

Soliton solution and conservation laws of the Zakharov equation in plasmas with power law nonlinearity

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Received: 26 July 2012 / **Revised:** 2 April 2013 / **Published online:** 18 April 2013

Abstract. This paper studies the Zakharov equation with power law nonlinearity. The traveling wave hypothesis is applied to obtain the 1-soliton solution of this equation. The multiplier method from Lie symmetries is subsequently utilized to obtain the conservation laws of the equations. Finally, using the exact 1-soliton solution, the conserved quantities are listed.

Keywords: solitons, integrability, conservation laws.

1 Introduction

There are several nonlinear evolution equations (NLEEs) that appears in various areas of applied mathematics and theoretical physics [1–13]. These NLEEs are a key to the understanding of various physical phenomena that governs the world today. Some of these commonly studied NLEEs are the nonlinear Schrödinger's equation (NLSE), Korteweg–de Vries (KdV) equation, sine–Gordon equation (SGE), just to name a few. Some of these equations appear in the real domain while others appear in the complex domain. The NLSE appears in nonlinear optics, while KdV equation is studied in fluid dynamics and the SGE is seen in theoretical physics. There are various vector valued coupled equations that describe many physical phenomena. One such equation is the Zakharov equation (ZE) that is studied in the context of plasma physics [1].

NLEEs are studied by several authors and there are several interesting aspects and issues that have been addressed in the past. A systematical discussion on the secant type function was conducted earlier [6]. Additionally, the traveling wave solutions to nonlinear evolution equations by the transformed rational function method was displayed in 2009 [8]. Moreover, a hierarchy of conservation can be easily generated from conserved densities of Hamiltonian structures behind Lax pairs. This aspect has been addressed on

numerous occasions. The details were already addressed in 2006 [9]. This paper will integrate one such NLEE, namely the ZE with power law nonlinearity, by the traveling wave hypothesis method and will obtain the conserved quantities after Lie symmetry analysis extracts the conserved densities.

The traveling wave hypothesis will lead to an exact 1-soliton solution of ZE. There are a few constraint conditions that will naturally fall out during the course of derivation of the soliton solution of this equation. These constraint conditions must remain valid in order for the soliton solutions to exist. Once integrated into closed form, there are various physical features that are naturally revealed and many physical and mathematical aspects become illustrious. The conserved quantities will be subsequently computed using the soliton solution by the aid of Lie symmetry analysis.

2 Governing equations

The dimensionless form of the Zakharov equation (ZE) with power law nonlinearity is given by [1]

$$iq_t + aq_{xx} + b|q|^{2n}q = qr, \quad (1)$$

$$r_{tt} - k^2r_{xx} = (|q|^{2n})_{xx}. \quad (2)$$

In (1) and (2) the dependent variables are $q(x, t)$ and $r(x, t)$ where the first dependent variable is on a complex valued, while the second dependent variable is real-valued. The independent variables are x and t , which are the spatial and temporal variables, respectively. The parameter n is the power law nonlinearity parameter. If however $n = 1$, the power law nonlinearity relaxes to cubic nonlinearity that is commonly studied in the literature. In (1), the first term is the evolution term, while the coefficient of a is the group velocity dispersion (GVD) and b represents the power law nonlinearity. The right hand side represents the coupling term. Then in (2), the left hand side represents the wave operator that couples nonlinearly with the complex valued function $q(x, t)$ by means of the term on its right hand side. Equations (1) and (2) form a coupled system of NLEEs.

This equation with $n = 1$ has been studied earlier by using several other mathematical techniques such as the variational method [5, 13]. However, for arbitrary n the semi-inverse variational principle was applied to solve the coupled system [1]. Then, the traveling wave hypothesis was applied in 2009 [9]. The bifurcation analysis is also carried out for this equation with $n = 1$ [11]. Several other solutions were also obtained in terms of Jacobi's elliptic functions in 2009 [3]. This paper will study equations (1) and (2) for arbitrary values of n .

3 Traveling wave solution

In order to solve (1) and (2) by the aid of traveling wave hypothesis, it is assumed that

$$q(x, t) = g(s)e^{i\phi}, \quad (3)$$

where $g(s)$ represents the solitary wave profile and

$$s = x - vt, \quad (4)$$

$$\phi = -\kappa x + \omega t + \theta. \quad (5)$$

Here, v is the velocity of the soliton, κ is the frequency while ω is the soliton wave number and θ is the phase constant. Again, for (2), it is assumed that

$$r(x, t) = h(s). \quad (6)$$

Substituting (3), (4) and (6) into (2) yields

$$(v^2 - k^2)h'' = (g^{2n})'', \quad (7)$$

which gives

$$h = \frac{g^{2n}}{v^2 - k^2} \quad (8)$$

after integrating (7) twice. The integration constant is taken to be zero, since the search is for a soliton solution. If however, the integration constant is taken to be non-zero, it will lead to cnoidal waves, which is outside the scope of this paper. Now equation (1), by virtue of (3)–(6) and (8), yields after decomposing into real and imaginary parts, the following pair of equations

