm{T}} \mathrm{e}^{\mathrm{i}w_1\tau_1\theta}.$$

The isotropic subgroup of $Z_2 \times S^1$ is $z_2(\rho)$, the center space associated to eigenvalues $\pm\mathrm{i}\omega_1$ is spanned by $q_1(\theta)$ and $\bar{q}_1(\theta)$, and the bifurcated periodic solutions are in-phase, taking the form

$$\big(u(t), v(t), u(t), v(t)\big).$$

Similarly, if ω_2 satisfies Eq. (18), then corresponding eigenvectors of $\Delta_2 = 0$ can be chosen as

$$q_2(\theta) = \left(\frac{a_2}{\mathrm{i}w_2 + a_1 - \frac{2D}{\varepsilon}},\ 1,\ -\frac{a_2}{\mathrm{i}w_2 + a_1 - \frac{2D}{\varepsilon}},\ -1\right)^{\mathrm{T}} \mathrm{e}^{\mathrm{i}w_2\tau_2\theta}.$$

$Z_2 \times S^1$ has another isotropic subgroup $z_2(\rho, \pi)$, the center space associated to eigenvalues $\pm\mathrm{i}\omega_2$ is spanned by $q_2(\theta)$, $\bar{q}_2(\theta)$ which implies that the bifurcated periodic solutions are anti-phase, i.e., taking the form

$$\left(u(t), v(t), u\left(t + \frac{T}{2}\right), v\left(t + \frac{T}{2}\right)\right)$$

where T is a period. □

4 Direction and stability of the Hopf bifurcation

In the previous section, we have obtained some conditions to ensure that the system (5) undergoes a single Hopf bifurcation at the origin $(x_{10}, z_{10}, x_{20}, z_{20})$ when $\tau = \tau_j$ passes through certain critical values. In this section, we shall study the direction, stability, and the period of the bifurcating periodic solutions. The method we used is based on the normal form method and the center manifold theory introduced by Hassard et al. [13].

We first focus on the case $\Delta_1 = 0$, because the other case can be dealt with analogously. We re-scale the time by $t \mapsto t/\tau$, to normalize the delay so that system (11) can be written as

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \frac{\tau}{\varepsilon}\left(\frac{uh(z_1 + z_0)}{u + (x_1 + x_0)} - \frac{h(x_1 + x_0)(z_1 + z_0)}{u + (x_1 + x_0)}\right. \\
&\qquad \left. + (x_1 + x_0)(1 - x_1 - x_0) + D(x_2 - x_1)\right), \\
\frac{\mathrm{d}z_1}{\mathrm{d}t} &= \tau x_1 - \tau z_1 + \tau k z_1(t-1), \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= \frac{\tau}{\varepsilon}\left(\frac{uh(z_2 + z_0)}{u + (x_2 + x_0)} - \frac{h(x_2 + x_0)(z_2 + z_0)}{u + (x_2 + x_0)}\right. \\
&\qquad \left. + (x_2 + x_0)(1 - x_2 - x_0) + D(x_1 - x_2)\right), \\
\frac{\mathrm{d}z_1}{\mathrm{d}t} &= \tau x_1 - \tau z_1 + \tau k z_1(t-1).
\end{aligned} \tag{21}$$

Letting $\tau = \tau_1 + a$ ($a \in R$), then $a = 0$ is Hopf bifurcation value of (21), Eq. (21) can be rewritten as:

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= (\tau_1 + a)\left(\left(a_1 - \frac{D}{\varepsilon}\right)x_1 + a_2 z_1 + \frac{D}{\varepsilon}x_2 + M_1\right), \\
\frac{\mathrm{d}z_1}{\mathrm{d}t} &= (\tau_1 + a)\big(x_1 - z_1 + k z_1(t-1)\big),
\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}x_2}{\mathrm{d}t} &= (\tau_1 + a)\left((a_1 - \frac{D}{\varepsilon}\right)x_2 + a_2 z_2 + \frac{D}{\varepsilon}x_1 + M_2\right),\\ \frac{\mathrm{d}z_2}{\mathrm{d}t} &= (\tau_1 + a)\big(x_2 - z_2 + kz_2(t-1)\big),\end{aligned}$$

where

$$\begin{aligned}M_1 &= \frac{1}{\varepsilon}\left[\frac{hz_0(u - x_0 + 1)}{(u + x_0)^2}\right]x_1^2 + \frac{1}{\varepsilon}\frac{-2uh}{(u + x_0)}x_1 z_1,\\ M_2 &= \frac{1}{\varepsilon}\left[\frac{hz_0(u - x_0 + 1)}{(u + x_0)^2}\right]x_2^2 + \frac{1}{\varepsilon}\frac{-2uh}{(u + x_0)}x_2 z_2.\end{aligned}$$

