Large deviations for weighted random sums

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Abstract. In the present paper we consider weighted random sums $Z_N = \sum_{j=1}^{N} a_j X_j$, where $0 \leqslant a_j < \infty$, N denotes a non-negative integer-valued random variable, and $\{X, X_j,\, j = 1, 2, \ldots\}$ is a family of independent identically distributed random variables with mean $\mathbf{E}X = \mu$ and variance $\mathbf{D}X = \sigma^2 > 0$. Throughout this paper N is independent of $\{X, X_j,\, j = 1, 2, \ldots\}$ and, for definiteness, it is assumed $Z_0 = 0$. The main idea of the paper is to present results on theorems of large deviations both in the Cramér and power Linnik zones for a sum $\tilde{Z}_N = (Z_N - \mathbf{E}Z_N)(\mathbf{D}Z_N)^{-1/2}$, exponential inequalities for a tail probability $\mathbf{P}(\tilde{Z}_N \geqslant x)$ in two cases: $\mu = 0$ and $\mu \neq 0$ pointing out the difference between them. Only normal approximation is considered. It should be noted that large deviations when $\mu \neq 0$ have been already considered in our papers [1, 2].

Keywords: cumulant, random sums, large deviation theorems, normal approximation.

1 Introduction

The biggest and, possibly, most important for applications part of modern probability theory consist of the limit theorems where large deviations occupy a significant place. As it is mentioned in [3], most often there have been considered these cases: when the Cramér's condition is satisfied, scilicet, characteristic functions of the terms are analytic in a neighbourhood of zero; the Linnik case when all the moments of summands are finite but their growth does not assure the analyticity of the characteristic functions in the neighbourhood of zero; the case of so-called moderate deviations – the summands have only the finite number of moments; the case then Cramér's and Linnik's conditions are not fulfilled, but the behaviour of the distribution tails of summands is regular enough.

Many of the basic ideas and results on theorems of large deviations for the sums $S_n = X_1 + \cdots + X_n$ of independent identically distributed (i.i.d.) random variables (r.vs.) have been presented by Ibragimov, Linnik (1965), Petrov (1972), Nagaev (1979), accordingly, in [4–6].

The next step in the problems of large deviation theorems was made when Statulevičius (1966) in the paper [7] proposed the method of cumulants to consider large deviation probabilities of various statistics. The method of cumulants rendered an opportunity to obtain large deviation theorems for sums of independent and dependent r.vs., polynomials forms, multiple stochastic integrals of random processes, polynomial statistics both in the Cramér and power Linnik zones. Statulevičius, Rudzkis and Saulis (1978) in the paper [8] proved general lemma of large deviations when a random variable (r.v.) X satisfies Statulevičius' condition (see, e.g., [3]) which is very convenient for asymptotic analysis of large deviations of various statistics. Later, the same authors demonstrated large deviation theorem for sums of independent non-identically distributed (i.n.d.) r.vs. (see [9]).

Frequently, instead of accurate equalities of large deviations, less accurate exponential inequalities are used. These were proved in [10] by Bentkus, Rudzkis (1980) under a requirement that any r.v. X should satisfy Statulevičius' condition. Applications of the general lemma of large deviations and exponential inequalities to prove large deviation theorems of various statistics are presented in the monograph [3].

Undoubtedly, there are a large amount of other strong literature on large deviations (see for more literature, for example, in [3]), as large deviation theory has been and still is rapidly developed, because of problems in various areas of mathematics which require development of large deviation theory.

The theory of large deviations offers interesting problems when the number of summands is itself a r.v. Let us assume that throughout this paper, N denotes a non-negative integer-valued r.v. in a broad sense with mean $\mathbf{E}N = \alpha$, variance $\mathbf{D}N = \beta^2$ and a distribution $\mathbf{P}(N = l) = q_l$, $l \in \mathbb{N}_0$. For example, N can obey binomial, negative binomial laws, or it can be a homogeneous Poisson process. In the last case $N = N_t$, $t \geqslant 0$. In addition, $\{X, X_j,\ j = 1, 2, \ldots\}$ is a family of i.i.d. r.vs. with mean $\mathbf{E}X = \mu$, variance $\mathbf{D}X = \sigma^2 > 0$ and with a distribution function $F_X(x) = \mathbf{P}(X < x)$ for all $x \in \mathbb{R}$. We assume that N is independent of $\{X, X_j,\ j = 1, 2, \ldots\}$.

The kth order moments and cumulants of X will be denoted by

$$\mathbf{E}X^k = \frac{1}{\mathrm{i}^k}\frac{\mathrm{d}^k}{\mathrm{d}u^k}\varphi_X(u)\bigg|_{u=0}, \quad \mathit{\Gamma}_k(X) = \frac{1}{\mathrm{i}^k}\frac{\mathrm{d}^k}{\mathrm{d}u^k}\ln\varphi_X(u)\bigg|_{u=0}, \quad k = 1, 2, \ldots,$$

respectively, where $\varphi_X(u) = \mathbf{E}\exp\{\mathrm{i}uX\}$, $u \in \mathbb{R}$ is the characteristic function of the r.v. X. Note that $\mathit{\Gamma}_1(X) = \mathbf{E}X$, $\mathit{\Gamma}_2(X) = \mathbf{D}X$.

Consider weighted random sum (r.s.)

$$Z_N = \sum_{j=1}^{N} a_j X_j, \tag{1}$$

where $0 \leqslant a_j < \infty$. For definiteness, we suppose $Z_0 = 0$.

