Global attractors for non-linear viscoelastic equation with strong damping*

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Abstract. In this paper, we consider the long-time dynamical behavior of the viscoelastic equations with strong damping and further prove the existence of global attractors for this system.

Keywords: attractors, viscoelastic, damping.

1 Introduction

In this paper, we discuss the long-time dynamical behavior for the nonlinear viscoelastic problem:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int\limits_0^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau - \gamma\Delta u_t = 0, \quad x \in \Omega, \tag{1}$$

together with initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x)$$

and boundary conditions

$$u(x,t) = 0, \quad x \in \partial\Omega,$$

where Ω is a bounded domain in R^n, $n \geqslant 1$, with a smooth boundary and $\rho, \gamma > 0$ are real numbers. Here, $u(x,t)$ represent displacement and g is a positive decaying function representing the kernel of memory term that will be specified below.

Problems relate to the equation

$$f(u_t)u_{tt} - \Delta u - \Delta u_{tt} = 0 \tag{2}$$

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are interesting not only from the point of view of PDE general theory, but also due to its applications in mechanics. For instance, when the material density, $f(u_t)$, is equal to 1, Eq. (2) describes the extensional vibrations of thin rods, see [1] for the physical details. When the material density $f(u_t)$ is not constant, we are dealing with a thin rod which possesses a rigid surface and whose interior is somehow permissive to slight deformations such that the material density varies according to the velocity. On the other hand, it is important to observe that similar equations to the one given in (2) arise in the study of viscoelastic plates with memory, more precisely

$$u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau = 0.$$

More recently, Cavalcanti et al. [2] considered this problem by assuming $0 < \rho \leqslant 2/(n-2)$ when $n \geqslant 3$, $\rho > 0$ when $n = 1, 2$ and g decays exponentially. They obtained the global existence result for $\gamma \geqslant 0$ and the uniform exponential decay of energy for $\gamma > 0$. Later, the decay result has been extended by Messaoudi and Tatar in [3] to the case $\gamma = 0$. Han and Wang [4] considered the following viscoelastic equation:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau + |u_t|^m u_t = 0, \quad x \in \Omega,$$

more recently, Ma [5] considered the attractors of this problem.

When there is no dispersion term, the related problems have been extensively studied and several results about existence, decay and blow-up been obtained. For instance, Cavalcanti et al. [6] deal with the equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau + a(x)u_t + |u|^\gamma u = 0, \quad x \in \Omega,\ t > 0, \tag{3}$$

with same boundary and initial conditions as that of system (1). Assuming that $a(x)$ is a nonnegative function that may vanish outside of a subset $\Omega_0 \subset \Omega$ of positive measure and g decays exponentially, they proved an exponential decay result of energy of (3). This result was later extended by Berrimi and Messaoudi [7] to the nonlinear damping case

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau + a(x)|u_t|^m u_t + b|u|^\gamma u = 0, \quad x \in \Omega,\ t > 0.$$

By introducing a new functional, they weakened the conditions on $a(x)$ and g and obtained the decay result. Motivated by the ideas of Messaoudi [8], the authors established the general uniform decay of the energy for this model.

The aim of this paper is to prove the existence of the global attractor for system (1).

2 Preliminaries

Assume that ρ satisfies

$$0 < \rho \leqslant \frac{2}{n-2} \quad \text{if } n \geqslant 3, \qquad \rho > 0 \quad \text{if } n = 1, 2. \tag{4}$$

For the kernel function g, we assume that it verifies:

(A1) $g : [0, \infty) \to (0, \infty)$ is a bounded C^1 function such that

$$1 - \int_0^\infty g(s)\,\mathrm{d}s = l > 0.$$

(A2) There exists a positive function $\xi(t)$ verifying

$$g'(t) \leqslant -\xi(t) g(t), \quad t \geqslant 0,$$

and

$$\xi(t) > 0, \quad \xi'(t) \leqslant 0, \quad \left|\frac{\xi'(t)}{\xi(t)}\right| \leqslant k, \quad t \geqslant 0, \qquad \int_0^\infty \xi(s)\,\mathrm{d}s = +\infty,$$

where k is a positive constant.