Select the phase space $C = C([-1, 0], R^4)$. For any $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^{\mathrm{T}} \in C$, setting

$$\begin{aligned}L_a(\phi) = (\tau_1 + a)&\begin{pmatrix} a_1 - \frac{D}{\varepsilon} & a_2 & \frac{D}{\varepsilon} & 0\\ 1 & -1 & 0 & 0\\ \frac{D}{\varepsilon} & 0 & a_1 - \frac{D}{\varepsilon} & a_2\\ 0 & 0 & 1 & -1\end{pmatrix}\begin{pmatrix}\phi_1(0)\\ \phi_2(0)\\ \phi_3(0)\\ \phi_4(0)\end{pmatrix}\\ &+ (\tau_1 + a)\begin{pmatrix}0&0&0&0\\0&k&0&0\\0&0&0&0\\0&0&0&k\end{pmatrix}\begin{pmatrix}\phi_1(-1)\\ \phi_2(-1)\\ \phi_3(-1)\\ \phi_4(-1)\end{pmatrix},\end{aligned}$$

and

$$f(a, \varphi) \stackrel{\mathrm{def}}{=} (\tau_1 + a)(M_1,\, 0,\, M_2,\, 0)^{\mathrm{T}}.$$

By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ $(0 \leqslant \theta \leqslant 1)$, whose elements are of bounded variation such that

$$L_a\phi = \int_{-1}^{0} \mathrm{d}\eta(\theta, a)\phi(\theta), \quad \phi \in C.$$

We choose

$$\eta(\theta, a) = (\tau_1 + a)A\delta(\theta) + (\tau_1 + a)B\delta(\theta + 1),$$

where δ is defined by

$$\delta(\theta) = \begin{cases}1, & \theta = 0,\\ 0, & \theta \neq 0.\end{cases}$$

For $\phi \in C^1([-1, 0], R^4)$, define

$$A(a)\phi = \begin{cases}\frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta \in [-1, 0),\\ \int_{-1}^{0} \mathrm{d}\eta(t, a)\phi(t), & \theta = 0,\end{cases}$$

and

$$R(a)\varphi = \begin{cases} 0, & \theta \in [-1,0), \\ f(a,\varphi), & \theta = 0. \end{cases}$$

Then, system (21) is equivalent to the following operator equation:

$$\dot{u}_t = A(\alpha)u_t + R(\alpha)u_t, \tag{22}$$

where $u_t = u(t+\theta)$ ($\theta \in [-1,0]$).

For $\psi \in C^1([0,1],(R^4)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s \in (0,1], \\ \int_{-1}^{0} \mathrm{d}\eta^T(s,a)\phi(-s), & s = 0, \end{cases}$$

and a bilinear form

$$\langle \psi(s), \phi(\theta) \rangle = \overline{\psi}(0)\phi(0) - \int\limits_{\theta=1}^{0} \int\limits_{\xi=0}^{\theta} \overline{\psi}(\xi-\theta)\,\mathrm{d}\eta(\theta)\phi(\xi)\,\mathrm{d}\xi,$$

where $\eta(\theta) = \eta(\theta,0)$, then $A(0)$ and A^* are adjoint operators. Setting $q(\theta)$ and $q^*(s)$ is the eigenvector of $A(0)$ and A^* corresponding to $\mathrm{i}\tau_1 w_1$ and $-\mathrm{i}\tau_1 w_1$. By direct calculation we have

$$q(\theta) = \left(\frac{a_2}{\mathrm{i}w_1 + a_1}, 1, \frac{a_2}{\mathrm{i}w_1 + a_1}, 1\right)^{\mathrm{T}} \mathrm{e}^{\mathrm{i}w_1\tau_1\theta},$$
$$q^*(s) = D\left(\frac{1}{\mathrm{i}w_1 + a_1}, 1, \frac{1}{\mathrm{i}w_1 + a_1}, 1\right)\mathrm{e}^{\mathrm{i}w_1\tau_1 s},$$

where

$$\overline{D} = \left(\left(\frac{2a_2}{a_1^2 + w_1^2} + 2\right) - 2k\tau_1\left(e^{-\mathrm{i}w_1\tau_1} + \mathrm{i}w_1\tau_1\mathrm{e}^{-\mathrm{i}w_1\tau_1}\right)\right)^{-1}.$$

Then $\langle q^*, q\rangle = 1$ and $\langle q^*, \overline{q}\rangle = 0$.

