Some exact solutions to the generalized Korteweg–de Vries equation and the system of shallow water wave equations

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Abstract. In this paper, we establish exact solutions for nonlinear evolution equations in mathematical physics. The exp-transform method is proposed to seek solitary solutions, periodic solutions and compaction-like solutions of nonlinear differential equations. The generalized KdV equation and the system of the shallow water wave equation are chosen to illustrate the effectiveness and convenience of the method.

Keywords: exp-transform method, non-linear PDE, system of the shallow water wave equation, generalized KdV equation.

1 Introduction

Many phenomena in physics and other fields are often described by non-linear partial differential equations (PDE's), such as fluid dynamics, plasma physics, mathematical biology, solid-state physics and chemical kinetics. When we want to understand the physical mechanism of natural phenomenon described by non-linear PDE, exact solutions for the nonlinear PDE have to be explored, thus the methods for deriving exact solutions for the governing equations have to be developed. Exploring exact solutions of nonlinear PDE's has become one of most important topics in mathematical physics.

Up to now, there exist many powerful methods to construct exact solutions of nonlinear PDE's. For example, the Wadati trace method [1–3], tanh-function method [4–7], F-expansion method [8–10], Lie group theory [11–13], Hirota's bilinear forms [14, 15], the inverse scattering method [16] and exp-function method [17–20], etc.

The rest of the paper is organized as follows. In Section 2, we describe the exp-transform method for finding travelling wave solutions of nonlinear evolution equations and give the main steps of the method here. In the subsequent sections, in Section 3 and Section 4, we illustrate the method in detail with the generalized KdV equation and the system of the shallow water wave equations. Finally, conclusions and discussion are given.

2 Exp-transform method

We consider a general nonlinear PDE in the form

$$P(u, u_t, u_x, u_{xx}, u_{tt}, \dots) = 0, \tag{1}$$

using a transformation

$$\eta = x - \lambda t,$$

where λ is constant, we can rewrite Eq. (1) in the following nonlinear ODE:

$$Q(u, u', u'', u''', \dots) = 0. \tag{2}$$

The exp-transform method is based on the assumption that travelling wave solutions can be expressed in the following form [17–24]:

$$f(\eta) = f\big(\omega(\eta)\big) = \frac{\sum_{i=-n}^{n} a_i \mathrm{e}^{i\omega(\eta)}}{\sum_{i=-m}^{m} b_i \mathrm{e}^{i\omega(\eta)}}, \tag{3}$$

where a_i, b_i are unknown constants. To determined values of m and n, we balance the linear term of highest order in Eq. (2) with the highest order nonlinear term.

$\omega(\eta)$ is a solution of the following first-order ordinary differential equations containing exponential functions (SET means set of exact solutions) in various combinations:

ODE A: $$\frac{\mathrm{d}\omega}{\mathrm{d}\eta} = \omega' = c\mathrm{e}^{\omega} + ad\mathrm{e}^{-\omega},$$

SET A: $$\omega_{A,1} = \ln\left[\sqrt{\frac{d}{c}} \tan\big[\sqrt{cd}(\eta - c_1)\big]\right] \quad \text{if } a = 1,$$

$$\omega_{A,2} = \ln\left[-\sqrt{\frac{d}{c}} \tanh\big[\sqrt{cd}(\eta - c_1)\big]\right] \quad \text{if } a = -1,$$

ODE B: $$\frac{\mathrm{d}\omega}{\mathrm{d}\eta} = \omega' = a\sqrt{ad^2\mathrm{e}^{2\omega(\eta)} - acd},$$

SET B: $$\omega_{B,1} = \ln\left[-\sqrt{\frac{c}{d}} \sec\big[\sqrt{cd}(\eta - c_1)\big]\right] \quad \text{if } a = 1,$$

$$\omega_{B,2} = \ln\left[-\sqrt{\frac{c}{d}} \operatorname{sech}\big[\sqrt{cd}(\eta - c_1)\big]\right] \quad \text{if } a = -1,$$

ODE C: $$\frac{\mathrm{d}\omega}{\mathrm{d}\eta} = \omega' = \sqrt{ad^2\mathrm{e}^{-2\omega(\eta)} - acd},$$

$$\text{SET C:} \qquad \omega_{C,1} = \ln\left[\sqrt{\frac{d}{c}}\sin\left[\sqrt{cd}(\eta - c_1)\right]\right] \quad \text{if } a = 1,$$
$$\omega_{C,2} = \ln\left[\sqrt{\frac{d}{c}}\cosh\left[\sqrt{cd}(\eta - c_1)\right]\right] \quad \text{if } a = -1,$$

where $cd > 0$.

