

A coupled common fixed point theorem for a family of mappings

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Abstract. In this paper we introduce the concept of coincidentally commuting pair in the context of coupled fixed point problems. It is established that an arbitrary family of mappings has a coupled common fixed point with two other functions under certain contractive inequality condition where two specific members of the family are assumed to be coincidentally commuting with these two functions respectively. The main result has certain corollaries. An example shows that the main theorem properly contains one of its corollaries.

Keywords: coupled common fixed point, coincidentally commuting mappings.

1 Introduction

In a recent work Bhaskar and Lakshmikantham have established a coupled contraction mapping principle [1]. Following this work many authors created coupled fixed point theorems for a variety of mappings and in various spaces. Some instances of this work are noted in [2–16]. It has a large share in the recent development of fixed point theory.

An important category in fixed point theory is the common fixed point problems. An early result was established by Jungck under commuting conditions [17]. The concept of commuting has been generalized in various directions and in several ways over the years. One such notion is “coincidentally commuting”, also known as “weak compatibility” which was introduced in [18]. Several works have been done on fixed points of “coincidentally commuting” mappings as, for instances, in [19–21].

Coupled common fixed point and coincidence point problems were first addressed by Lakshmikantham and Ćirić [9] in which the authors extended the work of Bhaskar and Lakshmikantham [1]. Following this result other coupled coincidence point results appeared in [3] and [5].

In this paper we define the concept of “coincidentally commuting” for two mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ and establish some coupled common fixed point results. Our main result is for family of mappings which is not necessarily countable. It has several corollaries and an illustrative example. This example shows that the corollaries are effectively included in the main theorem. We have also shown with an example that the concept of “coincidentally commuting” is strictly weaker than the prior concept of “commuting” given in [9].

2 Mathematical preliminaries

Definition 1. (See [1].) An element $(x, y) \in X \times X$, is called a coupled fixed point of the mapping $F : X \times X \rightarrow X$ if

$$F(x, y) = x \quad \text{and} \quad F(y, x) = y.$$

Definition 2. (See [9].) An element $(x, y) \in X \times X$, is called a coupled coincidence point of the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ if

$$F(x, y) = gx \quad \text{and} \quad F(y, x) = gy.$$

Definition 3. (See [9].) An element $(x, y) \in X \times X$, is called a coupled common fixed point of the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ if

$$F(x, y) = gx = x \quad \text{and} \quad F(y, x) = gy = y.$$

Definition 4. (See [9].) Let X be a non-empty set and $F : X \times X \rightarrow X$ and $g : X \rightarrow X$. We say F and g are commutative if

$$gF(x, y) = F(gx, gy) \quad \text{for all } x, y \in X.$$

Definition 5. Let X be a non-empty set and $F : X \times X \rightarrow X$ and $g : X \rightarrow X$. F and g are said to be coincidentally commuting if they commute at their coupled coincidence points; that is, if $gx = F(x, y)$ and $gy = F(y, x)$ for some $(x, y) \in X \times X$, then

$$gF(x, y) = F(gx, gy) \quad \text{and} \quad gF(y, x) = F(gy, gx).$$

Example 1. Let $X = [0, \infty)$. Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be defined respectively as follows:

$$F(x, y) = \begin{cases} 1/3 & \text{if } x > 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad gx = \begin{cases} 0 & \text{if } x = 0, \\ 100 & \text{if } 0 < x < 1, \\ 1 & \text{if } x = 1, \\ 20 & \text{if } x > 1. \end{cases}$$

Here, the functions g and F commute at their only coupled coincidence point $(0, 0)$. Therefore, the pair of functions (g, F) is coincidentally commuting. But the pair of functions (g, F) is not commuting.

In view of the above example we have the following observation.

Remark 1. Every commuting pair is a coincidentally commuting pair but its converse is not true.

3 Main result

Theorem 1. Let (X, d) be a complete metric space. Let $\{F_\alpha : X \times X \rightarrow X : \alpha \in \Lambda\}$ be a family of mappings and $h, g : X \rightarrow X$ be two self mappings such that:

- (A) $h(X)$ and $g(X)$ are closed subsets of X , and
 (B) there exist $\alpha_0, \beta_0 \in \Lambda$ such that:
- (i) $F_{\alpha_0}(X \times X) \subseteq g(X), F_{\beta_0}(X \times X) \subseteq h(X)$,
 - (ii) the pairs (h, F_{α_0}) and (g, F_{β_0}) are coincidentally commuting,
 - (iii) $d(F_{\alpha_0}(x, y), F_\alpha(u, v)) \leq ad(hx, gu) + bd(hy, gv) + L \min\{d(F_{\alpha_0}(x, y), gu), d(F_\alpha(u, v), hx), d(F_{\alpha_0}(x, y), hx), d(F_\alpha(u, v), gu)\}$ for all $\alpha \in \Lambda$ and for all $x, y, u, v \in X$, where a, b and L are non-negative real numbers with $a + b < 1$.

