Stabilizing uncertain steady states of some dynamical systems by means of proportional feedback

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Abstract. A simple zero-order proportional feedback technique for stabilizing unknown fixed points is described. The technique employs either artificially created or natural stable fixed point to find the coordinates of the unknown unstable fixed point. Four physical examples have been investigated, namely the mechanical pendulum, the autonomous Duffing damped oscillator, the van der Pol oscillator, and the Lorenz system have been considered both analytically and numerically.

Keywords: control, steady states, proportional feedback.

1 Introduction

Stability of any either natural or artificial system is a valuable and desired property. Stabilization in particular of unstable fixed points (UFPs) of dynamical systems is an important problem in basic science and engineering applications, if periodic or chaotic oscillations are unacceptable. Usual control methods, based on proportional feedback [1–3] require knowledge of a mathematical model of a system or at least the exact location of the UFP in the phase space for the reference point. However, in many real complex systems, especially in biology, physiology, economics, sociology, chemistry neither the full reliable models, nor the exact coordinates of the UFPs are known. Moreover, the position of the UFP may slowly vary with time because of external unknown and unpredictable forces. Therefore, in some cases model-independent and reference-free methods, automatically tracing unknown UFP, can be helpful.

A number of adaptive, reference-free methods using either low-pass, high-pass or notch stable filters have been described in literature (many of the references can be found in [4, 5]). However, they can stabilize only unstable nodes and unstable spirals, but fail to stabilize the saddle-type UFPs, more specifically, the UFPs with an odd number of real positive eigenvalues. To solve the problem of the odd number limitation Pyragas et al. [6] proposed to use an unstable filter. It was a bold idea to fight an instability with another

instability. The technique has been demonstrated to stabilize saddle steady states in mathematical models [6–8] also in the experiments with an electrochemical oscillator [6, 7], and the Duffing-type electrical circuit [8]. However, this advanced method is limited to dissipative dynamical systems. It is not applicable to conservative systems. The limitation of the unstable filter method can be proved analytically using the Routh–Hurwitz stability criteria: the necessary condition for stabilizing a saddle UFP is that the dimensionless cut-off frequency of the unstable filter is lower than the dimensionless damping coefficient of the system [5, 8]. In conservative systems damping is zero under definition. Formally, the cut-off frequency could be set to negative value. However, this would mean that the unstable filter should become a stable one and, therefore, inappropriate to stabilize a saddle-type UFP. To get around the problem, a conjoint filter, that involves unstable and stable subfilters, has been suggested and demonstrated for the Lagrange point L2 of the Sun–Earth conservative astrodynamical system [5].

The control methods described in [4–8] are focused on designing complex higher order controllers with several adjustable control parameters. Even linear analysis of the stability properties employs high rank Hurwitz matrixes for determining the threshold values of the feedback coefficients, while finding optimal control parameters requires numerical solution of high order characteristic equations. Therefore the developed techniques are somewhat complicated for practical applications.

In this paper, we suggest simple zero-order stable proportional feedback technique, which employs either artificially created or natural stable fixed points (SFPs) to find unknown coordinates of the UFP.

To illustrate the idea, we start with extremely simple mathematical examples. A dynamical system given by
$$\dot{x} = ax - c$$
has a single unstable steady state $x_0 = c/a$, which can be easily stabilized by means of a proportional feedback:
$$\dot{x} = ax - c + k(x_0 - x),$$
provided $k > a$. Note, that the control term $k(x_0 - x)$ vanishes, when the goal steady state $x \to x_0$ is achieved.

However, when the system's dynamics is not fully defined, e.g. is described by
$$\dot{x} = ax - \xi \tag{1}$$
with ξ as an unknown term, the corresponding UFP, $x_0 = \xi/a$ is also unknown and therefore the proportional feedback cannot be applied directly. Nevertheless, we demonstrate that this unknown UFP can be still stabilized by the two-step proportional feedback. In the first step we apply proportional feedback with an arbitrarily chosen reference point r_1:
$$\dot{x} = ax - \xi + k(r_1 - x), \tag{2}$$
where r_1 is any real, either positive or negative (zero value is also applicable) constant. For $k > a$ the feedback creates an artificial SFP:
$$x_1 = \frac{kr_1 - \xi}{k - a}. \tag{3}$$