Substituting (3) into Eq. (2) along with their derivations relating to the given ODEs A to C and yields a set of algebraic equations for $\mathrm{e}^{i\omega(\eta)}$. Setting the coefficients of $\mathrm{e}^{i\omega(\eta)}$ to zero yields a set of over-determined algebraic equations with respect to the parameters c, d, a_i, b_i, λ

3 The generalized Korteweg–de Vries equation (GKdV)

We next examine the GKdV equation:

$$u_t - 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} - u_{xxxxx} = 0. \tag{4}$$

The wave variable $u(x,t) = u(\eta)$, where $\eta = x - \lambda t$, carries the GKdV equation (4) into a system of ODEs

$$\lambda u'(\eta) + 30u(\eta)^2u'(\eta) - 20u'(\eta)u''(\eta) - 10u^{(3)}(\eta)u(\eta) + u^{(5)}(\eta) = 0. \tag{5}$$

Case 1. We suppose that the solution of Eq. (5) can be expressed as

$$u(\eta) = \frac{\sum_{i=-n}^{n} a_i\mathrm{e}^{i\omega(\eta)}}{\sum_{i=-m}^{m} b_i\mathrm{e}^{i\omega(\eta)}}, \tag{6}$$

where $\omega'(\eta) = c\mathrm{e}^{\omega} + ad\mathrm{e}^{-\omega}$. Eq. (6) can be re-written in an alternative form as follows:

$$u(\eta) = \frac{a_n\mathrm{e}^{n\mu} + \cdots + a_{-n}\mathrm{e}^{-n\mu}}{b_m\mathrm{e}^{m\mu} + \cdots + b_{-m}\mathrm{e}^{-m\mu}}$$

to determined values of m and n, we balance the linear term of highest order in Eq. (5) with the highest order nonlinear term. By simple calculation, we have

$$u^{(5)} = \frac{c_1\mathrm{e}^{(n+5m+5)\mu} + \cdots}{c_2\mathrm{e}^{6m\mu} + \cdots}$$

and

$$u^2u' = \frac{c_3\mathrm{e}^{(3n+m+1)\mu} + \cdots}{c_4\mathrm{e}^{4m\mu} + \cdots} = \frac{c_3\mathrm{e}^{(3n+3m+1)\mu} + \cdots}{c_4\mathrm{e}^{6m\mu} + \cdots}.$$

It is easy to find that $n = m + 2$ by balancing u^2u' with $u^{(5)}$.

Solutions for $m = 0$ and $\omega'(\eta) = c\mathrm{e}^{\omega} + ad\mathrm{e}^{-\omega}$:

$$u(\eta) = a_{-2}\mathrm{e}^{-2\omega(\eta)} + a_{-1}\mathrm{e}^{-\omega(\eta)} + a_0 + a_1\mathrm{e}^{\omega(\eta)} + a_2\mathrm{e}^{2\omega(\eta)}. \tag{7}$$

Substituting (7) into (5) and collecting the coefficients of $\mathrm{e}^{i\omega(\eta)}$ we obtain a system of algebraic equations for a_i and λ. By solving this system, we obtain the following solutions:

$$
\begin{aligned}
u_1 &= 2cd\bigl(3\coth^2\bigl[\sqrt{cd}\bigl(x+896c^2d^2t-c_1\bigr)\bigr] \\
&\qquad + 3\tanh^2\bigl[\sqrt{cd}\bigl(x+896c^2d^2t-c_1\bigr)\bigr]-2\bigr),\\
u_2 &= 2cd\bigl(3\cot^2\bigl[\sqrt{cd}\bigl(x+896c^2d^2t-c_1\bigr)\bigr] \\
&\qquad + 3\tan^2\bigl[\sqrt{cd}\bigl(x+896c^2d^2t-c_1\bigr)\bigr]+2\bigr).
\end{aligned}
$$

Solutions for $m=1$ and $\omega'(\eta)=c\mathrm{e}^{\omega}+ad\mathrm{e}^{-\omega}$:

$$
u(\eta)=\frac{a_{-3}\mathrm{e}^{-3\omega(\eta)}+a_{-2}\mathrm{e}^{-2\omega(\eta)}+\cdots+a_2\mathrm{e}^{2\omega(\eta)}+a_3\mathrm{e}^{3\omega(\eta)}}{b_{-1}\mathrm{e}^{-\omega(\eta)}+b_0+b_1\mathrm{e}^{\omega(\eta)}}. \tag{8}
$$