Let's introduce the following compound r.v. $T_{N,r}$:

$$T_{N,r} = \sum_{j=1}^{N} a_j^r, \quad T_{l,r} = \sum_{j=1}^{l} a_j^r, \quad r \in \mathbb{N},\ l \in \mathbb{N}_0. \tag{2}$$

we assume $T_{0,r} = 0$. It's easy to see that

$$\mathbf{E}T_{N,r} = \sum_{l=1}^{\infty} T_{l,r} q_l, \qquad \mathbf{D}T_{N,r} = \mathbf{E}T_{N,r}^2 - (\mathbf{E}T_{N,r})^2, \tag{3}$$

where $\mathbf{E}T_{N,r}^2 = \sum_{l=1}^{\infty} T_{l,r}^2 q_l$. Accordingly, with reference to (8) in [2, p. 257] we have

$$\mathbf{E}Z_N = \mu \mathbf{E}T_{N,1}, \qquad \mathbf{D}Z_N = \sigma^2 \mathbf{E}T_{N,2} + \mu^2 \mathbf{D}T_{N,1}, \tag{4}$$

on the understanding that $\mu \neq 0$.

Considering the importance of the main probability characteristics we separate expressions of $\mathbf{E}Z_N$ and $\mathbf{D}Z_N$ in the case when $\mu = 0$. In set terms,

$$\mathbf{E}Z_N = 0, \qquad \mathbf{D}Z_N = \sigma^2 \mathbf{E}T_{N,2}. \tag{5}$$

The paper is designated to the research of large deviation theorems both in the Cramér and power Linnik zones for the following standardized sum:

$$\tilde{Z}_N = \frac{Z_N - \mathbf{E}Z_N}{\sqrt{\mathbf{D}Z_N}}, \tag{6}$$

exponential inequalities for a tail probability $\mathbf{P}(\tilde{Z}_N \geqslant x)$ under some assumptions for the r.v.'s X, $T_{N,1}$, $T_{N,2}$ kth order moments and cumulants. In particular, we say that the r.v. X with $\sigma^2 > 0$ satisfies condition ($\bar{\mathrm{B}}_\gamma$) if there exist constants $\gamma \geqslant 0$ and $K > 0$ such that

$$\left|\mathbf{E}(X-\mu)^k\right| \leqslant (k!)^{1+\gamma} K^{k-2} \sigma^2, \quad k = 3, 4, \ldots. \tag{$\bar{\mathrm{B}}_\gamma$}$$

Condition ($\bar{\mathrm{B}}_\gamma$) is a generalization of Bernstein's familiar condition

$$|\mathbf{E}X|^k \leqslant \frac{1}{2} k! K^{k-2} \sigma^2, \quad k = 2, 3, \ldots, \tag{B_0}$$

as $\mu = 0$. Taking into consideration that $\Gamma_k(X) = \Gamma_k(X-\mu)$ and according to Lemma 3.1 in [3, p. 42], we take up the position that

Proposition 1. *If for the r.v. X condition ($\bar{B}_\gamma$) is satisfied, then*

$$\left|\Gamma_k(X)\right| \leqslant (k)!^{1+\gamma} M^{k-2} \sigma^2, \quad M = 2\max\{\sigma, K\},\ k = 3, 4, \ldots. \tag{7}$$

Under condition ($\bar{\mathrm{B}}_\gamma$) we have that the r.v. X has finite moments of all orders.

We suppose that the r.vs. $T_{N,1}$, $T_{N,2}$ satisfy conditions (L) and (L_0), respectively, if there exist constants $K_1 > 0$, $K_2 > 0$ and $p \geqslant 0$ such that

$$\left|\Gamma_k(T_{N,1})\right| \leqslant \frac{1}{2} k! K_1^{k-2} (\mathbf{D}T_{N,1})^{1+(k-2)p}, \quad k = 2, 3, \ldots, \tag{L}$$

$$\left|\Gamma_k(T_{N,2})\right| \leqslant k! K_2^{k-1} (\mathbf{E}T_{N,2})^{1+(k-1)p}, \qquad k = 1, 2, \ldots. \tag{L_0}$$

Throughout the rest paper, the first condition (L) we use as $\mu \neq 0$, and the second one as $\mu = 0$.

It is easy to show that by virtue of conditions (L) and (L_0) with $0 \leqslant p < 1/2$, the cumulants $\Gamma_k(T_{N,1}/\sqrt{\mathbf{D}T_{N,1}})$, $\Gamma_k(T_{N,2}/\sqrt{\mathbf{E}T_{N,2}})$ gradually decrease, accordingly, as $\mathbf{D}T_{N,1} \to \infty$, $\mathbf{E}T_{N,2} \to \infty$.

We would like to remark that from expression (18) (see Section 3) of the kth order cumulants of the r.s. (1) follows why we imposed conditions not only for the kth order cumulants of the r.v. X but of the compound r.vs. $T_{N,1}$ and $T_{N,2}$ as well.

In this paper, the upper estimates of the normal approximation to the sum (6) is considered as well, but the main role of the paper goes to application of the cumulant method which was developed in [8] by Rudzkis, Saulis and Statulevičius (1978), to theorems of large deviations for standardized r.s. (6) in two cases: $\mu = 0$ and $\mu \neq 0$ by pointing out the difference between them. It should be noted that large deviations when $\mu \neq 0$ have been already considered in our papers [1, 2], thus in this instance we pointed only some results without proofs.