Note, that the control term $k(r_1 - x)$ in Eq. (2), in general, does not vanish, because the r_1 is not the natural UFP of the original Eq. (1). An exception is a "resonant" value $r_1 = x_0$. It means that we are lucky to guess the right reference point x_0 and the procedure is accomplished in one step. Otherwise the unknown term ξ should be derived from (3):

$$\xi = ax_1 + k(r_1 - x_1). \tag{4}$$

In the second step we simply replace r_1 in Eq. (2) with ξ/a found from (4):

$$\dot{x} = ax - \xi + k\left(\frac{\xi}{a} - x\right)$$

and readily stabilize the initially unknown UPF $x_0 = \xi/a$.

If a dynamical system has two fixed points, specifically an UFP and a SFP, the latter can be employed to find the position of the first one. In this case stabilization can be achieved in one step only, without creating an artificial SFP. The following nonlinear equation is an example:

$$\dot{x} = ax - x^2 - \xi. \tag{5}$$

For $\xi < a^2/4$ it has two real fixed points:

$$x_{01} = \frac{a}{2} - \sqrt{\frac{a^2}{4} - \xi}, \quad x_{02} = \frac{a}{2} + \sqrt{\frac{a^2}{4} - \xi}.$$

The x_{01} is an UFP, while x_{02} is a SFP. Note an important feature (independent on ξ):

$$x_{01} + x_{02} = a.$$

Thus, the natural SFP, x_{02} can be immediately used to find the UFP, $x_{01} = a - x_{02}$ and inserted in the feedback term:

$$\dot{x} = ax - x^2 - \xi + k(x_{01} - x).$$

Now we can generalize the above specific examples in the following form:

$$\dot{x} = F(x) - \xi, \tag{6}$$

where $F(x)$ is either linear or nonlinear function. Depending on $F(x)$ the Eq. (6) can have several fixed points, which satisfy the steady-state equation $F(x_{0i}) = \xi$. The fixed points are either UFPs or SFPs depending on the derivative of $F(x)$ with respect to x, the $F'(x)$ at $x = x_{0i}$. If $F'(x_{0i}) > 0$ the x_{0i} is an UFP, and if $F'(x_{0i}) < 0$ the x_{0i} is a SFP. We recall here that all the fixed points are unknown because of the unknown term ξ. Let us consider an UFP and apply the two-step procedure. The first step similarly to Eq. (2) is given by

$$\dot{x} = F(x) - \xi + k(r_1 - x). \tag{7}$$

The first step yields an artificial SFP x_1. The unknown term ξ is found from the steady-state case of Eq. (7):

$$\xi = F(x_1) + k(r_1 - x_1)$$

and then is inserted into the Eq. (6) for the uncontrolled system. Its steady-state case reads:

$$F(x_0) - F(x_1) - k(r_1 - x_1) = 0. \tag{8}$$

If the $F(x)$ is well defined the Eq. (8) can be solved with respect to x_0 and, finally, the second step is applied:
$$\dot{x} = F(x) - \xi + k(x_0 - x).$$

2 Mechanical pendulum

The first physical example is a mechanical pendulum given by

$$\ddot{\varphi} + \beta\dot{\varphi} + \sin\varphi = \xi. \tag{9}$$

In Eq. (9) φ is the angle between the downward vertical and the rod, β is the damping coefficient, and ξ is a constant, but generally unknown torque. For small torque $\xi < 1$, the system has two fixed points $(\varphi_{01,02}, \dot{\varphi}_{01,02}) = (\varphi_{01,02}, 0)$, where

$$\varphi_{01} = \arcsin\xi, \quad \varphi_{02} = \pi - \arcsin\xi.$$

The φ_{01} is a SFP (lower position of the pendulum), the φ_{02} is a saddle-type UFP (upper position of the pendulum). One can see that independently on ξ the sum of the two angles is a constant value:

$$\varphi_{01} + \varphi_{02} = \pi.$$

Thus we can apply a simplified one-step procedure, similarly to the first-order nonlinear mathematical example given by Eq. (5). Here we exploit the existing natural SFP of the pendulum to determine the position of the UFP, without creating any artificial SFP. The coordinate of the unknown UFP is readily obtained from the coordinate of the known (observed) SFP, $\varphi_{02} = \pi - \varphi_{01}$. Then we apply the proportional feedback:

$$\ddot{\varphi} + \beta\dot{\varphi} + \sin\varphi = \xi + k(\varphi_{02} - \varphi). \tag{10}$$