Substituting (8) into (5) and collecting the coefficients of $\mathrm{e}^{i\omega(\eta)}$ we obtain a system of algebraic equations for a_i, b_j and λ. By solving this system, we obtain the following solutions:

$$
\begin{aligned}
u_3 &= 2cd\bigl(1+3\,\mathrm{cosech}^2\bigl[\sqrt{cd}\bigl(x+336c^2d^2t-c_1\bigr)\bigr] \\
&\qquad + \tanh^2\bigl[\sqrt{cd}\bigl(x+336c^2d^2t-c_1\bigr)\bigr]\bigr),\\
u_4 &= 2cd\bigl(-2+3\,\mathrm{cosec}^2\bigl[\sqrt{cd}(x+336c^2d^2t-c_1)\bigr] \\
&\qquad + \sec^2\bigl[\sqrt{cd}\bigl(x+336c^2d^2t-c_1\bigr)\bigr]\bigr),\\
u_5 &= 2cd\bigl(2+\mathrm{cosech}^2\bigl[\sqrt{cd}(x+336c^2d^2t-c_1)\bigr] \\
&\qquad - 3\,\mathrm{sech}^2\bigl[\sqrt{cd}\bigl(x+336c^2d^2t-c_1\bigr)\bigr]\bigr).
\end{aligned}
$$

Case 2. We suppose that the solution of Eq. (5) can be expressed as

$$
u(\eta)=\frac{\sum_{i=-n}^{n} a_i\mathrm{e}^{i\omega(\eta)}}{\sum_{i=-m}^{m} b_i\mathrm{e}^{i\omega(\eta)}},
$$

where $\omega'(\eta)=a\sqrt{ad^2\mathrm{e}^{2\omega(\eta)}-acd}$. By balancing u^2u' with $u^{(5)}$:

$$
n=m+2.
$$

Solution for $m=0$ and $\omega'(\eta)=a\sqrt{ad^2\mathrm{e}^{2\omega(\eta)}-acd}$:

$$
u(\eta)=a_{-2}\mathrm{e}^{-2\omega(\eta)}+a_{-1}\mathrm{e}^{-\omega(\eta)}+a_0+a_1\mathrm{e}^{\omega(\eta)}+a_2\mathrm{e}^{2\omega(\eta)}. \tag{9}
$$

Substituting (9) into (5), and collecting the coefficients of $\mathrm{e}^{i\omega(\eta)}$ we obtain a system of algebraic equations for a_i and λ. By solving this system, we obtain the following solutions:

$$
u_6=2cd\bigl(\tan^2\bigl[\sqrt{cd}\bigl(x+2\bigl(8c^2d^2+20cda_0+15a_0^2\bigr)t-c_1\bigr)\bigr]\bigr).
$$

Solutions for $m = 1$ and $\omega'(\eta) = a\sqrt{ad^2\mathrm{e}^{2\omega(\eta)} - acd}$:

$$u(\eta) = \frac{a_{-3}\mathrm{e}^{-3\omega(\eta)} + a_{-2}\mathrm{e}^{-2\omega(\eta)} + \cdots + a_2\mathrm{e}^{2\omega(\eta)} + a_3\mathrm{e}^{3\omega(\eta)}}{b_{-1}\mathrm{e}^{-\omega(\eta)} + b_0 + b_1\mathrm{e}^{\omega(\eta)}}. \tag{10}$$

Substituting (10) into (5) and collecting the coefficients of $\mathrm{e}^{i\omega(\eta)}$ we obtain a system of algebraic equations for a_i, b_i and λ. By solving this system, we obtain the following solutions:

$$\begin{aligned}
u_7 &= \frac{a_{-1}}{b_{-1}}\left(1 - 3\,\mathrm{sech}^2\left[\frac{1}{\sqrt{2}}\frac{a_{-1}}{b_{-1}}\left(x + \frac{14a_{-1}^2}{b_{-1}^2}t - c_1\right)\right]\right),\\
u_8 &= \frac{4c^2b_1}{b_{-1}}\left(\mathrm{cosech}^2\left[4c\sqrt{\frac{b_1}{b_{-1}}}\left(x + \frac{896c^4b_1^2}{b_{-1}^2}t - c_1\right)\right] - 5\right)\\
&\quad\times \mathrm{sech}^2\left[2c\sqrt{\frac{b_1}{b_{-1}}}\left(x + \frac{896c^4b_1^2}{b_{-1}^2}t - c_1\right)\right],
\end{aligned}$$

where $b_1b_{-1} > 0$, $a_{-1}b_{-1} > 0$ and $b_{-1} \neq 0$.