The convergence, asymptotic behaviour of the r.s. in case $a_j \equiv 1$ have been investigated in the theory probability for a while, as the results by Robbins (1984) in the paper [11] appears. For more detailed list of literature see, e.g., in [12] or in the monograph [13] by Kruglov and Korolev where limit theorems for the r.s. when the summands are i.n.d. r.vs. are detailed discussed.

The r.s. appears as models in many applied problems, for instance, in stochastic processes, stochastic modelling, random walk, queue theory, theory of network or theory of estimation, biology, nuclear physics, insurance, economic theory, finance mathematics and is essential in other fields too (see, e.g., [12–15]).

Some strong results for the asymptotic behaviour of the r.s., in case N has concrete probability laws: Poisson, Bernoulli, binomial or geometry, have been presented in the paper [16]. In 1997 Bening, Korolev and Shorgin considered three methods of the construction of approximations to the generalized Poisson distributions: approximation by a normal law, approximations by asymptotic distributions and approximation with the help of asymptotic expansions where uniform and non-uniform estimates are given (see [14]). Later, Korolev and Shevtsova (2012) presented sharpened upper bounds for the absolute constant in the Berry–Esseen inequality for Poisson and mixed Poisson r.s. (see [17]).

Undoubtedly, there are a large amount of literature on theorems of large deviations for the r.s. under different assumptions and with various applications, for example, [1, 2, 18–24]. Unfortunately, as far as we know, without our papers (see [1,2,21]) there are only few papers, e.g., [18, 23, 24], accordingly, by Aksomaitis (1965), Statulevičius (1967), Saulis and Deltuvienė (2007) on large deviations for the r.s. in the Cramér zone when all $a_j \equiv 1$ and the cumulant method is used, although as it was already mentioned at the beginning of the Introduction, the cumulant method is an effective way of studying large deviation probabilities of various statistics.

In the present paper, we generalized theorems on large deviations for sums of non-random number of summands (see, e.g., [3, 25]) and for sums of random number of summands presented in the papers [18, 23, 24], besides developed results presented in our papers [1, 2, 21] by investigating large deviation theorems both in the Cramér and power Linnik zones for weighted r.s. (1) in two cases: $\mu \neq 0$, $\mu = 0$.

The outline of the paper is as follows: Section 2 lists main results and some instances of large deviations. The last section is devoted for the proofs of results presented in Section 2.

2 Large deviation theorems and exponential inequalities

As was mentioned in the Introduction, we restrict our attention to the cumulant method, as it is good in the investigation for large deviation probabilities for sums of independent or dependent r.vs., polynomials forms, multiple stochastic integrals of random processes, polynomial statistics.

Since we are interested not only in the convergence to the normal distribution, but also in a more precise asymptotic analysis of the distribution for the r.s. (6), first we must find the accurate upper bounds for $\Gamma_k(\tilde{Z}_N)$, $k = 3, 4, \dots$, and after that, we can use general lemmas on large deviations and exponential inequalities presented in [3, pp. 18–19]: Lemma 2.3 (Rudzkis, Saulis, Statulevičius, 1978) and Lemma 2.4 (Bentkus, Rudzkis, 1980).

Recall that $0 \leqslant a_j < \infty$ and denote $a = \sup\{a_j,\, j = 1, 2, \dots\} < \infty$, $(b \vee c) = \max\{b, c\}$, $b, c \in \mathbb{R}$.

Lemma 1 below presents the accurate upper estimate for $\Gamma_k(\tilde{Z}_N)$ in two cases: $\mu = 0$ and $\mu \neq 0$.

Lemma 1. *If for the r.v. X with variance $\sigma^2 > 0$, condition $(\bar{\mathrm{B}}_\gamma)$ is fulfilled and the r.vs. $T_{N,1}$, $T_{N,2}$, defined by (2) satisfy conditions (L), (L_0), respectively, then*

$$\left|\Gamma_k(\tilde{Z}_N)\right| \leqslant \frac{(k!)^{1+\gamma}}{\Delta_*^{k-2}}, \quad k = 3, 4, \dots, \tag{8}$$

where

$$\Delta_* = \begin{cases} \Delta_N & \text{if } \mu \neq 0, \\ \Delta_{N,0} & \text{if } \mu = 0. \end{cases} \tag{9}$$

Here

$$\Delta_N = \frac{\sqrt{\mathbf{D}Z_N}}{L_N}, \quad L_N = 2\biggl(K_1|\mu|(\mathbf{D}T_{N,1})^p \vee \biggl(1 \vee \frac{\sigma}{2|\mu|}\biggr) aM\biggr), \tag{10}$$

where $\mathbf{D}Z_N$ defined by (4),

$$\Delta_{N,0} = \frac{\sqrt{\mathbf{D}Z_N}}{L_{N,0}}, \quad L_{N,0} = 2\bigl(1 \vee K_2(\mathbf{E}T_{N,2})^p\bigr)\biggl(\frac{1}{2} \vee a\biggr)M, \tag{11}$$

where $\mathbf{D}Z_N$ defined by (5). Constants K_1, K_2, p, M are defined by conditions (L), (L_0), (7) and $\mathbf{D}T_{N,1}, \mathbf{E}T_{N,2}$ defined by (3).