In the following, we follow the ideas in Hassard et al. [13] and by using the same notations as there to compute the coordinates describing the center manifold C_0 at $a = 0$. Let u_t be the solution of (22) when $a = 0$. Define $z(t) = \langle q^*, u_t\rangle$, $W(t,\theta) = u_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}$.

On the center manifold C_0 we have $W(t,\theta) = W(\gamma(t), \bar{\gamma}(t), \theta)$, where

$$W(\gamma, \bar{\gamma}, \theta) = W_{20}(\theta)\frac{\gamma^2}{2} + W_{11}(\theta)\gamma\bar{\gamma} + W_{02}(\theta)\frac{\gamma^2}{2} + W_{30}\frac{\gamma^3}{6} + \cdots,$$

γ and $\overline{\gamma}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q}^*$. Note that W is real if u_t is real. We only consider real solutions. For solution $u_t \in C_0$ of (15), since $a = 0$, we have

$$\gamma'(t) = \mathrm{i}w_1 z + \left\langle q^*(\theta), f\left(W + 2\,\mathrm{Re}\left\{\gamma(t)q(\theta)\right\}\right)\right\rangle$$
$$\stackrel{\text{def}}{=} \mathrm{i}w_1 + \overline{q}^*(0)f_0(\gamma, \bar{\gamma}).$$

We rewrite this equation as

$$\gamma'(t) = \mathrm{i}w_1\gamma(t) + g(\gamma, \bar{\gamma}), \tag{23}$$

with

$$\begin{aligned} g(\gamma, \bar{\gamma}) &= \bar{q}^*(0) f\big(W(\gamma, \bar{\gamma}, 0) + 2\,\mathrm{Re}\big\{\gamma(t) q(0)\big\}\big) \\ &= g_{20}\frac{\gamma^2}{2} + g_{11}\gamma\bar{\gamma} + g_{02}\frac{\bar{\gamma}^2}{2} + g_{21}\frac{\gamma^2\bar{\gamma}}{2} + \cdots . \end{aligned} \tag{24}$$

It follows from (23) and (24) that

$$W' = u'_t - \gamma' q - \bar{\gamma}'\bar{q} = \begin{cases} AW - 2\,\mathrm{Re}\{\bar{q}^*(0) f_0 q(\theta)\}, & \theta \in [-1, 0), \\ AW - 2\,\mathrm{Re}\{\bar{q}^*(0) f_0 q(\theta)\} + f_0, & \theta = 0. \end{cases}$$

Comparing of coefficients we have:

$$\begin{aligned} g_{20} &= 2q_1^*(0)\frac{1}{\varepsilon}\frac{hz_0(u - x_0 + 1)}{(u + x_0)^2} + 2q_3^*(0)\frac{1}{\varepsilon}\frac{hz_0(u - x_0 + 1)}{(u + x_0)^2}, \\ g_{11} &= 2q_1^*(0)\frac{1}{\varepsilon}\frac{-2uh}{(u + x_0)} + 2q_3^*(0)\frac{1}{\varepsilon}\frac{-2uh}{(u + x_0)}, \\ g_{02} &= 0, \\ g_{21} &= 2q_1^*(0)\frac{1}{\varepsilon}\frac{hz_0(u - x_0 + 1)}{(u + x_0)^2}\left(\frac{2a_1a_2}{a_1^2 + w_1^2}w_{11}^1(0) + \frac{2a_1a_2}{a_1^2 + w_1^2}w_{20}^1(0)\right) \\ &\quad + \frac{1}{\varepsilon}\frac{-2uh}{(u + x_0)}\left(\frac{1}{2}w_{20}^1 + \frac{1}{2}\frac{2a_1a_2}{a_1^2 + w_1^2}w_{20}^2(0) + \frac{2a_1a_2}{a_1^2 + w_1^2}w_{11}^2(0)\right) \\ &\quad + 2q_3^*(0)\frac{1}{\varepsilon}\frac{hz_0(u - x_0 + 1)}{(u + x_0)^2}\left(\frac{2a_1a_2}{a_1^2 + w_1^2}w_{11}^1(0) + \frac{2a_1a_2}{a_1^2 + w_1^2}w_{20}^1(0)\right), \end{aligned}$$