Then, there exists a unique $(x, y) \in X \times X$ such that $x = hx = gx = F_\alpha(x, y)$ and $y = hy = gy = F_\alpha(y, x)$ for all $\alpha \in \Lambda$, that is, h, g and $\{F_\alpha : \alpha \in \Lambda\}$ have a unique coupled common fixed point in X . Moreover, any coupled common fixed point of h, g, F_{α_0} and F_{β_0} is a coupled common fixed point of h, g and $\{F_\alpha : \alpha \in \Lambda\}$.

Proof. Suppose that h, g and $\{F_\alpha : \alpha \in \Lambda\}$ have two coupled common fixed points (x, y) and (p, q) . Then, for all $\alpha \in \Lambda$,

$$x = hx = gx = F_\alpha(x, y) \quad \text{and} \quad y = hy = gy = F_\alpha(y, x), \quad (1)$$

and

$$p = hp = gp = F_\alpha(p, q) \quad \text{and} \quad q = hq = gq = F_\alpha(q, p). \quad (2)$$

From the condition (iii), using (1) and (2), we have

$$\begin{aligned} d(x, p) &= d(F_{\alpha_0}(x, y), F_\alpha(p, q)) \\ &\leq ad(hx, gp) + bd(hy, gq) \\ &\quad + L \min\{d(F_{\alpha_0}(x, y), gp), d(F_\alpha(p, q), hx), d(F_{\alpha_0}(x, y), hx), d(F_\alpha(p, q), gp)\} \\ &= ad(x, p) + bd(y, q). \end{aligned} \quad (3)$$

Again, from the condition (iii), using (1) and (2), we have

$$\begin{aligned} d(y, q) &= d(F_{\alpha_0}(y, x), F_\alpha(q, p)) \\ &\leq ad(hy, gq) + bd(hx, gp) \\ &\quad + L \min\{d(F_{\alpha_0}(y, x), gq), d(F_\alpha(q, p), hy), d(F_{\alpha_0}(y, x), hy), d(F_\alpha(q, p), gp)\} \\ &= ad(y, q) + bd(x, p). \end{aligned} \quad (4)$$

From (3) and (4), we have

$$d(x, p) + d(y, q) \leq (a + b)[d(x, p) + d(y, q)].$$

Since $a + b < 1$, it follows from the above inequality that

$$d(x, p) + d(y, q) = 0,$$

which implies $d(x, p) = 0$ and $d(y, q) = 0$, that is, $x = p$ and $y = q$, that is, $(x, y) = (p, q)$. Hence, the coupled common fixed point of h, g and $\{F_\alpha: \alpha \in \Lambda\}$, if it exists, is unique.

Now suppose that $(w, z) \in X \times X$ is a coupled common fixed point of h, g, F_{α_0} and F_{β_0} . Then

$$\begin{aligned} w &= hw = gw = F_{\alpha_0}(w, z) = F_{\beta_0}(w, z), \\ z &= hz = gz = F_{\alpha_0}(z, w) = F_{\beta_0}(z, w). \end{aligned} \tag{5}$$

For any $\alpha \in \Lambda$, from (iii) using (5), we have

$$\begin{aligned} &d(w, F_\alpha(w, z)) \\ &= d(F_{\alpha_0}(w, z), F_\alpha(w, z)) \\ &\leq ad(hw, gw) + bd(hz, gz) + L \min\{d(F_{\alpha_0}(w, z), gw), d(F_\alpha(w, z), hw), \\ &\qquad\qquad\qquad d(F_{\alpha_0}(w, z), hw), d(F_\alpha(w, z), gw)\} = 0, \end{aligned}$$