Note that by Leonov (1964) (see [26]), for the convergence to standard normal distribution under conditions $\mathit{\Gamma}_1(\tilde{Z}_N) = 0$, $\mathit{\Gamma}_2(\tilde{Z}_N) = 1$, it is sufficient that $\mathit{\Gamma}_k(\tilde{Z}_N) \to 0$ for every $k = 3, 4, \dots$ if $\mathit{\Delta}_* \to \infty$.

Denote

$$\mathit{\Delta}_{*,\gamma} = c_\gamma \mathit{\Delta}_*^{1/(1+2\gamma)}, \quad c_\gamma = \frac{1}{6}\left(\frac{\sqrt{2}}{6}\right)^{1/(1+2\gamma)}, \ \gamma \geqslant 0. \tag{12}$$

Let's say θ (with or without an index) denote some variable, not always the same, not exceeding 1 in absolute value.

By the following Theorems 1, 2, 3 and Corollaries 1, 2 we present the exact large deviations equivalent for the tails (left and right tails) of (6), asymptotic convergence to a unit of large deviation relations, non-asymptotic exponential inequalities for the probability of large deviations and normal approximation with an explicit non-asymptotic estimate of the "distance" between the distribution of $\tilde{Z}_N$ and the standard Gaussian distribution $\Phi(x)$.

Theorem 1. *If the r.v. X with variance $\sigma^2 > 0$, satisfies condition* ($\bar{\mathrm{B}}_\gamma$)*, and the r.vs. $T_{N,1}$, $T_{N,2}$ satisfy conditions* (L), (L$_0$)*, respectively, then in the interval $0 \leqslant x < \mathit{\Delta}_{*,\gamma}$ the relations of large deviations*

$$\frac{1 - F_{\tilde{Z}_N}(x)}{1 - \Phi(x)} = \exp\bigl\{L_{*,\gamma}(x)\bigr\}\left(1 + \theta_1 f(x)\frac{x+1}{\mathit{\Delta}_{*,\gamma}}\right),$$

$$\frac{F_{\tilde{Z}_N}(-x)}{\Phi(-x)} = \exp\bigl\{L_{*,\gamma}(-x)\bigr\}\left(1 + \theta_2 f(x)\frac{x+1}{\mathit{\Delta}_{*,\gamma}}\right)$$

are valid, where

$$f(x) = \frac{60\bigl(1 + 10\mathit{\Delta}_{*,\gamma}^2 \exp\{-(1 - x/\mathit{\Delta}_{*,\gamma})\sqrt{\mathit{\Delta}_{*,\gamma}}\}\bigr)}{1 - x/\mathit{\Delta}_{*,\gamma}},$$

$$L_{*,\gamma}(x) = \sum_{3 \leqslant k < s} \lambda_{*,k} x^k + \theta_3 \left(\frac{x}{\mathit{\Delta}_{*,\gamma}}\right)^3, \quad s = \begin{cases} 2 + 1/\gamma, & \gamma > 0, \\ \infty, & \gamma = 0. \end{cases} \tag{13}$$

The coefficients $\lambda_{*,k}$ (*expressed by cumulants of the standardized r.s.* (6)) *coincide with the coefficients of the Cramér–Petrov series* [5] *given by the formula*

$$\lambda_{*,k} = -\frac{b_{*,k-1}}{k}, \tag{14}$$

where $b_{*,k}$ are determined successively from the equations

$$\sum_{r=1}^{j} \frac{1}{r!} \mathit{\Gamma}_{r+1}(\tilde{Z}_N) \sum_{\substack{j_1 + \cdots + j_r = j \\ j_i \geqslant 1}} \prod_{i=1}^{r} b_{*,j_i} = \begin{cases} 1, & j = 1, \\ 0, & j = 2, 3, \dots. \end{cases}$$

For $\lambda_{*,k}$ the estimate

$$|\lambda_{*,k}| \leqslant \frac{2}{k}\left(\frac{16}{\Delta_*}\right)^{k-2}\big((k+1)!\big)^{\gamma}, \quad k = 2, 3, \ldots,$$

is valid. Therefore,

$$L_{*,\gamma}(x) \leqslant \frac{x^3}{2(x + 8\Delta_{*,\gamma})}, \qquad L_{*,\gamma}(-x) \geqslant -\frac{x^3}{3\Delta_{*,\gamma}}.$$

Theorem 2. *Under conditions of Theorem* 1 *relations*

$$\frac{1 - F_{\tilde{Z}_N}(x)}{1 - \Phi(x)} \to 1, \qquad \frac{F_{\tilde{Z}_N}(-x)}{\Phi(-x)} \to 1 \tag{15}$$

hold for $x \geqslant 0$ such that

$$x = \begin{cases} o\big((\mathbf{D}T_{N,1})^{(1/2-p)\nu(\gamma)}\big) & \text{as } \mu \neq 0,\\ o\big((\mathbf{E}T_{N,2})^{(1/2-p)\nu(\gamma)}\big) & \text{as } \mu = 0 \end{cases} \tag{16}$$

if $\mathbf{D}T_{N,1} \to \infty$ or $\mathbf{E}T_{N,2} \to \infty$ (belongs on considered case: $\mu \neq 0$, $\mu = 0$) when $0 \leqslant p < 1/2$. Here $\nu(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$.

Remark 1. It follows from the Theorem 2 that in case $\gamma = 0$ relations (15) hold for $x \geqslant 0$ such that

$$x = \begin{cases} o\big((\mathbf{D}T_{N,1})^{(1/2-p)/3}\big) & \text{as } \mu \neq 0,\\ o\big((\mathbf{E}T_{N,2})^{(1/2-p)/3}\big) & \text{as } \mu = 0 \end{cases}$$

if $\mathbf{D}T_{N,1} \to \infty$ or $\mathbf{E}T_{N,2} \to \infty$ when $0 \leqslant p < 1/2$.