where

$$\begin{aligned} &q_1^*(0) = \frac{1}{a_1 + w_1}, \quad q_3^*(0) = \frac{1}{a_1 + w_1}, \\ &W_{20}(\theta) = -\frac{g_{20}}{\mathrm{i}\omega_1}q(0)\mathrm{e}^{\mathrm{i}\omega_1\theta} - \frac{\bar{g}_{20}}{3\mathrm{i}\omega_1}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_1\theta} + E_1\mathrm{e}^{2\mathrm{i}\omega_1\theta}, \\ &W_{11}(\theta) = \frac{g_{11}}{\mathrm{i}\omega_1}q(0)\mathrm{e}^{\mathrm{i}\omega_1\theta} - \frac{\bar{g}_{11}}{\mathrm{i}\omega_1}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_1\theta} + E_2. \end{aligned}$$

Moreover E_1 and E_2 satisfies the following equations, respectively:

$$\begin{aligned} &\begin{pmatrix} 2\mathrm{i}w_1 - (a_1 - \frac{D}{\varepsilon}) & -a_2 & -\frac{D}{\varepsilon} & 0 \\ -1 & 2\mathrm{i}w_1 + 1 - k\mathrm{e}^{-2\mathrm{i}w_1} & 0 & 0 \\ -\frac{D}{\varepsilon} & 0 & 2\mathrm{i}w_1 - (a_1 - \frac{D}{\varepsilon}) & -a_2 \\ 0 & 0 & -1 & 2\mathrm{i}w_1 + 1 - k\mathrm{e}^{-2\mathrm{i}w_1} \end{pmatrix} E_1 \\ &\quad = \left(\frac{1}{\varepsilon}\frac{hz_0(u - x_0 + 1)}{(u + x_0)^2},\ 0,\ \frac{1}{\varepsilon}\frac{hz_0(u - x_0 + 1)}{(u + x_0)^2},\ 0\right)^{\mathrm{T}}, \end{aligned}$$

$$\begin{pmatrix} -(a_1-\frac{D}{\varepsilon}) & -a_2 & -\frac{D}{\varepsilon} & 0\\ -1 & 1-k & 0 & 0\\ -\frac{D}{\varepsilon} & 0 & -(a_1-\frac{D}{\varepsilon}) & -a_2\\ 0 & 0 & -1 & 1-k \end{pmatrix} E_2 = \begin{pmatrix} \frac{1}{\varepsilon}\frac{-2uh}{(u+x_0)}\\ 0\\ \frac{1}{\varepsilon}\frac{-2uh}{(u+x_0)}\\ 0 \end{pmatrix}.$$

Then we can compute the following quantities:

$$\begin{aligned} c_{11}(0) &= \frac{\mathrm{i}}{2\omega_1}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \qquad v_{21} = -\frac{\operatorname{Re} c_1(0)}{\operatorname{Re}\lambda'(\tau_1)},\\ T_{21} &= -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_1)\}}{\omega_1}, \qquad \beta_{21} = 2\operatorname{Re} c_1(0). \end{aligned} \tag{25}$$