that is,

$$F_\alpha(w, z) = w \quad \text{for all } \alpha \in \Lambda. \tag{6}$$

Again, for any $\alpha \in \Lambda$, from (iii) using (5), we have

$$\begin{aligned} &d(z, F_\alpha(z, w)) \\ &= d(F_{\alpha_0}(z, w), F_\alpha(z, w)) \\ &\leq ad(hz, gz) + bd(hw, gw) + L \min\{d(F_{\alpha_0}(z, w), gz), d(F_\alpha(z, w), hz), \\ &\qquad\qquad\qquad d(F_{\alpha_0}(z, w), hz), d(F_\alpha(z, w), gz)\} = 0, \end{aligned}$$

that is,

$$F_\alpha(z, w) = z \quad \text{for all } \alpha \in \Lambda. \tag{7}$$

From (5), (6) and (7), we have $w = hw = gw = F_\alpha(w, z)$ and $z = hz = gz = F_\alpha(z, w)$ for all $\alpha \in \Lambda$, that is, (w, z) is a coupled common fixed point of h, g and $\{F_\alpha: \alpha \in \Lambda\}$. Hence, any coupled common fixed point of h, g, F_{α_0} and F_{β_0} is a coupled common fixed point of h, g and $\{F_\alpha: \alpha \in \Lambda\}$. The converse part is trivial.

Let x_0, y_0 be two points in X . Since $F_{\alpha_0}(X \times X) \subseteq g(X), F_{\beta_0}(X \times X) \subseteq h(X)$, we define the sequences $\{x_n\}$ and $\{y_n\}$ in X as follows for all $n \geq 0$:

$$p_{2n} = gx_{2n+1} = F_{\alpha_0}(x_{2n}, y_{2n}), \quad p_{2n+1} = hx_{2n+2} = F_{\beta_0}(x_{2n+1}, y_{2n+1}), \tag{8}$$

$$q_{2n} = gy_{2n+1} = F_{\alpha_0}(y_{2n}, x_{2n}), \quad q_{2n+1} = hy_{2n+2} = F_{\beta_0}(y_{2n+1}, x_{2n+1}). \tag{9}$$

From the condition (iii), using (8) and (9), we have

$$\begin{aligned}
d(p_{2n}, p_{2n+1}) &= d(F_{\alpha_0}(x_{2n}, y_{2n}), F_{\beta_0}(x_{2n+1}, y_{2n+1})) \\
&\leq ad(hx_{2n}, gx_{2n+1}) + bd(hy_{2n}, gy_{2n+1}) \\
&\quad + L \min\{d(F_{\alpha_0}(x_{2n}, y_{2n}), gx_{2n+1}), d(F_{\beta_0}(x_{2n+1}, y_{2n+1}), hx_{2n}), \\
&\quad\quad d(F_{\alpha_0}(x_{2n}, y_{2n}), hx_{2n}), d(F_{\beta_0}(x_{2n+1}, y_{2n+1}), gx_{2n+1})\} \\
&= ad(p_{2n-1}, p_{2n}) + bd(q_{2n-1}, q_{2n}). \tag{10}
\end{aligned}$$

From the condition (iii), using (8) and (9), we have

$$\begin{aligned}
d(q_{2n}, q_{2n+1}) &= d(F_{\alpha_0}(y_{2n}, x_{2n}), F_{\beta_0}(y_{2n+1}, x_{2n+1})) \\
&\leq ad(hy_{2n}, gy_{2n+1}) + bd(hx_{2n}, gx_{2n+1}) \\
&\quad + L \min\{d(F_{\alpha_0}(y_{2n}, x_{2n}), gy_{2n+1}), d(F_{\beta_0}(y_{2n+1}, x_{2n+1}), hy_{2n}), \\
&\quad\quad d(F_{\alpha_0}(y_{2n}, x_{2n}), hy_{2n}), d(F_{\beta_0}(y_{2n+1}, x_{2n+1}), gy_{2n+1})\} \\
&= ad(q_{2n-1}, q_{2n}) + bd(p_{2n-1}, p_{2n}). \tag{11}
\end{aligned}$$

From (10) and (11), we have

$$d(p_{2n}, p_{2n+1}) + d(q_{2n}, q_{2n+1}) \leq (a + b)[d(p_{2n-1}, p_{2n}) + d(q_{2n-1}, q_{2n})]. \tag{12}$$