Theorem 3. *Let X with $\sigma^2 > 0$, and $T_{N,1}$, $T_{N,2}$ satisfy conditions* ($\bar{\mathrm{B}}_\gamma$), (L), (L_0)*, respectively. Then, for all $x \geqslant 0$,*

$$\mathbf{P}(\pm\tilde{Z}_N \geqslant x) \leqslant \exp\left\{-\frac{x^2}{2(2^{1+\gamma} + (x/\Delta_*^{1/(1+2\gamma)}))^{(1+2\gamma)/(1+\gamma)}}\right\}.$$

Corollary 1. *Under conditions of Theorem* 3 *exponential inequalities*

$$\mathbf{P}(\pm\tilde{Z}_N \geqslant x) \leqslant \begin{cases} \exp\{-x^2/4\}, & 0 \leqslant x \leqslant (2^{(1+\gamma)^2}\Delta_*)^{1/(1+2\gamma)},\\ \exp\{-(x\Delta_*)^{1/(1+\gamma)}/4\}, & x \geqslant (2^{(1+\gamma)^2}\Delta_*)^{1/(1+2\gamma)}, \end{cases}$$

are valid.

Corollary 2. *If for X with $\sigma^2 > 0$, and for $T_{N,1}$, $T_{N,2}$, accordingly, conditions* ($\bar{\mathrm{B}}_\gamma$), (L), (L_0) *are fulfilled, then*

$$\sup_x \big|F_{\tilde{Z}_N}(x) - \Phi(x)\big| \leqslant \frac{4.4}{\Delta_{*,\gamma}}.$$

Now let's consider some instances of large deviations.

Example 1. Assume N is non-random: $N = n \in \mathbb{N}$. Then $T_{N,r} = T_{n,r} = \sum_{j=1}^{n} a_j^r$, $r \in \mathbb{N}$, thus $\mathbf{E}T_{N,r} = T_{n,r}$, $\Gamma_k(T_{n,r}) = 0$, $k = 2, 3, \dots$. Accordingly, taking into account (4), we have

$$\mathbf{E}Z_n = \mu T_{n,1}, \qquad \mathbf{D}Z_n = \sigma^2 T_{n,2}.$$

Equality (18) (see Section 3) and condition (7) yield

$$\left|\Gamma_k(\tilde{Z}_n)\right| \leqslant \frac{(k!)^{1+\gamma}}{\Delta^{k-2}}, \quad \Delta = \frac{\sqrt{\mathbf{D}Z_n}}{aM}, \ k = 3, 4, \dots.$$

Obtained estimate coincide with estimate (15) presented in [25, p. 280] if we suppose that considered r.vs. in mentioned paper are not only independent, but also identically distributed r.vs. In this instance, estimate (15) holds with the parameters $\Delta_n = \Delta$, $\bar{B}_n^2 = \mathbf{D}Z_n$, $\gamma_n = a$.

Note that $\Delta \geqslant C\sqrt{T_{n,2}}$, where $C > 0$. Therefore, considering on Theorem 2 proof (see Section 3) relations (15) are valid for $x \geqslant 0$ such that $x = o(T_{n,2}^{\nu(\gamma)/2})$ if $T_{n,2} \to \infty$. Here $\nu(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$, $\gamma \geqslant 0$.

For the examples when N obey Poisson, binomial, negative binomial distributions, and for discount instance of large deviations we refer to our papers [2, 21].

From now, in the rest of section let us assume that $a_j \equiv 1$, $j = 1, 2, 3, \dots$, i.e. instead of the sum Z_N defined by (1) we consider the sum of the random number of i.i.d. r.vs. $\{X, X_j, \ j = 1, 2, \dots\}$

$$S_N = \sum_{j=1}^{N} X_j, \qquad S_0 = 0,$$

where as was mentioned in the Introduction, the non-negative integer-valued r.v. N with mean $\mathbf{E}N = \alpha$ and variance $\mathbf{D}N = \beta^2$ is independent of $\{X, X_j, \ j = 1, 2, \dots\}$.

It should be noted that some remarks on theorems of large deviations both in the Cramér and power Linnik zones when $\mu \neq 0$ and $a_j \equiv 1$ we made in our paper [2]. Now let's consider both cases: $\mu \neq 0$, $\mu = 0$.

If $a_j \equiv 1$, then (3) we can rewrite in the following way:

$$\mathbf{E}T_{N,r} = \alpha, \qquad \mathbf{D}T_{N,r} = \mathbf{E}N^2 - \alpha^2 = \beta^2$$

due to $T_{N,r} = N$, $\mathbf{E}T_{N,r}^2 = \mathbf{E}N^2$, $r \in \mathbb{N}$. Hence, it follows from (4) that

$$\mathbf{E}S_N = \mu\alpha, \qquad \mathbf{D}S_N = \sigma^2\alpha + \mu^2\beta^2.$$

In addition, conditions (L), (L_0) hold with $\mathbf{E}T_{N,2} = \alpha$, $\mathbf{D}T_{N,1} = \beta^2$.