$$ag'' - (\omega + a\kappa^2)g + bg^{2n+1} - \frac{g^{2n+1}}{v^2 - k^2} = 0 \quad (9)$$

and

$$v = -2a\kappa, \quad (10)$$

respectively, where the notations $g' = dg/ds$ and $g'' = d^2g/ds^2$ are being used. While (10) gives the velocity of the soliton, (9) is the ordinary differential equation (ODE) that can be integrated. Therefore, multiplying (9) by g' and integrating once yields

$$a(g')^2 - (\omega + a\kappa^2)g^2 + \frac{1}{n+1} \left(b - \frac{1}{v^2 - k^2} \right) g^{2n+2} = 0. \quad (11)$$

Now, separating variables and integration of the ODE leads to

$$g(s) = A_1 \operatorname{sech}^{1/n} [B(x - vt)], \quad (12)$$

where the amplitude A_1 of the soliton is given by

$$A_1 = \left[\frac{(n+1)(v^2 - k^2)(\omega + a\kappa^2)}{b(v^2 - k^2) - 1} \right]^{1/2n} \quad (13)$$

while the width of the soliton is given by

$$B = \frac{n}{\sqrt{a}}. \quad (14)$$

Then, from (8), the solution $h(s)$ is given by

$$h(s) = A_2 \operatorname{sech}^2 [B(x - vt)], \quad (15)$$

where the amplitude A_2 is

$$A_2 = \frac{(n+1)(v^2 - k^2)(\omega + a\kappa^2)}{b(v^2 - k^2) - 1}. \quad (16)$$

The constraint conditions that immediately fall out of relations (13) and (14) are

$$a > 0 \quad (17)$$

and

$$(v^2 - k^2)(\omega + a\kappa^2) \{b(v^2 - k^2) - 1\} > 0 \quad (18)$$

if n is even. Thus, finally, the 1-soliton solution to the system (1) and (2) is given by

$$q(x, t) = A_1 \operatorname{sech}^{1/n} [B(x - vt)] e^{i(-\kappa x + \omega t + \theta)} \quad (19)$$

and

$$r(x, t) = A_2 \operatorname{sech}^2 [B(x - vt)], \quad (20)$$

and the velocity of the solitons is given by (10). The solitons will exist provided the constraint conditions given by (17) and (18) hold.

4 Symmetries and conservation laws

In order to determine conserved densities and fluxes, we resort to the invariance and multiplier approach based on the well known result that the Euler-Lagrange operator annihilates a total divergence (see [2]). Firstly, if (T, S) is a conserved vector corresponding to a conservation law, then

$$D_t T + D_x S = 0$$

along the solutions of the differential equation $E(t, x, q, q_{(1)}, q_{(2)}, \dots) = 0$, where $q_{(i)}$ represents all the possible i th derivatives of q .

Moreover, if there exists a nontrivial differential function Q , called a ‘‘multiplier’’ such that

$$\mathcal{E}_q[QE] = 0,$$

then QE is a total divergence, i.e.,

$$QE = D_t T^t + D_x T^x,$$

for some (conserved) vector (T, S) and \mathcal{E}_q is the respective Euler–Lagrange operator. Thus, a knowledge of each multiplier Q leads to a conserved vector determined by, inter alia, a Homotopy operator. See details and references in [2, 4].

For a system of three equations $E^j(t, x, q, q_{(1)}, q_{(2)}, \dots) = 0$, where $q = (u, v, w)$ say, we require

$$fE^1 + kE^2 + hE^3 = D_t T + D_x S$$

so that

$$\mathcal{E}(fE^1 + kE^2 + hE^3) = \mathbf{0},$$

where \mathcal{E} is the Euler operator and the vector (f, k, h) is the equivalent of the multiplier Q above. In each case, T^t is called the *conserved density*.

The system

$$\begin{aligned} iq_t + aq_{xx} + b|q|^{2n}q - qw &= 0, \\ r_{tt} - k^2 r_{xx} &= (|q|^{2n})_{xx} \end{aligned} \quad (21)$$

becomes, with $q = u + iv$,

$$\begin{aligned} u_t + av_{xx} + b(u^2 + v^2)^n v - vw &= 0, \\ -v_t + au_{xx} + b(u^2 + v^2)^n u - uw &= 0, \\ r_{tt} - k^2 r_{xx} - [(u^2 + v^2)^n]_{xx} &= 0. \end{aligned} \quad (22)$$

For $n = 1$, the detail calculations reveal the following results for the multipliers and corresponding conserved vectors for the system (22).