Again, from the condition (iii), using (8) and (9), we have

$$\begin{aligned}
d(p_{2n+1}, p_{2n+2}) &= d(p_{2n+2}, p_{2n+1}) = d(F_{\alpha_0}(x_{2n+2}, y_{2n+2}), F_{\beta_0}(x_{2n+1}, y_{2n+1})) \\
&\leq ad(hx_{2n+2}, gx_{2n+1}) + bd(hy_{2n+2}, gy_{2n+1}) \\
&\quad + L \min\{d(F_{\alpha_0}(x_{2n+2}, y_{2n+2}), gx_{2n+1}), d(F_{\beta_0}(x_{2n+1}, y_{2n+1}), hx_{2n+2}), \\
&\quad\quad d(F_{\alpha_0}(x_{2n+2}, y_{2n+2}), hx_{2n+2}), d(F_{\beta_0}(x_{2n+1}, y_{2n+1}), gx_{2n+1})\} \\
&= ad(p_{2n+1}, p_{2n}) + bd(q_{2n+1}, q_{2n}) = ad(p_{2n}, p_{2n+1}) + bd(q_{2n}, q_{2n+1}). \tag{13}
\end{aligned}$$

From the condition (iii), using (8) and (9), we have

$$\begin{aligned}
d(q_{2n+1}, q_{2n+2}) &= d(q_{2n+2}, q_{2n+1}) = d(F_{\alpha_0}(y_{2n+2}, x_{2n+2}), F_{\beta_0}(y_{2n+1}, x_{2n+1})) \\
&\leq ad(hy_{2n+2}, gy_{2n+1}) + bd(hx_{2n+2}, gx_{2n+1}) \\
&\quad + L \min\{d(F_{\alpha_0}(y_{2n+2}, x_{2n+2}), gy_{2n+1}), d(F_{\beta_0}(y_{2n+1}, x_{2n+1}), hy_{2n+2}), \\
&\quad\quad d(F_{\alpha_0}(y_{2n+2}, x_{2n+2}), hy_{2n+2}), d(F_{\beta_0}(y_{2n+1}, x_{2n+1}), gy_{2n+1})\} \\
&= ad(q_{2n+1}, q_{2n}) + bd(p_{2n+1}, p_{2n}) = ad(q_{2n}, q_{2n+1}) + bd(p_{2n}, p_{2n+1}). \tag{14}
\end{aligned}$$

From (13) and (14), we have

$$d(p_{2n+1}, p_{2n+2}) + d(q_{2n+1}, q_{2n+2}) \leq (a+b)[d(p_{2n}, p_{2n+1}) + d(q_{2n}, q_{2n+1})]. \quad (15)$$

It follows from (12) and (15) that

$$d(p_n, p_{n+1}) + d(q_n, q_{n+1}) \leq (a+b)[d(p_{n-1}, p_n) + d(q_{n-1}, q_n)].$$

Set $r_n = d(p_n, p_{n+1}) + d(q_n, q_{n+1})$ and $\delta = a+b$, then

$$0 \leq r_n \leq \delta r_{n-1} \leq \delta^2 r_{n-2} \leq \dots \leq \delta^n r_0.$$

Now, we show that both $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences. For each $m \geq n$, we have

$$d(p_m, p_n) \leq d(p_m, p_{m-1}) + d(p_{m-1}, p_{m-2}) + \dots + d(p_{n+1}, p_n)$$

and

$$d(q_m, q_n) \leq d(q_m, q_{m-1}) + d(q_{m-1}, q_{m-2}) + \dots + d(q_{n+1}, q_n).$$

Therefore,

$$\begin{aligned} & d(p_m, p_n) + d(q_m, q_n) \\ & \leq r_{m-1} + r_{m-2} + \dots + r_n \leq (\delta^{m-1} + \delta^{m-2} + \dots + \delta^n) r_0 \leq \frac{\delta^n}{1-\delta} r_0. \end{aligned}$$

Since $\delta < 1$, we have

$$\lim_{n,m \rightarrow \infty} [d(p_m, p_n) + d(q_m, q_n)] = 0,$$

which implies that

$$\lim_{n,m \rightarrow \infty} d(p_m, p_n) = 0 \quad \text{and} \quad \lim_{n,m \rightarrow \infty} d(q_m, q_n) = 0.$$

It follows that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in X . From the completeness of X , there exist $x, y \in X$ such that

$$\lim_{n \rightarrow \infty} p_n = x \quad \text{and} \quad \lim_{n \rightarrow \infty} q_n = y. \quad (16)$$