Linearization of Eq. (10) around φ_{02} gives the characteristic equation:

$$\lambda^2 + \beta\lambda + k + \cos(\pi - \varphi_{01}) = 0.$$

For small ξ the angle $\varphi_{01} \ll \pi$, thus $\lambda_{1,2} = -\beta/2 \pm \sqrt{\beta^2/4 - (k-1)}$. The threshold value of the feedback coefficient is $k_{th} = 1$ for which the largest eigenvalue λ_1 crosses zero from positive to negative values. The optimal value of the feedback coefficient $k_{opt} = 1 + \beta^2/4$; the eigenvalues are both negative and equal to each other, $\lambda_1 = \lambda_2 = -\beta/2$. Further increase of k makes the eigenvalues complex, but does not change their real parts. So, for higher feedback coefficients the convergence rate saturates with k and is fully determined by the damping coefficient β. Result of numerical integration of Eq. (10), shown in Fig. 1, demonstrates dynamics of stabilization (including transient process) of the saddle-type UFP.
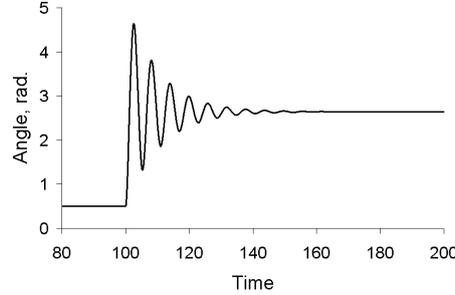
Fig. 1. One-step stabilization of the upper position of mechanical pendulum given by Eq. (10). The control is switched on at $t = 100$. The parameters are $\beta = 0.2$, $k = 2$. The stable angle observed before switching the control $\varphi_{01} = 0.5$, extracted unknown term $\xi = \sin \varphi_{01} = 0.47943$, stabilized UFP and angle calculated from the relationship $\varphi_{02} = \pi - \varphi_{01} = 2.64$.

3 Duffing damped oscillator

The second physical example is the Duffing nonlinear damped oscillator, which, in contrast to the classical Duffing system [9, 10], lacks external periodic driving force, but includes an unknown term ξ:

$$\ddot{x} + b\dot{x} - x + x^3 = \xi. \tag{11}$$

Here b is the damping coefficient. For $|\xi| < 2/\sqrt{27}$ Eq. (11) has three fixed points: $(x_{01,02,03}, \dot{x}_{01,02,03}) = (x_{01,02,03}, 0)$. The two side points are SFPs, while the middle one is a saddle-type UFP. Their coordinates, in general, are rather cumbersome:

$$x_{01} = -\frac{2}{\sqrt{3}} \cos \frac{\pi - \theta}{3}, \quad x_{02} = -\frac{2}{\sqrt{3}} \cos \frac{\pi + \theta}{3}, \quad x_{03} = \frac{2}{\sqrt{3}} \cos \frac{\theta}{3}, \tag{12}$$

where the formal parameter θ is given by

$$\theta = \arccos \frac{\xi \sqrt{27}}{2}. \tag{13}$$

While for $\xi = 0$ they become: $x_{01} = -1$, $x_{02} = 0$, $x_{03} = 1$. There is a simple relationship between the three coordinates:

$$x_{01} + x_{02} + x_{03} = 0,$$

which is valid for the non-zero ξ as well. Therefore one can think about the one-step algorithm ($x_{02} = -x_{01} - x_{03}$), similarly to the case of the pendulum. From a practical point of view the procedure is not convenient, since one needs to find (to observe) two remote SFPs, separated by an UFP. So, if a system is located at one of the SFP, say

x_{01}, we have to switch it to another SFP (x_{03}) by applying some rather strong external force. Formally, we can use only one SFP, either x_{01} or x_{03}. From the corresponding formulas (12), (13) we can extract ξ and to use it for finding x_{02}, again from the formulas (12), (13). However, this formal way requires rather long and complicated calculations. There is a shorter way. Indeed, the SFP x_{01} satisfies the steady-state equation:

$$x_{01}^3 - x_{01} - \xi = 0.$$

From here the unknown term ξ is readily derived as $\xi = x_{01}^3 - x_{01}$ and is used to calculate x_{02} from the appropriate formulas (12), (13). Finally, this coordinate is employed in the proportional feedback:

$$\ddot{x} + b\dot{x} - x + x^3 = \xi + k(x_{02} - x). \tag{14}$$

Linearization of Eq. (14) around x_{02} provides the characteristic equation:

$$\lambda^2 + b\lambda + k - 1 + 3x_{02}^2 = 0.$$

Its two eigenvalues are given by $\lambda_{1,2} = -b/2 \pm \sqrt{b^2/4 - (k - 1 + 3x_{02}^2)}$. For small ξ the coordinate of the UFP $|x_{02}| \ll 1$. Then stabilization parameters are the same as that for the pendulum: the threshold coefficient $k_{th} = 1$, the optimal value $k_{\mathrm{opt}} = 1 + b^2/4$, and the best pair of the real negative eigenvalues $\lambda_{1,2} = -b/2$. Numerical results for the Duffing oscillator obtained by integrating Eq. (14) are presented in Fig. 2.
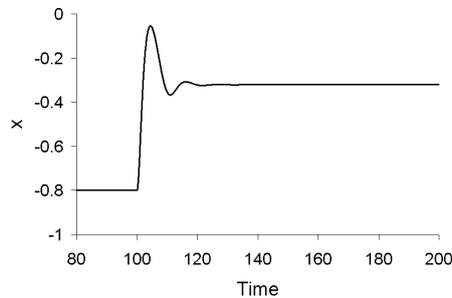

Fig. 2. One-step stabilization of the UFP of the Duffing oscillator given by Eq. (14). The control is switched on at $t = 100$. The parameters are $b = 0.5$, $k = 1.1$. SFP observed before switching the control $x_{01} = -0.8$, extracted unknown term $\xi = x_{01}^3 - x_{01} = 0.288$, stabilized UFP and coordinate calculated from the formula (25) $x_{02} = -0.321$.

4   Van der Pol oscillator

The next physical example is the well-known van der Pol oscillator, but with an additionally applied unknown force ξ:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = \xi. \tag{15}$$

Eq. (15) can be presented in the form of two coupled the 1st order equations:

$$\dot{x} = -y + \mu\left(x - \frac{x^3}{3}\right),$$
$$\dot{y} = x - \xi.$$

The van der Pol oscillator for any $\mu > 0$ and $|\xi| < 1$ has a single UFP (x_0, y_0):

$$x_0 = \xi, \quad y_0 = \mu\left(\xi - \frac{\xi^3}{3}\right), \tag{16}$$

which is either a spiral, if $\mu(1 - \xi^2) < 2$, or a node, if $\mu(1 - \xi^2) > 2$; however the both coordinates are unknown because of the unknown force ξ. In contrast to the two previous examples, there are no SFPs. Therefore, we need to apply the two-step stabilization technique:

$$\dot{x} = -y + \mu\left(x - \frac{x^3}{3}\right) + k_1(r_1 - x),$$
$$\dot{y} = x - \xi. \tag{17}$$

The proportional feedback with $k_1 > \mu(1 - \xi^2)$ creates an artificial SFP (x_1, y_1):

$$x_1 = \xi, \quad y_1 = \mu\left(x_1 - \frac{x_1^3}{3}\right) + k_1(r_1 - x_1).$$

The second coordinate y_1 is not important in this specific case, since the unknown parameter ξ is found immediately from the first coordinate x_1: $\xi = x_1$. Then, in the second step we simply replace the auxiliary reference point r_1 with the ξ, found in the first step:

$$\dot{x} = -y + \mu\left(x - \frac{x^3}{3}\right) + k_2(\xi - x),$$
$$\dot{y} = x - \xi \tag{18}$$

and stabilize the natural UFP (x_0, y_0), given by (16). Linearization around the steady state (x_0, y_0) yields the characteristic equation:

$$\lambda^2 + \left[k_2 - \mu\left(1 - x_0^2\right)\right]\lambda + 1 = 0.$$

For $x_0^2 \ll 1$ the $\lambda_{1,2} = -(k_2 - \mu)/2 \pm \sqrt{(k_2 - \mu)^2/4 - 1}$. Thus, the threshold feedback coefficient $k_{2th} = \mu$, when $\mathrm{Re}\,\lambda_1$ becomes negative. The optimal value is $k_{2\mathrm{opt}} = \mu + 2$, when the both eigenvalues are negative and equal to each other, $\lambda_1 = \lambda_2 = -1$.