The application of (18) for all $k = 1, 2, \dots$ leads to

$$\Gamma_k(S_N) = k! \sum\nolimits_1^* \frac{\Gamma_m(N)}{m_1! \cdots m_k!} \prod_{s=1}^{k} \left(\frac{1}{s!}\Gamma_s(X)\right)^{m_s}, \tag{17}$$

where $\sum_1^*$ denotes a summation over all the non-negative integer solutions $m_1 + 2m_2 + \cdots + km_k = k$, $m_1 + \cdots + m_k = m$. Here $0 \leqslant m_1, \dots, m_k \leqslant k$ and $1 \leqslant m \leqslant k$.

Equality (17) is presented in the paper [24] where probabilities of large deviations in the Cramér zone for the sum S_N when $\mu = 0$ have been obtained. From the expression of the equality mentioned above follows that if we need to estimate upper bounds for $\Gamma_k(\tilde{S}_N)$, $k = 3, 4, \dots$, we must impose not only conditions for the kth order cumulants of the r.v. X but of N, too.

With reference to (17), in a similar way as in proof of Lemma 1, we obtain that the upper estimate $|\Gamma_k(\tilde{S}_N)|$, $k = 3, 4, \dots$, where $\tilde{S}_N = (S_N - \mathbf{E}S_N)/(\mathbf{D}S_N)^{1/2}$ is defined by (8), where Δ_* defined by (9) holds with $a_j \equiv 1$, $\mathbf{E}T_{N,2} = \alpha$, $\mathbf{D}T_{N,1} = \beta^2$. Apparently, the application of Theorem 2 proof implies that relations (15) hold for $x \geqslant 0$ defined by (16) if $\beta \to \infty$ or $\alpha \to \infty$ (it belongs on considered case: $\mu \neq 0$, $\mu = 0$).

We recall that large deviation theorems in the Cramér zone for the sum S_N have been investigated in the paper [23] as well, where conditions (L), ($\bar{\mathrm{B}}_\gamma$) when $\gamma = 0$ and $\mu \neq 0$ were used. Our results in this instance coincides with those obtained in [23, p. 534–535].

Example 2. Assume that the number of summands in the sum S_N is non-random: $N = n \in \mathbb{N}$. Then $\alpha = n$, $\Gamma_m(N) = 0$, $m = 2, 3, \dots$.

Wherefore,

$$\mathbf{E}S_n = n\mu, \qquad \mathbf{D}S_n = n\sigma^2.$$

Taking into account (17) and (7), we arrive at

$$\left|\Gamma_k(\tilde{S}_n)\right| \leqslant \frac{(k!)^{1+\gamma}}{\Delta^{k-2}}, \quad \Delta = \frac{\sqrt{\mathbf{D}S_n}}{M}, \ k = 3, 4, \dots.$$

It is clear that $\Delta \geqslant C_1\sqrt{n}$, $C_1 > 0$. Obviously, it goes from Theorem 2 that relations (15) hold for $x \geqslant 0$ such that $x = o(n^{\nu(\gamma)/2})$ if $n \to \infty$.

From Examples 1, 2 follows that obtained theorems on large deviations for r.s. can be regarded as refinements of the theorems on large deviations for sums of non-random number of summands.

In stochastic theory N are often assumed to follow Poisson law.

Example 3. Let us look at the case where $N = N_t$, $t \geqslant 0$, is a homogeneous Poisson process with the probability $q_l = e^{-\lambda t}(\lambda t)^l/l!$, $l \in \mathbb{N}_0$, and intensity $\lambda > 0$. Then the sum $S_{N_t} = \sum_{j=1}^{N_t} X_j$ is the compound Poisson process with

$$\mathbf{E}S_{N_t} = \mu\lambda t, \qquad \mathbf{D}S_{N_t} = \lambda t\bigl(\mu^2 + \sigma^2\bigr)$$

as $\alpha = \beta^2 = \lambda t$. Furthermore,

$$\varphi_{N_t}(u) = \exp\bigl\{\lambda t\bigl(\exp\{\mathrm{i}u\} - 1\bigr)\bigr\}, \qquad \Gamma_k(N_t) = \lambda t, \quad k = 1, 2, \dots.$$

Afterwards, conditions (L), (L_0) hold with $p = 0$, $K_1 = 1$, $K_2 = 1$. Therefore, the use of Lemma 1 in directly gives that estimate (8) is valid with

$$\Delta_{N_t} = \frac{\sqrt{\lambda t(\mu^2 + \sigma^2)}}{L_{N_t}}, \quad L_{N_t} = 2\left(|\mu| \vee \left(1 \vee \frac{\sigma}{2|\mu|}\right)M\right),$$

$$\Delta_{N_t,0} = \frac{\sqrt{\lambda t}\sigma}{L_{N_t,0}}, \quad L_{N_t,0} = 2M,$$

where M defined by (7).

Undoubtedly, more accurate upper estimate for the kth order cumulants of the standardized compound Poisson process is valid. Indeed, based on remarks for compound Poisson process when $\mu \neq 0$ made in [2, p. 262], we conclude that

$$\left|\Gamma_k(\tilde{S}_{N_t})\right| \leqslant \frac{(k!)^{1+\gamma}}{\tilde{\Delta}_t}, \quad \tilde{\Delta}_t = \frac{\sqrt{\lambda t(\mu^2+\sigma^2)}}{K},\ k = 3, 4, \ldots,$$

where $K > 0$ defined by $(\bar{B}_\gamma)$. Certainly in both cases: $\mu \neq 0$, $\mu = 0$ relations (15) are valid for $x \geqslant 0$ such that $x = o(t^{\nu(\gamma)/2})$ if $t \to \infty$.