Therefore, from (8), (9) and (16), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} p_{2n} &= \lim_{n \rightarrow \infty} g x_{2n+1} = \lim_{n \rightarrow \infty} F_{\alpha_0}(x_{2n}, y_{2n}) = \lim_{n \rightarrow \infty} p_{2n+1} \\ &= \lim_{n \rightarrow \infty} h x_{2n+2} = \lim_{n \rightarrow \infty} F_{\beta_0}(x_{2n+1}, y_{2n+1}) = x \end{aligned} \quad (17)$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} q_{2n} &= \lim_{n \rightarrow \infty} g y_{2n+1} = \lim_{n \rightarrow \infty} F_{\alpha_0}(y_{2n}, x_{2n}) = \lim_{n \rightarrow \infty} q_{2n+1} \\ &= \lim_{n \rightarrow \infty} h y_{2n+2} = \lim_{n \rightarrow \infty} F_{\beta_0}(y_{2n+1}, x_{2n+1}) = y. \end{aligned} \quad (18)$$

Since $F_{\alpha_0}(X \times X) \subseteq g(X)$, $F_{\beta_0}(X \times X) \subseteq h(X)$ and $h(X)$, $g(X)$ are closed subsets of X , from (17) and (18), it is clear that $x, y \in h(X) \cap g(X)$. Then there exist $r, s \in X$ such that $hr = x$ and $hs = y$ and there exist $w, z \in X$ such that $gw = x$ and $gz = y$.

From the condition (iii), we have

$$\begin{aligned} & d(F_{\alpha_0}(r, s), F_{\beta_0}(x_{2n+1}, y_{2n+1})) \\ & \leq ad(hr, gx_{2n+1}) + bd(hs, gy_{2n+1}) \\ & \quad + L \min\{d(F_{\alpha_0}(r, s), gx_{2n+1}), d(F_{\beta_0}(x_{2n+1}, y_{2n+1}), hr), \\ & \quad d(F_{\alpha_0}(r, s), hr), d(F_{\beta_0}(x_{2n+1}, y_{2n+1}), gx_{2n+1})\}. \end{aligned}$$

Taking $n \rightarrow \infty$ in the above inequality, using (17) and (18), we have

$$d(F_{\alpha_0}(r, s), x) = 0; \quad \text{that is, } x = F_{\alpha_0}(r, s).$$

Therefore, we have

$$x = hr = F_{\alpha_0}(r, s). \quad (19)$$

Similarly, we can prove

$$y = hs = F_{\alpha_0}(s, r). \quad (20)$$

Again, from the condition (iii), we have

$$\begin{aligned} & d(F_{\alpha_0}(x_{2n}, y_{2n}), F_{\beta_0}(w, z)) \\ & \leq ad(hx_{2n}, gw) + bd(hy_{2n}, gz) \\ & \quad + L \min\{d(F_{\alpha_0}(x_{2n}, y_{2n}), gw), d(F_{\beta_0}(w, z), hx_{2n}), \\ & \quad d(F_{\alpha_0}(x_{2n}, y_{2n}), hx_{2n}), d(F_{\beta_0}(w, z), gw)\}. \end{aligned}$$

Taking $n \rightarrow \infty$ in the above inequality, using (17) and (18), we have

$$d(x, F_{\beta_0}(w, z)) = 0; \quad \text{that is, } x = F_{\beta_0}(w, z).$$

Therefore, we have

$$x = gw = F_{\beta_0}(w, z). \quad (21)$$

Similarly, we can prove

$$y = gz = F_{\beta_0}(z, w). \quad (22)$$

From (19), (21) and (20), (22), we have respectively

$$x = hr = F_{\alpha_0}(r, s) = gw = F_{\beta_0}(w, z) \quad (23)$$

and

$$y = hs = F_{\alpha_0}(s, r) = gz = F_{\beta_0}(z, w). \quad (24)$$

From (23) and (24), it follows that (r, s) is a coupled coincidence point of h and F_{α_0} . Since the pair (h, F_{α_0}) is coincidentally commuting, we have

$$hF_{\alpha_0}(r, s) = F_{\alpha_0}(hr, hs) \quad \text{and} \quad hF_{\alpha_0}(s, r) = F_{\alpha_0}(hs, hr),$$

that is,

$$hx = F_{\alpha_0}(x, y) \quad \text{and} \quad hy = F_{\alpha_0}(y, x). \quad (25)$$