Case 3. We suppose that the solution of Eq. (5) can be expressed as

$$u(\eta) = \frac{\sum_{i=-n}^{n} a_i\mathrm{e}^{i\omega(\eta)}}{\sum_{i=-m}^{m} b_i\mathrm{e}^{i\omega(\eta)}},$$

where $\omega'(\eta) = \sqrt{ad^2\mathrm{e}^{-2\omega(\eta)} - acd}$. It is easy to find that $n = m + 2$ by balancing u^2u' with $u^{(5)}$.

Solutions for $m = 1$ and $\omega'(\eta) = \sqrt{ad^2\mathrm{e}^{-2\omega(\eta)} - acd}$:

$$u(\eta) = \frac{a_{-1}\mathrm{e}^{-\omega(\eta)} + a_0 + a_1\mathrm{e}^{\omega(\eta)}}{b_{-1}\mathrm{e}^{-\omega(\eta)} + b_0 + b_1\mathrm{e}^{\omega(\eta)}}. \tag{11}$$

Substituting (11) into (5) and collecting the coefficients of $\mathrm{e}^{i\omega(\eta)}$ we obtain a system of algebraic equations for a_i, b_i and λ. By solving this system, we obtain the following solutions:

$$\begin{aligned}
u_9 &= \frac{cb_{-1}^2(7 - \sin[\frac{cb_{-1}}{b_0}(x + \frac{7c^4b_{-1}^4}{32b_0^4}t - c_1)])}{8b_0^2(1 + \sin[\frac{cb_{-1}}{b_0}(x + \frac{7c^4b_{-1}^4}{32b_0^4}t - c_1)])}, \quad b_0 \neq 0,\\
u_{10} &= \frac{c^2b_{-1}^2(1 - 7\,\mathrm{sech}[\frac{b_{-1}c}{b_0}(x + \frac{7c^4b_{-1}^4}{32b_0^4}t - c_1)])}{8b_0^2(1 + \mathrm{sech}[\frac{b_{-1}c}{b_0}(x + \frac{7c^4b_{-1}^4}{32b_0^4}t - c_1)])}, \quad b_0 \neq 0.
\end{aligned}$$

4 The system of the shallow water wave equation

We first consider the system of the shallow water wave equation in order to demonstrate the exponential transform method

$$\begin{aligned} &u_t + (uv)_x + v_{xxx} = 0, \\ &v_t + u_x + vv_x = 0. \end{aligned} \tag{12}$$

The wave variable $\eta = x - \lambda t$ carries Eq. (12) into the ODE

$$\begin{aligned} &-\lambda u' + vu' + uv' + v^{(3)} = 0, \\ &u' - \lambda v' + vv' = 0, \end{aligned} \tag{13}$$

where by integrating once the second equation and the constant integrations is zero, we find

$$u = \lambda v - \frac{v^2}{2}. \tag{14}$$

Substituting Eq. (14) into the first equation of Eq. (13) we obtain

$$v^{(3)}(\eta) + \left(3\lambda v(\eta) - \frac{3v(\eta)^2}{2} - \lambda^2\right)v'(\eta) = 0. \tag{15}$$

Case 1. We suppose that the solution of Eq. (15) can be expressed as

$$v(\eta) = \frac{\sum_{i=-n}^{n} a_i \mathrm{e}^{i\omega(\eta)}}{\sum_{i=-m}^{m} b_i \mathrm{e}^{i\omega(\eta)}}, \tag{16}$$

where $\omega'(\eta) = c\mathrm{e}^{\omega} + ad\mathrm{e}^{-\omega}$. Eq. (16) can be re-written in an alternative form as follows:

$$v(\eta) = \frac{a_n \mathrm{e}^{n\mu} + \cdots + a_{-n}\mathrm{e}^{-n\mu}}{b_m \mathrm{e}^{m\mu} + \cdots + b_{-m}\mathrm{e}^{-m\mu}}.$$