Moreover, in view of Corollary 2 we can assert that in the Cramér zone ($\gamma = 0$) the upper estimate of the normal approximation to the the r.s. $\tilde{S}_{N_t}$ is

$$\sup_x \left|F_{\tilde{S}_{N_t}}(x) - \Phi(x)\right| \leqslant \frac{4.4K}{\sqrt{\lambda t(\mu^2+\sigma^2)}}.$$

Note that absolute constant 4.4 in the above upper bound may be sharpened. Really, it was already noted in the introduction that in [17, p. 97] when $N = N_\lambda$ is Poisson r.v. the following theorem is proved.

Theorem 4. *Under conditions* $\mathbf{E}X = \mu$, $\mathbf{D}X = \sigma^2$, $\beta^3 = \mathbf{E}|X|^3 < \infty$ *for any* $\lambda > 0$ *the following inequality holds*:

$$\sup_x \left|F_{\tilde{S}_{N_\lambda}}(x) - \Phi(x)\right| \leqslant \frac{0.3041\beta^3}{(\mu^2+\sigma^2)^{3/2}\sqrt{\lambda}}.$$

Thereby, according to this upper estimate together with condition $(\bar{\mathrm{B}}_\gamma)$ with $\gamma = 0$, we can state that

$$\sup_x \left|F_{\tilde{S}_{N_t}}(x) - \Phi(x)\right| \leqslant \frac{1.8246K}{\sqrt{(\mu^2+\sigma^2)\lambda t}}.$$

Numerous examples of applied problems from diverse areas in which Poisson r.s. arise are given in [14]. For instance in the continuous dynamic models of an insurance stock we can express the surplus R_t at the moment $t \geqslant 0$ by $R_t = R_0 + P_t - S_{N_t}$ (see [12]). R_0 is the initial reserve, P_t – the total premium obtained up to time t, and S_{N_t} – the total claims in time interval $[0, t]$. Here X_j, $j = 1, 2, \ldots$, express the jth claim, and N_t – the number of claims by time t.

3 Proofs of results

At first, it should be recalled that Lemma 1, Theorems 1, 2 and Corollaries 1, 2 in case $\mu \neq 0$ are proved in our paper [2]. Thus, in this section in the proofs that are obtained in both cases: $\mu \neq 0$, $\mu = 0$ we give some results (that are already presented in the paper [2]) without proofs.

Let us begin with the proof of Lemma 1.

Proof of Lemma 1. In our paper inequality (18) [2, p. 259] for all $k = 1, 2, \dots$ was proved

$$\Gamma_k(Z_N) = k! \sum\nolimits_1^* \frac{(-1)^{m-1}(m-1)!}{m_1! \cdots m_k!} \times \prod_{s=1}^{k} \left(\sum\nolimits_2^* \mathbf{E}\bigl(T_{N,1}^{\eta_1} \cdots T_{N,s}^{\eta_s}\bigr) \prod_{r=1}^{s} \frac{1}{\eta_r!} \left(\frac{1}{r!} \Gamma_r(X) \right)^{\eta_r} \right)^{m_s}, \tag{18}$$

where $\sum_1^*$ denotes a summation over all the non-negative integer solutions $m_1 + 2m_2 + \cdots + km_k = k$, $m_1 + \cdots + m_k = m$. Here $0 \leqslant m_1, \dots, m_k \leqslant k$, and $1 \leqslant m \leqslant k$. In addition, $\sum_2^*$ is taken over all the non-negative integer solutions $\eta_1 + 2\eta_2 + \cdots + s\eta_s = s$, where $0 \leqslant \eta_1, \dots, \eta_s \leqslant s$. Moreover, $T_{N,r}^{\eta_r} = \bigl(\sum_{j=1}^{N} a_j^r \bigr)^{\eta_r}$, $r \in \mathbb{N}$, where $0 \leqslant a_j < \infty$.

Recall that $a = \sup\{a_j,\ j = 1, 2, \dots\} < \infty$ and suppose that $\mu = 0$, $0^0 = 1$. Obviously

$$T_{l,r} \leqslant a^{r-2} T_{l,2}, \quad \text{so} \quad \mathbf{E}T_{N,r}^{\eta_r} \leqslant a^{\eta_r(r-2)} \mathbf{E}T_{N,2}^{\eta_r},$$

where $\mathbf{E}T_{N,r}^{\eta_r} = \sum_{l=1}^{\infty} T_{l,r}^{\eta_r} q_l$, and $T_{l,r}$ is defined by (2).

Having in mind that $\mu = 0$, by the above inequality after evaluations, we obtain the estimate of (18)

$$\bigl|\Gamma_k(Z_N)\bigr| \leqslant k! \sum\nolimits_3^* \frac{|\Gamma_{\bar{m}}(T_{N,2})|}{m_2! \cdots m_k!} \prod_{s=2}^{k} \left(\frac{1}{s!} a^{s-2} \bigl|\Gamma_s(X)\bigr| \right)^{m_s}, \tag{19}$$

where $\sum_3^*$ denote a summation over all the non-negative integer solutions $2m_2 + \cdots + km_k = k$, $m_2 + \cdots + m_k = \bar{m}$, $1 \leqslant \bar{m} < k$. Here, in general case

$$\Gamma_m(T_{N,2}) = m! \sum\nolimits_4^* \frac{(-1)^{\tau-1}(\tau-1)!}{\tau_1! \cdots \tau_m!} \prod_{n=1}^{m} \left(\frac{1}{n!} \mathbf{E}T_{N,2}^n \right)^{\tau_n},$$

where $\sum_4^*$ denotes a summation over all the non-negative integer solutions $\tau_1 + 2\tau_2 + \cdots + m\tau_m = m$, and $\tau_1 + \cdots + \tau_m = \tau$. And $0 \leqslant \tau_1, \dots, \tau_m \leqslant m$, $1 \leqslant \tau \leqslant m$.