0. $(f, k, h) = (u, -v, 1)$:

$$\begin{aligned} T &= \frac{1}{2}(u^2 + v^2 + 2r_t), \\ S &= -2uu_x - avu_x + auv_x - 2vv_x - k^2 r_x. \end{aligned}$$

1. $(f, k, h) = (u, -v, t)$:

$$\begin{aligned} T &= \frac{1}{2}(u^2 + v^2 - 2r + 2tr_t), \\ S &= -2tuu_x - avu_x + auv_x - 2tvv_x - tk^2 r_x. \end{aligned}$$

2. $(f, k, h) = (-2tu, 2tv, (1/2)k^2 t^2 + (1/2)x^2)$:

$$\begin{aligned} T &= \frac{1}{2}(-2tu^2 - 2tv^2 - 2tk^2 r + x^2 r_t + t^2 k^2 r_t), \\ S &= \frac{1}{2}(2xu^2 + 2xv^2 - 2u((x^2 + t^2 k^2)u_x + 2atv_x) \\ &\quad - v(-4atu_x + 2(x^2 + t^2 k^2)v_x) - k^2(-2xr + (x^2 + t^2 k^2)r_x)). \end{aligned}$$

3. $(f, k, h) = (-t^2 u, t^2 v, (1/6)k^2 t^3 + (1/2)tx^2)$:

$$\begin{aligned} T &= \frac{1}{6}(-3t^2 u^2 - 3t^2 v^2 - 3x^2 r - 3t^2 k^2 r + 3tx^2 r_t + t^3 k^2 r_t), \\ S &= -\frac{1}{6}t(-6xu^2 - 6xv^2 + 2u((3x^2 + t^2 k^2)u_x + 3atv_x) \\ &\quad + v(-6atu_x + 2(3x^2 + t^2 k^2)v_x) + k^2(-6xr + (3x^2 + t^2 k^2)r_x)). \end{aligned}$$

Also, the one parameter Lie groups of transformations, in vector field form, that leave invariant (22) are

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= u\partial_v - v\partial_u, \\ X_3 &= 2tu\partial_v - 2tv\partial_u + \partial_r, \\ X_4 &= t^2u\partial_v - t^2v\partial_u + t\partial_r, \\ X_5 &= \partial_x, \end{aligned}$$

whose commutator table is

[,]	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$2X_2$	X_3	0
X_2	0	0	0	0	0
X_3	$-2X_2$	0	0	0	0
X_4	$-X_3$	0	0	0	0
X_5	0	0	0	0	0

Finally, the conserved densities of (21) are

$$\begin{aligned} \Phi_0 &= \frac{1}{2}(|q|^2 + 2r_t), \\ \Phi_1 &= \frac{1}{2}(|q|^2 - 2r + 2tr_t), \\ \Phi_2 &= \frac{1}{2}(-2t|q|^2 - 2tk^2r + x^2r_t + t^2k^2r_t), \\ \Phi_3 &= \frac{1}{6}(-3t^2|q|^2 - 3x^2r - 3t^2k^2r + 3tx^2r_t + t^3k^2r_t). \end{aligned}$$

Therefore, the conserved quantities using the 1-soliton solution given by (19) and (20) for $n = 1$ are given by

$$I_0 = \int_{-\infty}^{\infty} \Phi_0 dx = \frac{1}{2} \int_{-\infty}^{\infty} (|q|^2 + 2r_t) dx = \frac{A_1^2}{B}, \quad (23)$$

$$I_1 = \int_{-\infty}^{\infty} \Phi_1 dx = \frac{1}{2} \int_{-\infty}^{\infty} (|q|^2 - 2r + 2tr_t) dx = \frac{1}{B}(A_1^2 - 2A_2) \quad (24)$$

$$I_2 = \int_{-\infty}^{\infty} \Phi_2 dx = \frac{1}{2} \int_{-\infty}^{\infty} (-2t|q|^2 - 2tk^2r + x^2r_t + t^2k^2r_t) dx = -\frac{2t}{B}(A_1^2 + k^2A_2),$$

which will be conserved quantity provided

$$k^2 = -\frac{A_1^2}{A_2}.$$

The fourth conserved density, however, does not lead to a conserved quantity since it is a divergent integral.

5 Conclusions

This paper obtains the exact 1-soliton solution to the Zakharov equation by the aid of traveling wave hypothesis. This equation very commonly appears in the study of plasma physics. The constraint conditions or the parameter domains are obtained and listed in order for the soliton solution to exist. Subsequently, the Lie symmetry analysis is also carried out to extract the conserved densities of the equation. The commutator table is also given. Finally, the conserved quantities are obtained for the first three conserved densities. These results will serve as a very important milestone in the study of plasma physics.

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