To determine values of m and n, we balance the linear term of higest order in Eq. (15) with the highest order nonlinear term. By simple calculation, we have

$$v^{(3)} = \frac{c_1 \mathrm{e}^{(n+3m+3)\mu} + \cdots}{c_2 \mathrm{e}^{4m\mu} + \cdots}, \quad v^2 v' = \frac{c_3 \mathrm{e}^{(3n+m+1)\mu} + \cdots}{c_4 \mathrm{e}^{4m\mu} + \cdots}.$$

By balancing $v^{(3)}$ with $v^2 v'$:

$$n = m + 1.$$

Solutions for $m = 0$ and $\omega'(\eta) = c\mathrm{e}^{\omega} + ad\mathrm{e}^{-\omega}$:

$$v(\eta) = a_{-1}\mathrm{e}^{-\omega(\eta)} + a_0 + a_1 \mathrm{e}^{\omega(\eta)}. \tag{17}$$

Substituting (17) into (15) and collecting the coefficients of $\mathrm{e}^{i\omega(\eta)}$ we obtain a system of algebraic equations for a_i and λ. By solving this system, we obtain the following solutions:

$$u_1 = -8cd\,\mathrm{cosech}^2\left[2\sqrt{cd}\big(x-4\sqrt{cd}t-c_1\big)\right],$$
$$v_1 = 2\sqrt{cd}\big(2-\coth\left[\sqrt{cd}\big(x-4\sqrt{cd}t-c_1\big)\right]-\tanh\left[\sqrt{cd}\big(x-4\sqrt{cd}t-c_1\big)\right]\big),$$
$$u_2 = -2cd\big(\tan^2\left[\sqrt{cd}\big(x-2\sqrt{2cd}t-c_1\big)\right]+\cot^2\left[\sqrt{cd}\big(x-2\sqrt{2cd}t-c_1\big)\right]\big),$$
$$v_2 = 2\sqrt{cd}\big(2\csc\left[2\sqrt{cd}\big(x-2\sqrt{2cd}t-c_1\big)\right]+\sqrt{2}\big).$$

Solutions for $m=1$ and $\omega'(\eta)=c\mathrm{e}^{\omega}+ad\mathrm{e}^{-\omega}$:

$$v(\eta)=\frac{a_{-2}\mathrm{e}^{-2\omega(\eta)}+a_{-1}\mathrm{e}^{-\omega(\eta)}+a_0+a_1\mathrm{e}^{\omega(\eta)}+a_2\mathrm{e}^{2\omega(\eta)}}{b_{-1}\mathrm{e}^{-\omega(\eta)}+b_0+b_1\mathrm{e}^{\omega(\eta)}}. \tag{18}$$

Substituting (18) into (15) and collecting the coefficients of $\mathrm{e}^{i\omega(\eta)}$ we obtain a system of algebraic equations for a_i, b_i and λ. By solving this system, we obtain the following solutions:

$$u_3 = -\frac{8ca_0}{3b_1}\,\mathrm{cosech}^2\left[\frac{2\sqrt{a_0c}(x-8\sqrt{cd}t-c_1)}{\sqrt{3b_1}}\right],$$
$$v_3 = -\frac{4\sqrt{a_0c}\,(\coth[\frac{\sqrt{a_0c}(x-8\sqrt{cd}t-c_1)}{\sqrt{3b_1}}]-1)^2}{\sqrt{3b_1}[\tanh[\frac{\sqrt{a_0c}\,(x-8\sqrt{cd}t-c_1)}{2\sqrt{3b_1}}]+\coth[\frac{\sqrt{a_0c}(x-8\sqrt{cd}t-c_1)}{2\sqrt{3b_1}}]]},$$

where $b_1\neq 0$,

$$u_4 = -\frac{a_0^2}{b_0^2}\left(\cosh\left[\frac{a_0(x+\frac{8db_0}{b_{-1}}t-c_1)}{b_0}\right]-1\right)^{-1},$$
$$v_4 = -\frac{a_0}{b_0}\left(\coth\left[\frac{a_0(x+\frac{8db_0}{b_{-1}}t-c_1)}{b_0}\right]+1\right),$$

where $\lambda=8cdb_0/(a_0-2db_1)$, $b_0\neq 0$, $M=2+\sqrt{3}+\sqrt{3+2\sqrt{3}}$ and $N=1+\sqrt{3}$.