Again, from (23) and (24), it follows that (w, z) is a coupled coincidence point of g and F_{β_0} . Since the pair (g, F_{β_0}) is coincidentally commuting, we have

$$gF_{\beta_0}(w, z) = F_{\beta_0}(gw, gz) \quad \text{and} \quad gF_{\beta_0}(z, w) = F_{\beta_0}(gz, gw),$$

that is,

$$gx = F_{\beta_0}(x, y) \quad \text{and} \quad gy = F_{\beta_0}(y, x).$$

From the condition (iii), using (23), (24) and (25), we have

$$\begin{aligned} d(F_{\alpha_0}(x, y), x) &= d(F_{\alpha_0}(x, y), F_{\beta_0}(w, z)) \\ &\leq ad(hx, gw) + bd(hy, gz) + L \min\{d(F_{\alpha_0}(x, y), gw), d(F_{\beta_0}(w, z), hx), \\ &\quad d(F_{\alpha_0}(x, y), hx), d(F_{\beta_0}(w, z), gw)\} \\ &= ad(F_{\alpha_0}(x, y), x) + bd(F_{\alpha_0}(y, x), y). \end{aligned} \quad (26)$$

Again, from the condition (iii), using (23), (24) and (25), we have

$$\begin{aligned} d(F_{\alpha_0}(y, x), y) &= d(F_{\alpha_0}(y, x), F_{\beta_0}(z, w)) \\ &\leq ad(hy, gz) + bd(hx, gw) + L \min\{d(F_{\alpha_0}(y, x), gz), d(F_{\beta_0}(z, w), hy), \\ &\quad d(F_{\alpha_0}(y, x), hy), d(F_{\beta_0}(z, w), gz)\} \\ &= ad(F_{\alpha_0}(y, x), y) + bd(F_{\alpha_0}(x, y), x). \end{aligned} \quad (27)$$

From (26) and (27), we have

$$d(F_{\alpha_0}(x, y), x) + d(F_{\alpha_0}(y, x), y) \leq (a + b)[d(F_{\alpha_0}(x, y), x) + d(F_{\alpha_0}(y, x), y)].$$

Since $a + b < 1$, it follows from the above inequality that

$$d(F_{\alpha_0}(x, y), x) + d(F_{\alpha_0}(y, x), y) = 0,$$

which implies that

$$d(F_{\alpha_0}(x, y), x) = 0 \quad \text{and} \quad d(F_{\alpha_0}(y, x), y) = 0,$$

that is,

$$F_{\alpha_0}(x, y) = x \quad \text{and} \quad F_{\alpha_0}(y, x) = y. \quad (28)$$

From (25) and (28), we have

$$hx = F_{\alpha_0}(x, y) = x \quad \text{and} \quad hy = F_{\alpha_0}(y, x) = y. \quad (29)$$

Similarly, we can prove that

$$gx = F_{\beta_0}(x, y) = x \quad \text{and} \quad gy = F_{\beta_0}(y, x) = y. \quad (30)$$

From (29) and (30), we have

$$x = hx = gx = F_{\alpha_0}(x, y) = F_{\beta_0}(x, y)$$

and

$$y = hy = gy = F_{\alpha_0}(y, x) = F_{\beta_0}(y, x).$$

Therefore, (x, y) is a coupled common fixed point of h, g, F_{α_0} and F_{β_0} .

By what we have already proved, (x, y) is the unique coupled common fixed point of h, g and $\{F_\alpha: \alpha \in \Lambda\}$. \square

Note. The contractive inequality (iii) of Theorem 1 is an analogue of Condition B which was introduced by Babu, Sandhya and Kameswari [22] in 2008.

Since every commuting pair of functions is a coincidentally commuting pair, we have the following corollary.

Corollary 1. *Let (X, d) be a complete metric space. Let $\{F_\alpha : X \times X \rightarrow X: \alpha \in \Lambda\}$ be a family of mappings and $h, g : X \rightarrow X$ be two self mappings such that:*

(A) $h(X)$ and $g(X)$ are closed subsets of X , and

(B) there exist $\alpha_0, \beta_0 \in \Lambda$ such that:

(i) $F_{\alpha_0}(X \times X) \subseteq g(X), F_{\beta_0}(X \times X) \subseteq h(X)$,

(ii) the pairs (h, F_{α_0}) and (g, F_{β_0}) are commuting,

(iii) $d(F_{\alpha_0}(x, y), F_\alpha(u, v)) \leq ad(hx, gu) + bd(hy, gv) + L \min\{d(F_{\alpha_0}(x, y), gu), d(F_\alpha(u, v), hx), d(F_{\alpha_0}(x, y), hx), d(F_\alpha(u, v), gu)\}$ for all $\alpha \in \Lambda$ and for all $x, y, u, v \in X$, where a, b and L are non-negative real numbers with $a + b < 1$.