Then we focus on the case $\Delta_2 = 0$. As the same, we have

$$\begin{aligned} g_{20} &= 2q_1^*(0)\frac{1}{\varepsilon}\frac{hz_0(u-x_0+1)}{(u+x_0)^2} + 2q_3^*(0)\frac{1}{\varepsilon}\frac{hz_0(u-x_0+1)}{(u+x_0)^2},\\ g_{11} &= 2q_1^*(0)\frac{1}{\varepsilon}\frac{-2uh}{(u+x_0)} + 2q_3^*(0)\frac{1}{\varepsilon}\frac{-2uh}{(u+x_0)},\\ g_{02} &= 0,\\ g_{21} &= 2q_1^*(0)\frac{1}{\varepsilon}\frac{hz_0(u-x_0+1)}{(u+x_0)^2}\left(\frac{2a_1a_2}{a_1^2+w_2^2}w_{11}^1(0) + \frac{2a_1a_2}{a_1^2+w_2^2}w_{20}^1(0)\right)\\ &\quad + \frac{1}{\varepsilon}\frac{-2uh}{(u+x_0)}\left(\frac{1}{2}w_{20}^1(0) + \frac{1}{2}\frac{2a_1a_2}{a_1^2+w_2^2}w_{20}^2(0) + \frac{2a_1a_2}{a_1^2+w_2^2}w_{11}^2(0)\right)\\ &\quad + 2q_3^*(0)\frac{1}{\varepsilon}\frac{hz_0(u-x_0+1)}{(u+x_0)^2}\left(\frac{2a_1a_2}{a_1^2+w_2^2}w_{11}^1(0) + \frac{2a_1a_2}{a_1^2+w_2^2}w_{20}^1(0)\right), \end{aligned}$$

where

$$\begin{aligned} q_1^*(0) &= \frac{1}{a_1 - \frac{2D}{\varepsilon} + w_2}, \qquad q_3^*(0) = -\frac{1}{a_1 - \frac{2D}{\varepsilon} + w_2},\\ W_{20}(\theta) &= -\frac{g_{20}}{\mathrm{i}\omega_2}q(0)\mathrm{e}^{\mathrm{i}\omega_2\theta} - \frac{\bar{g}_{20}}{3\mathrm{i}\omega_2}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_2\theta} + E_1\mathrm{e}^{2\mathrm{i}\omega_2\theta},\\ W_{11}(\theta) &= \frac{g_{11}}{\mathrm{i}\omega_2}q(0)\mathrm{e}^{\mathrm{i}\omega_2\theta} - \frac{\bar{g}_{11}}{\mathrm{i}\omega_2}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_2\theta} + E_2, \end{aligned}$$

moreover E_1 and E_2 satisfies the following equations, respectively:

$$\begin{pmatrix} 2\mathrm{i}w_2 - (a_1-\frac{D}{\varepsilon}) & -a_2 & -\frac{D}{\varepsilon} & 0\\ -1 & 2\mathrm{i}w_2 + 1 - k\mathrm{e}^{-2\mathrm{i}w_2} & 0 & 0\\ -\frac{D}{\varepsilon} & 0 & 2\mathrm{i}w_2 - (a_1-\frac{D}{\varepsilon}) & -a_2\\ 0 & 0 & -1 & 2\mathrm{i}w_2 + 1 - k\mathrm{e}^{-2\mathrm{i}w_2} \end{pmatrix} E_1$$
$$= \left(\frac{1}{\varepsilon}\frac{hz_0(u-x_0+1)}{(u+x_0)^2},\ 0,\ \frac{1}{\varepsilon}\frac{hz_0(u-x_0+1)}{(u+x_0)^2},\ 0\right)^{\mathrm{T}},$$

$$\begin{pmatrix} -(a_1 - \frac{D}{\varepsilon}) & -a_2 & -\frac{D}{\varepsilon} & 0 \\ -1 & 1-k & 0 & 0 \\ -\frac{D}{\varepsilon} & 0 & -(a_1 - \frac{D}{\varepsilon}) & -a_2 \\ 0 & 0 & -1 & 1-k \end{pmatrix} E_2 = \begin{pmatrix} \frac{1}{\varepsilon}\frac{-2uh}{(u+x_0)} \\ 0 \\ \frac{1}{\varepsilon}\frac{-2uh}{(u+x_0)} \\ 0 \end{pmatrix}.$$

And

$$\begin{aligned} c_{12}(0) &= \frac{\mathrm{i}}{2\omega_2}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \qquad v_{22} = -\frac{\operatorname{Re} c_1(0)}{\operatorname{Re} \lambda'(\tau_2)}, \\ T_{22} &= -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_2)\}}{\omega_2}, \qquad \beta_{22} = 2\operatorname{Re} c_1(0). \end{aligned} \tag{26}$$

Hence we have the following theorem by the result of Hassard et al. [13].