Case 2. We suppose that the solution of Eq. (15) can be expressed as

$$v(\eta)=\frac{\sum_{i=-n}^{n}a_i\mathrm{e}^{i\omega(\eta)}}{\sum_{i=-m}^{m}b_i\mathrm{e}^{i\omega(\eta)}},$$

where $\omega'(\eta)=a\sqrt{ad^2\mathrm{e}^{2\omega(\eta)}-acd}$. By balancing $v^{(3)}$ with v^2v':

$$n=m+1.$$

Solution for $m=0$ and $\omega'(\eta)=a\sqrt{ad^2\mathrm{e}^{2\omega(\eta)}-acd}$:

$$u(\eta)=a_{-1}\mathrm{e}^{-\omega(\eta)}+a_0+a_1\mathrm{e}^{\omega(\eta)}. \tag{19}$$

Substituting (19) into (15) and collecting the coefficients of $\mathrm{e}^{i\omega(\eta)}$ we obtain a system of algebraic equations for a_i and λ. By solving this system, we obtain the following solutions:

$$u_5 = \frac{a_0^2}{2}\left(1 - 2\sec^2\left[\frac{a_0(x - a_0t - c_1)}{\sqrt{2}}\right]\right),$$
$$v_5 = a_0\left(1 + \sqrt{2}\sec^2\left[\frac{a_0(x - a_0t - c_1)}{\sqrt{2}}\right]\right).$$

Case 3. We suppose that the solution of Eq. (15) can be expressed as

$$v(\eta) = \frac{\sum_{i=-n}^{n} a_i \mathrm{e}^{i\omega(\eta)}}{\sum_{i=-m}^{m} b_i \mathrm{e}^{i\omega(\eta)}},$$

where $\omega'(\eta) = \sqrt{ad^2\mathrm{e}^{-2\omega(\eta)} - acd}$. By balancing $v^{(3)}$ with v^2v':

$$n = m.$$

Solutions for $m = 1$ and $\omega'(\eta) = \sqrt{ad^2\mathrm{e}^{-2\omega(\eta)} - acd}$:

$$u(\eta) = \frac{a_{-1}\mathrm{e}^{-\omega(\eta)} + a_0 + a_1\mathrm{e}^{\omega(\eta)}}{b_{-1}\mathrm{e}^{-\omega(\eta)} + b_0 + b_1\mathrm{e}^{\omega(\eta)}}. \tag{20}$$

Substituting (20) into (15) and collecting the coefficients of $\mathrm{e}^{i\omega(\eta)}$ we obtain a system of algebraic equations for a_i, b_i and λ. By solving this system, we obtain the following solutions:

$$u_6 = -\frac{a_{-1}c}{4b_0}\left(\csc^2\left[\frac{\sqrt{a_{-1}c}\,(x - \frac{a_1}{b_1}t - c_1)}{2\sqrt{2}\sqrt{b_0}}\right] + \sec^2\left[\frac{\sqrt{a_{-1}c}\,(x - \frac{a_1}{b_1}t - c_1)}{2\sqrt{2}\sqrt{b_0}}\right] - 2\right),$$
$$v_6 = \left(\sqrt{a_{-1}c}\,b_1\left(\sqrt{2}\sin\left[\frac{\sqrt{a_{-1}c}\,(x - \frac{a_1}{b_1}t - c_1)}{\sqrt{2}\sqrt{b_0}}\right] + 2\right)\right.$$
$$\left. + 2b_0^{3/2}c\left(\sqrt{2}\csc\left[\frac{\sqrt{a_{-1}c}\,(x - \frac{a_1}{b_1}t - c_1)}{\sqrt{2}\sqrt{b_0}}\right] + 1\right)\right)$$
$$\times\left(\sqrt{2}b_1\sqrt{b_0}\sin\left[\frac{\sqrt{a_{-1}c}\,(x - \frac{a_1}{b_1}t - c_1)}{\sqrt{2}\sqrt{b_0}}\right] + 2b_0^2\sqrt{a_{-1}c}\right)^{-1},$$

where $a_{-1}c > 0$, $b_0 > 0$.

5 Conclusions

The exp-transform method is proposed to obtain more general solutions of nonlinear evolution equations in mathematical physics. The generalized KdV equation and the system of the shallow water wave equation was used as a vehicle to achieve our goal. These exact solutions include hyperbolic function solution, trigonometric solution and exponential function solution. It was formally derived that the solutions leads to both solitary solutions and periodic solutions for the first case. The performance of the exp-transform method is reliable, effective. The applied method, with the aid of MATHEMATCA or MATLAB, can be easily extended to all kinds of nonlinear evolution equations in mathematical physics.

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