Then, there exists a unique $(x, y) \in X \times X$ such that $x = hx = gx = F_\alpha(x, y)$ and $y = hy = gy = F_\alpha(y, x)$ for all $\alpha \in \Lambda$, that is, h, g and $\{F_\alpha: \alpha \in \Lambda\}$ have a unique coupled common fixed point in X . Moreover, any coupled common fixed point of h, g, F_{α_0} and F_{β_0} is a coupled common fixed point of h, g and $\{F_\alpha: \alpha \in \Lambda\}$.

Corollary 2. *Let (X, d) be a complete metric space. Let $\{F_\alpha : X \times X \rightarrow X: \alpha \in \Lambda\}$ be a family of mappings and $h, g : X \rightarrow X$ be two self mappings such that:*

(A) $h(X)$ and $g(X)$ are closed subsets of X , and

(B) there exist $\alpha_0, \beta_0 \in \Lambda$ such that:

(i) $F_{\alpha_0}(X \times X) \subseteq g(X), F_{\beta_0}(X \times X) \subseteq h(X)$,

(ii) the pairs (h, F_{α_0}) and (g, F_{β_0}) are coincidentally commuting,

(iii) $d(F_{\alpha_0}(x, y), F_\alpha(u, v)) \leq ad(hx, gu) + bd(hy, gv)$ for all $\alpha \in \Lambda$ and for all $x, y, u, v \in X$, where a and b are non-negative real numbers with $a + b < 1$.

Then, there exists a unique $(x, y) \in X \times X$ such that $x = hx = gx = F_\alpha(x, y)$ and $y = hy = gy = F_\alpha(y, x)$ for all $\alpha \in \Lambda$, that is, h, g and $\{F_\alpha: \alpha \in \Lambda\}$ have a unique coupled common fixed point in X . Moreover, any coupled common fixed point of h, g, F_{α_0} and F_{β_0} is a coupled common fixed point of h, g and $\{F_\alpha: \alpha \in \Lambda\}$.

Proof. Taking $L = 0$ in Theorem 1, we have the required proof. □

Corollary 3. *Let (X, d) be a complete metric space. Let $\{F_\alpha : X \times X \rightarrow X : \alpha \in \Lambda\}$ be a family of mappings and $h, g : X \rightarrow X$ be two self mappings such that:*

- (A) $h(X)$ and $g(X)$ are closed subsets of X , and
- (B) there exist $\alpha_0, \beta_0 \in \Lambda$ such that:
 - (i) $F_{\alpha_0}(X \times X) \subseteq g(X), F_{\beta_0}(X \times X) \subseteq h(X)$,
 - (ii) the pairs (h, F_{α_0}) and (g, F_{β_0}) are coincidentally commuting,
 - (iii) $d(F_{\alpha_0}(x, y), F_{\alpha_0}(u, v)) \leq k/2[d(hx, gu) + d(hy, gv)]$ for all $\alpha \in \Lambda$ and for all $x, y, u, v \in X$, where $k \in [0, 1)$.

Then, there exists a unique $(x, y) \in X \times X$ such that $x = hx = gx = F_\alpha(x, y)$ and $y = hy = gy = F_\alpha(y, x)$ for all $\alpha \in \Lambda$, that is, h, g and $\{F_\alpha : \alpha \in \Lambda\}$ have a unique coupled common fixed point in X . Moreover, any coupled common fixed point of h, g, F_{α_0} and F_{β_0} is a coupled common fixed point of h, g and $\{F_\alpha : \alpha \in \Lambda\}$.

Proof. Taking $a = b = k/2$, where $k \in [0, 1)$ and $L = 0$ in Theorem 1, we have the required proof. □

Corollary 4. *Let (X, d) be a complete metric space. Let $h : X \rightarrow X, g : X \rightarrow X, F : X \times X \rightarrow X$ and $G : X \times X \rightarrow X$ be four mappings such that the following conditions are satisfied:*

- (i) $F(X \times X) \subseteq g(X), G(X \times X) \subseteq h(X)$,
- (ii) $h(X), g(X)$ are closed subsets of X ,
- (iii) (h, F) and (g, G) are coincidentally commuting pairs,
- (iv) $d(F(x, y), G(u, v)) \leq ad(hx, gu) + bd(hy, gv) + L \min\{d(F(x, y), gu), d(G(u, v), hx), d(F(x, y), hx), d(G(u, v), gu)\}$ for all $x, y, u, v \in X$, where a, b and L are non-negative real numbers with $a + b < 1$.