Theorem 3. *In* (25) (*in* (26)), *the sign of* v_{21} (v_{22}) *determined the direction of Hopf bifurcation*: *if* $v_{21} > 0$ ($v_{22} > 0$)*, then the Hopf bifurcation is supercritical and the bifurcating periodic solution exist for* $\tau > \tau_{21}$ ($\tau > \tau_{22}$); *if* $v_{21} < 0$ ($v_{22} < 0$)*, then the Hopf bifurcation is subcritical and the bifurcating periodic solution exist for* $\tau < \tau_{21}$ ($\tau < \tau_{22}$). β_{21} (β_{22}) *determined the stability of the bifurcating periodic solution*: *the bifurcating periodic solution is stable if* $\beta_{21} < 0$ ($\beta_{22} < 0$); *the bifurcating periodic solution is unstable if* $\beta_{21} > 0$ ($\beta_{22} > 0$). T_{21} (T_{22}) *determines the period of the bifurcating periodic solution*: *the period increase if* $T_{21} > 0$ ($T_{22} > 0$); *the period decrease if* $T_{21} < 0$ ($T_{22} < 0$).

5 Numerical simulations

Let us now give some numerical simulations to illustrate the above results.

Set $u = 8 \times 10^{-4}$, $h = 2/3$, $k = -2.5$, $\varepsilon = 0.04$, $D = 1$, we consider following system:

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= \frac{1}{0.04}\left(x_1(1-x_1) - \frac{2}{3}z_1\frac{x_1 - 8\times 10^{-4}}{x_1 + 8\times 10^{-4}} + (x_2 - x_1)\right), \\ \frac{\mathrm{d}z_1}{\mathrm{d}t} &= x_1 - z_1 - 2.5z_1(t-\tau), \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= \frac{1}{0.04}\left(x_2(1-x_2) - \frac{2}{3}z_2\frac{x_2 - 8\times 10^{-4}}{x_2 + 8\times 10^{-4}} + (x_1 - x_2)\right), \\ \frac{\mathrm{d}z_2}{\mathrm{d}t} &= x_2 - z_2 - 2.5z_2(t-\tau). \end{aligned}$$

We have the equilibrium point $(0.8075, 0.2307, 0.8075, 0.2307)$ and $\tau_{10} = 1.6918$, $\tau_{20} = 0.9653$, than $\tau_0 = \min\{\tau_{10}, \tau_{20}\} = 0.9653$.

Figs. 1, 2 show that zero solution is asymptotical stable. Figs. 3, 4 depict that anti-phased periodic solutions are bifurcated from the trivial solution. Figs. 5, 6 illustrate that two waveforms are completely symmetrical.

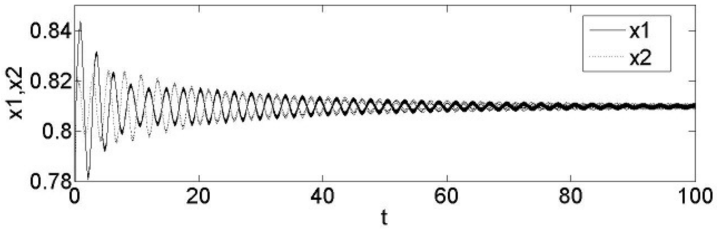

Fig. 1. Equilibrium point is asymptotical stable with $\tau = 0.9 < \tau_0$, initial value $x_1 = 0.7$, $x_2 = 0.5$.

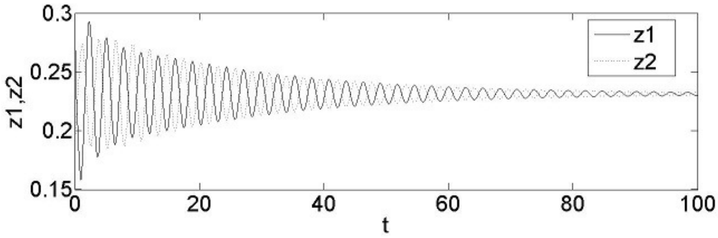

Fig. 2. Equilibrium point is asymptotical stable with $\tau = 0.9 < \tau_0$, initial value $z_1 = 0.3$, $z_2 = 0.2$.