The energy associated with problem (1) can be written as

$$E(t) = \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t),$$

where

$$(g \circ v)(t) = \int_0^t g(t-s)\big\|v(t) - v(s)\big\|_2^2\,\mathrm{d}s.$$

Let us denote by $\mathcal{H}$ the Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and denote by $\|\cdot\|_{\mathcal{H}}$ the induced norm of $\mathcal{H}$. Let us introduce the history space

$$\mathcal{M} = L_g^2\big(0, +\infty; H_0^1(\Omega)\big)$$

which is a Hilbert space. Let us denote by $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ the operator given by

$$\mathcal{F}\eta = -\eta_s \quad \eta \in D(\mathcal{F}) = \big\{\eta \in \mathcal{M};\ \eta_s \in \mathcal{M} \text{ and } \eta(0) = 0\big\},$$

with

$$\langle \eta, \eta_1\rangle_{\mathcal{M}} = \int_0^\infty g(s)\big\langle \eta(s), \eta_1(s)\big\rangle \quad \forall \eta, \eta_1 \in D(\mathcal{F}).$$

Let us introduce the summed past history as

$$\eta^t(s) = \int_0^s u(t-y)\,\mathrm{d}y, \quad (t,s) \in R^+ \times R^+.$$

It is not difficult to see that η satisfies

$$\eta_t^t + \eta_s^t = u, \quad t \in R^+.$$

Moreover

$$\eta^0(s) = \int_0^s \psi(y)\,\mathrm{d}y, \quad s \in R^+, \qquad \eta^t(0) = 0 \quad \forall t \geqslant 0.$$

Denoting by $\Phi = (u, v, \eta^t)^{\mathrm{T}}$, $\Phi_0 = (u_0, v_0, \eta^0)^{\mathrm{T}}$, $v = u_t$, the semigroup formulation of the system (1) is given by

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = A\Phi, \quad \Phi(0) = \Phi_0,$$

where $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is defied as

$$A\left(u, \frac{1}{\rho+1}v^{\rho+1}, \eta\right) = \left(v,\ \Delta u + \Delta v_t - \int_0^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau + \gamma\Delta v,\ \eta_s\right)$$

with domain

$$D(A) = \bigl\{\Phi \in \mathcal{H};\ v \in H_0^1 \cap H^2,\ g * \Delta\eta \in L^2,\ \eta \in D(\mathcal{F})\bigr\},$$

where $\mathcal{H} = H_0^1 \cap H^2 \times H_0^1 \times \mathcal{M}$.

Firstly, we have the following result:

Lemma 1. *The operator A is the infinitesimal generator of the C_0-semigroup of contractions on $\mathcal{H}$ that we denote as $S(t)$.*

Proof. See, e.g., [9]. □

Lemma 2. *Let $u_0, u_1 \in H_0^1$ and $\gamma \geqslant 0$. Assume that the kernel g satisfies assumptions* (A1)*,* (A2)*. Then, problem* (1) *possesses at least a weak solution $u : \Omega \times (0,\infty) \to R$ in the class*

$$u \in L^\infty\bigl(0,\infty; H_0^1(\Omega)\bigr), \quad u' \in L^\infty\bigl(0,\infty; H_0^1(\Omega)\bigr), \quad u'' \in L^2\bigl(0,\infty; H_0^1(\Omega)\bigr).$$

Moreover, assuming that $\gamma > 0$, the energy determined by the solution u possesses the following decay:

$$E(t) \leqslant 3l^{-1}E(0)\mathrm{e}^{-(\varepsilon/2)C_2 t}, \quad t \geqslant 0,\ \ \varepsilon \in (0,\varepsilon_0],$$

where $C_2 = C_2(\rho, E(0), \|g\|_{L^1_{(0,\infty)}})$ and $\varepsilon_0 = \varepsilon_0(\rho, E(0), \|g\|_{L^1_{(0,\infty)}})$ are positive constants.

Proof. See, e.g., [2]. □

3 Proof of the main result

In this section we prove our main result. Eqs. (1) can be transformed into the system

$$u_t = v, \tag{5}$$

$$|v|^\rho v_t - l\Delta u - \Delta v_t + \int_0^t g(t-\tau)\Delta\eta(\tau)\,\mathrm{d}\tau - \gamma\Delta v = 0, \tag{6}$$

$$\eta_t^t = u - \eta_s^t \tag{7}$$

in $\Omega \times (0,\infty)$.