Then there exists a unique $(x, y) \in X \times X$ such that $x = hx = gx = F(x, y) = G(x, y)$ and $y = hy = gy = F(y, x) = G(y, x)$; that is, h, g, F and G have a unique coupled common fixed point in X .

Proof. Considering $\{F_\alpha : \alpha \in \Lambda\} = \{F, G\}$ in Theorem 1, we have the required proof. □

Corollary 5. *Let (X, d) be a complete metric space. Let $F : X \times X \rightarrow X$ and $G : X \times X \rightarrow X$ be two mappings. Suppose there exist non-negative real numbers a, b and L with $a + b < 1$ such that*

$$\begin{aligned}
 & d(F(x, y), G(u, v)) \\
 & \leq ad(x, u) + bd(y, v) \\
 & + L \min\{d(F(x, y), u), d(G(u, v), x), d(F(x, y), x), d(G(u, v), u)\}
 \end{aligned}$$

for all $x, y, u, v \in X$. Then there exists a unique $(x, y) \in X \times X$ such that $x = F(x, y) = G(x, y)$ and $y = F(y, x) = G(y, x)$; that is, F and G have a unique common coupled fixed point in X .

Proof. Considering $\{F_\alpha: \alpha \in \Lambda\} = \{F, G\}$ and h, g to be the identity mappings in Theorems 1, we have the required proof. \square

Corollary 6. Let (X, d) be a complete metric space. Let $F : X \times X \rightarrow X$ be a mapping. Suppose there exist a constant $k \in [0, 1)$ such that

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)]$$

for all $x, y, u, v \in X$. Then there exists a unique $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$; that is, F has a unique coupled fixed point in X .

Proof. Considering $\{F_\alpha: \alpha \in \Lambda\} = \{F\}$, $a = b = k/2$, where $k \in [0, 1)$, $L = 0$ and h, g to be the identity mappings in Theorems 1, we have the required proof. \square

Example 2. Let $X = [0, \infty)$ and the metric d on X be defined as $d(x, y) = |x - y|$ for $x, y \in X$.

Then (X, d) is a complete metric space.

Let $\Lambda = \{1, 2, 3, \dots\}$ and for every $\alpha \in \Lambda$, $F_\alpha : X \times X \rightarrow X$ be defined as follows: for $\alpha \in \Lambda$ with $\alpha \neq 1, 2$,

$$F_\alpha(x, y) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1, \\ 2\alpha/(\alpha + 1) & \text{if } x > 1, \end{cases}$$

$$F_1(x, y) = \begin{cases} 1/3 & \text{if } x > 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$F_2(x, y) = \begin{cases} 1 & \text{if } x > 1 \text{ and } y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $h, g : X \rightarrow X$ be defined as follows:

$$hx = \begin{cases} 0 & \text{if } x = 0, \\ 100 & \text{if } 0 < x < 1, \\ 1 & \text{if } x = 1, \\ 20 & \text{if } x > 1 \end{cases} \quad \text{and} \quad gx = \begin{cases} x/2 & \text{if } 0 \leq x \leq 1, \\ 200 & \text{if } x > 1. \end{cases}$$

Then $F_1(X \times X) \subseteq g(X)$, $F_2(X \times X) \subseteq h(X)$, the pairs (h, F_1) and (g, F_2) are coincidentally commuting and also $h(X), g(X)$ are closed subsets of X . Let $a = 1/2$, $b = 1/3$ and L is any arbitrary non-negative real number. Then the inequality (iii) of Theorem 1 is satisfied. Hence all the required conditions of Theorems 1 are satisfied and it is seen that $(0, 0) \in X \times X$ is the unique coupled common fixed point of h, g and $\{F_\alpha: \alpha \in \Lambda\}$.

Remark 2. Corollary 5 is an extension of Theorem 2.1 [10] to a pair of maps, and Corollary 6 is just Theorem 2.1 and 2.2 [1] in metric space setting.

Remark 3. In Example 2, $F_\alpha \subseteq g(X)$ for only $\alpha = 1$ but the pair (h, F_1) is not commuting so that this example is not applicable to Corollary 1. Hence Theorem 1 properly contains Corollary 1.

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