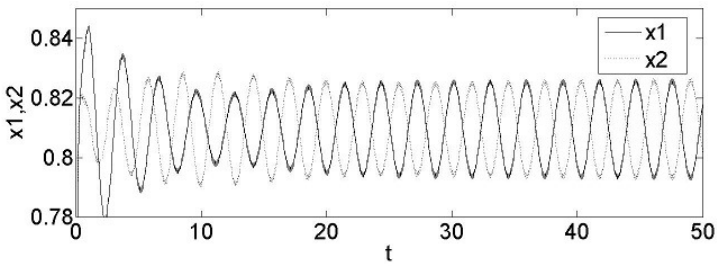

Fig. 3. A branch of anti-phased periodic solutions is bifurcated from the trivial solution with $\tau = 0.97 \approx \tau_0$, initial value $x_1 = 0.7$, $x_2 = 0.5$.

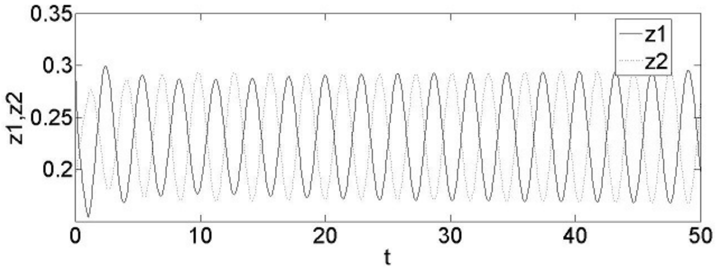

Fig. 4. A branch of anti-phased periodic solutions is bifurcated from the trivial solution with $\tau = 0.97 \approx \tau_0$, initial value $z_1 = 0.3$, $z_2 = 0.2$.

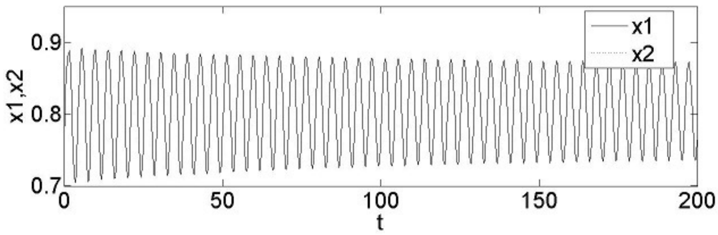

Fig. 5. A branch of synchronous periodic solutions is bifurcated from the trivial solution with $\tau = \tau_{10} = 1.69$, initial value $x_1 = 0.7$, $x_2 = 0.7$.

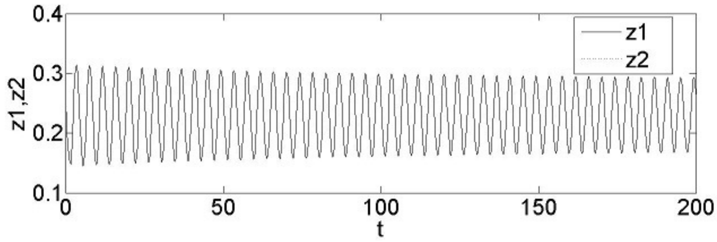

Fig. 6. A branch of synchronous periodic solutions is bifurcated from the trivial solution with $\tau = \tau_{10} = 1.69$, initial value $z_1 = 0.3$, $z_2 = 0.3$.

6 Conclusions

For a coupled Oregonator model with delay, an important issue is how delays change the stability of Oregonator model states, steady or oscillatory, causing further oscillations or significantly altering existing ones and hence inducing delay-controlled periodic behavior. In this paper, experimental and numerical investigations on the effect of electrical feedback in the oscillating Belousov–Zhabotinsky reaction are studied. By analyzing the associated characteristic equation and means of space decomposition, we subtly discuss the distribution of zeros of the characteristic equation, and then derive some sufficient conditions ensuring that all the characteristic roots have negative real parts. By regarding the eigenvalues of the connection matrix of the system as bifurcation parameters, we discuss Hopf bifurcation of the equilibria. Meanwhile, with the help of center manifold reduction and normal form theory, we study Hopf bifurcation of the equilibria, and obtain the detailed information about the bifurcation direction and stability of various bifurcated periodic solutions. Finally, numerical simulations have demonstrated the correctness of the theoretical results.

From a chemical viewpoint, both means that time delay could cause a stable equilibrium to become unstable and cause the properties in a coupled Oregonator model to fluctuate: if $\tau < \tau_j$, the density of various elements reach an equilibrium. If τ increases and crosses the value τ_j, then this equilibrium becomes unstable: the density of various elements oscillates around the unstable equilibrium.

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