We shall consider the problem (5)–(7) in the following Hilbert space

$$\mathcal{H} = H_0^1 \cap H^2 \times H_0^1 \times \mathcal{M}.$$

Recall that the global attractor of $S(t)$ acting on $\mathcal{H}$ is a compact set $\mathcal{A} \subset \mathcal{H}$ enjoying the following properties:

(i) $\mathcal{A}$ is fully invariant for $S(t)$, that is, $S(t)\mathcal{A} = \mathcal{A}$ for every $t \geqslant 0$;

(ii) $\mathcal{A}$ is an attracting set, namely, for any bounded set $\mathcal{R} \subset \mathcal{H}$,

$$\lim_{t\to\infty} \delta_{\mathcal{H}}\big(S(t)\mathcal{R}, \mathcal{A}\big) = 0,$$

where $\delta_{\mathcal{H}}$ denotes the Hausdorff semi-distance on $\mathcal{H}$.

More details on the subject of global attractors can be found in the books [10, 11].

Remark 1. Lemma 2 implies that the existence of a bounded absorbing set $\mathcal{R}^* \subset \mathcal{H}$ for the C_0-semigroup $S(t)$. Indeed, if $\mathcal{R}^*$ is any absorbing ball of $\mathcal{H}$, then for any bounded set $\mathcal{R} \subset \mathcal{H}$, it is immediate to see that there exists $t(\mathcal{R}) \geqslant 0$ such that

$$S(t)\mathcal{R} \subset \mathcal{R}^*$$

for every $t \geqslant t(\mathcal{R})$.

Moreover, if we define

$$\mathcal{R}_0 = \bigcup_{t\geqslant 0} S(t)\mathcal{R}^*,$$

it is clear that $\mathcal{R}_0$ is still a bounded absorbing set which is also invariant for $S(t)$, that is, $S(t)\mathcal{R}_0 \subset \mathcal{R}_0$ for every $t \geqslant 0$.

In the sequel, we define the operator A as $Af = -\Delta f$. It is well known that A is a positive operator on L^2 with domain $\mathcal{D}(A) = H^2 \cap H_0^1$. Moreover, we can define the powers A^s of A for $s \in R$. The space $V_{2s} = \mathcal{D}(A^s)$ turns out to be a Hilbert space with the inner product

$$\langle u, v\rangle_{V_{2s}} = \big\langle A^s u, A^s v\big\rangle,$$

where $\langle\cdot\rangle$ stands for L^2-inner product on L^2.

In particular, $V_{-1} = H^{-1}$, $V_0 = L^2$, $V_1 = H_0^1$. The injection $V_{s_1} \hookrightarrow V_{s_2}$ is compact whenever $s_1 > s_2$. For further convenience, for $s \in R$, introduce the Hilbert space

$$\mathcal{H}_s = V_{1+s} \cap V_{2+s} \times V_{1+s} \times L_g^2\big(R^+, V_{1+s}\big).$$

Clearly, $\mathcal{H}_0 = \mathcal{H}$.

Now, let $\Phi_0 = (u_0, 1/(\rho+1)v_0^{\rho+1}, \eta^0) \in \mathcal{R}_0$, where $\mathcal{R}_0$ is the invariant, bounded absorbing set of $S(t)$ given by Remark 1, take the inner product in $\mathcal{H}_0$ of (5)–(7) and $(A^\sigma u, A^\sigma v, A^\sigma \eta^t)$ to get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho+2}\|u_t\|_\sigma^{\rho+2} + \frac{1}{2}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|u\|_{1+\sigma}^2 + \frac{1}{2}\|v\|_{1+\sigma}^2 + \frac{1}{2}\|\eta^t\|_{1+\sigma,g}\right)\\
&\quad = \frac{1}{2}\int_0^\infty g'(s)\big\|\eta^t(s)\big\|_{1+\sigma}^2\,\mathrm{d}s\,\mathrm{d}x - \gamma\|v\|_{1+\sigma}^2 \leqslant 0.
\end{aligned}$$

Here, the boundary term of integration by parts is neglected since we are working with more regular functions, similar application, please refer to [12].