In accordance with (L_0) and (7) the estimate of the right hand side of (19) is

$$\bigl|\Gamma_k(Z_N)\bigr| \leqslant k! \mathbf{E}T_{N,2} \sum\nolimits_3^* \frac{\bar{m}!}{m_2! \cdots m_k!} \times \bigl(K_2 (\mathbf{E}T_{N,2})^p \bigr)^{\bar{m}-1} \prod_{s=2}^{k} \bigl((s!)^{\gamma} (aM)^{s-2} \sigma^2 \bigr)^{m_s}, \tag{20}$$

where $\mathbf{E}T_{N,2}$, M are defined by (3) and (7), respectively. Here $K_2 > 0$ together with $p \geqslant 0$ are defined by (L_0).

For further evaluations inequality and equality

$$a!b! \leqslant (a+b)!, \qquad \sum\nolimits_1^* \frac{(m_1 + \cdots + m_k)!}{m_1! \cdots m_k!} = 2^{k-1}, \quad k = 1, 2, \dots,$$

[2, p. 260–261] are needed. They yield the following inequalities:

$$\prod_{s=2}^{k}(s!)^{m_s} \leqslant k!, \quad \sum\nolimits_3^* \frac{\bar{m}!}{m_2!\cdots m_k!} \leqslant 2^{k-2}, \quad k=2,3,\ldots. \tag{21}$$

It is assumed by convention that $g_0 = 1$. Next, it's easy to derive that

$$\prod_{s=2}^{k}\bigl((aM)^{s-2}\sigma^2\bigr)^{m_s} = \sigma^{2\bar{m}}(aM)^{k-2\bar{m}}, \quad k=2,3,\ldots. \tag{22}$$

Finally, by substituting (21), (22) into (20), we imply that in case $\mu = 0$ estimate

$$\bigl|\Gamma_k(Z_N)\bigr| \leqslant (k!)^{1+\gamma}\sigma^2\mathbf{E}T_{N,2}L_{N,0}^{k-2}, \quad k=3,4,\ldots, \tag{23}$$

holds. Here $L_{N,0}$ defined by (11).

Based on Lemma 1 proof [2, p. 258], we derive

$$\bigl|\Gamma_k(Z_N)\bigr| \leqslant (k!)^{1+\gamma}L_N^{k-2}\bigl(\sigma^2\mathbf{E}T_{N,2} + \mu^2\mathbf{D}T_{N,1}\bigr), \quad k=3,4,\ldots, \tag{24}$$

on the understanding that $\mu \neq 0$. Here $\mathbf{D}T_{N,1}$, L_N are defined by (3) and (10), respectively.

To complete the proof of Lemma 1, it is sufficient to use (23), (24), and then by noticing that

$$\Gamma_k(\tilde{Z}_N) = \frac{\Gamma_k(Z_N - \mathbf{E}Z_N)}{(\mathbf{D}Z_N)^{k/2}} = \frac{\Gamma_k(Z_N)}{(\mathbf{D}Z_N)^{k/2}}, \quad k=2,3,\ldots,$$

with (4) or (5) (belongs on considered case: $\mu \neq 0$, $\mu = 0$), we arrive at (8). □

Proof of Theorem 2. The statement of Theorem 2 follows immediately if we use the definition of $L_{*,\gamma}(x)$, $\gamma \geqslant 0$ by relation (13).

We have to prove that $L_{*,\gamma}(x) \to 0$, $x/\Delta_{*,\gamma} \to 0$ as $\Delta_* \to \infty$, where Δ_*, $\Delta_{*,\gamma}$ defined, accordingly, by (9) and (12).

Recalling the definitions of $\mathbf{E}T_{N,2}$, $\Delta_{N,0}$ by (3), (11), respectively, we get $\Delta_{N,0} \geqslant C_2(\mathbf{E}T_{N,2})^{1/2-p}$, where $C_2 > 0$. This proves that $\Delta_{N,0} \to \infty$ if $\mathbf{E}T_{N,2} \to \infty$ when $0 \leqslant p < 1/2$. Subsequently, on the ground of Theorem 4 proof in [2, p. 265–266], together with (8), (12), (14), we imply that $L_{*,\gamma}(x) \to 0$ for all $x \geqslant 0$ defined by (16) if $\mathbf{D}T_{N,1} \to \infty$ or $\mathbf{E}T_{N,2} \to \infty$ when $0 \leqslant p < 1/2$. Here $\nu(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$. □

Proof of Theorem 3. The proof of Theorem 3 is obtained thanks to Lemma 2.4 (Bentkus, Rudzkis, 1980) in [3, p. 19], where relations (2.12), (2.13) hold with $H = 2^{1+\gamma}$, $\Delta = \Delta_*$.

It should be observed that the proofs of Theorem 1 and Corollaries 1, 2 follow almost directly from the proofs of Theorems 2, 3, 1 placed in our paper [2]. □

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