Next, let

$$\begin{aligned}
E_1(t) &= \frac{1}{\rho+2}\|u_t\|_\sigma^{\rho+2} + \frac{1}{2}\left\|\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)u\right\|_{1+\sigma}^2 + \frac{1}{2}\|v\|_{1+\sigma}^2 + \frac{1}{2}\big\|\eta^t\big\|_{1+\sigma,g},\\
\Phi(t) &= \xi(t)\left\{\frac{1}{\rho+1}\int_\Omega A^\sigma |u_t|^\rho u_t u\,\mathrm{d}x + \int_\Omega A^\sigma \nabla u_t \nabla u\,\mathrm{d}x\right\},\\
\Psi(t) &= \xi(t)\int_\Omega A^\sigma\left(\Delta u_t - \frac{1}{\rho+1}|u_t|^\rho u_t\right)\int_0^t g(t-s)\big(u(t) - u(s)\big)\,\mathrm{d}s\,\mathrm{d}x,
\end{aligned}$$

and

$$F(t) = ME_1(t) + \varepsilon\Phi(t) + \Psi(t),$$

where M and ε are positive constants to be determined later.

By assumption (A2), we find that $\xi(t)$ is a positive non-increasing function, then $\xi(t) \leqslant \xi(0) = L$ for every $t \geqslant 0$. Repeating the similar arguments to those of the proof of Theorem 2.1 of Cavalcanti et al. [2], choosing our constants very carefully and properly, we get

$$F(t) \leqslant 3l^{-1}F(0)\mathrm{e}^{-(\varepsilon/2)C_2 t}, \quad t \geqslant 0,\ \ \varepsilon \in (0, \varepsilon_0]. \tag{8}$$

Finally (see Lemma 5.5 in [13]), we have the compact embedding

$$\mathcal{B}(t) = \bigcup_{\Phi_0 \in \mathcal{R}_0} \eta^t \hookrightarrow L_\mu^2\big(R^+, H_0^1\big), \tag{9}$$

where η^t is defined in (4). Denote the closure of $\mathcal{B}$ in $L^2_g(R^+, H^1_0)$ by $\bar{\mathcal{B}}$. With reference to (8) and (9) for $t \geqslant 0$, let $\mathcal{R}(t)$ be the ball of $V_{5/2} \times V_{3/2}$ and introduce the set

$$\mathcal{G}(t) = \mathcal{R}(t) \times \bar{\mathcal{B}} \subset \mathcal{H}.$$

From the compact embedding $V_{5/2} \times V_{1/2} \hookrightarrow H^1_0 \cap H^2 \times H^1_0$ and (9), $\mathcal{G}(t)$ is compact in $\mathcal{H}$. Then, due to the compactness of $\mathcal{G}(t)$ for every fixed $t \geqslant 0$ and every $d > 3l^{-1}F(0)\mathrm{e}^{-(\varepsilon/2)C_2 t}$, there exist finitely many balls of $\mathcal{H}$ of radius d such that $\Phi(t)$ belongs to the union of such balls for every $\Phi_0 \in \mathcal{R}_0$. This implies that

$$\alpha_{\mathcal{H}}(S(t)\mathcal{R}_0) \leqslant 3l^{-1}F(0)\mathrm{e}^{-(\varepsilon/2)C_2 t} \quad \forall t \geqslant 0, \tag{10}$$

where $\alpha_{\mathcal{H}}$ is the Kuratowski measure of non-compactness, defined by

$$\alpha_{\mathcal{H}}(\mathcal{R}) = \inf\{d\colon\ \mathcal{R} \text{ has a finite cover of balls of } \mathcal{H} \text{ of diameter less than } d\}.$$

Since the invariant, connected, bounded absorbing set $\mathcal{R}_0$ fulfills (10), exploiting a classical result of the theory of attractors of semigroups (see, e.g., [14]), we conclude that the ω-limit set of $\mathcal{R}_0$, that is,

$$\mathcal{A} \equiv \omega(\mathcal{R}_0) = \bigcap_{t \geqslant t_0} \overline{\bigcup_{s \geqslant t} S(s)\mathcal{R}_0},$$

is a connected and compact global attractor of $S(t)$. Therefore we have proved the following result.

Theorem 1. *Under assumptions of* (A1)–(A2)*, problem* (5)–(7) *possesses a unique global attractor* $\mathcal{A}$*.*

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