The two-step technique applied to the van der Pol oscillator to stabilize the unknown UFP is illustrated by numerical results in Fig. 3.
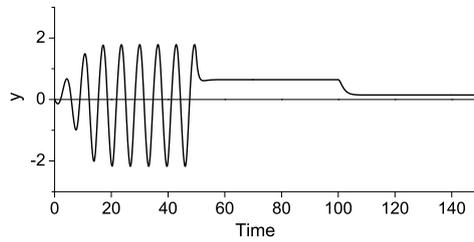
Fig. 3.   Two-step stabilization of the UFP of the van der Pol oscillator given by Eqs. (17), (18). The first step is switched on at $t = 50$, the second step is applied at $t = 100$. The parameters are $\mu = 0.5$, $k_1 = k_2 = 2.5$, $r_1 = 0.5$. The x-coordinate of the artificial SFP $x_1 = 0.3$, the extracted unknown force $\xi = 0.3$, the stabilized UFP $(x_0, y_0) = (0.3, 0.1455)$.

5   Lorenz system

Finally we consider the famous Lorenz system [9, 11], which for certain sets of the parameters exhibits chaotic behaviour. The system is given by three coupled differential equations:

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= \rho x - y - xz, \\ \dot{z} &= xy - \beta z. \end{aligned} \tag{19}$$

Two parameters are usually fixed at $\sigma = 10$ and $\beta = 8/3$, while the third parameter ρ is considered as a control parameter to observe various kinds of bifurcations. For $\rho < 1$ the system has a single SFP at the origin $(x_{01}, y_{01}, z_{01}) = (0, 0, 0)$. For $\rho > 1$ it looses stability and two additional SFPs $(x_{02,03}, y_{02,03}, z_{02,03})$ appear at

$$\left(\pm\sqrt{\beta(\rho - 1)},\, \pm\sqrt{\beta(\rho - 1)},\, \rho - 1 \right).$$

Linearization of Eq. (19) around these fixed points leads to the following characteristic equation:

$$\lambda^3 + (\sigma + \beta + 1)\lambda^2 + \beta(\sigma + \rho)\lambda + 2\sigma\beta(\rho - 1) = 0. \tag{20}$$

Using the Routh–Hurwitz criteria we find that this pair becomes unstable for

$$\rho > \sigma \frac{\sigma + \beta + 3}{\sigma - \beta - 1} \approx 24.74$$

giving rise to chaotic oscillations, e.g. at $\rho = 28$, which is the most popular parameter value used in literature [7, 9, 11].

Now we assume that the exact value of the parameter ρ is unknown, i.e. $\rho = \xi$. Consequently, the coordinates of the fixed points are also unknown:

$$\left(\pm\sqrt{\beta(\xi - 1)},\, \pm\sqrt{\beta(\xi - 1)},\, \xi - 1 \right). \tag{21}$$

Therefore, to stabilize the UFPs we apply the two-step procedure:

$$\begin{aligned}
\dot{x} &= -\sigma x + \sigma y, \\
\dot{y} &= \xi x - y - xz + k_1(r_1 - y), \\
\dot{z} &= xy - \beta z.
\end{aligned} \tag{22}$$

For simplicity we set $r_1 = 0$. The feedback term $-k_1 y$ creates a pair of artificial SFPs $(x_{12,13}, y_{12,13}, z_{12,13})$:

$$\left(\pm\sqrt{\beta(\xi - k_1 - 1)}, \pm\sqrt{\beta(\xi - k_1 - 1)}, \xi - k_1 - 1 \right).$$

We can extract the unknown parameter ξ from any coordinate of the artificial SFP, most conveniently from the z_{12}: $\xi = z_{12} + k_1 + 1$. The other coordinates of the natural UFP can be calculated as $(x_{02,03}, y_{02,03}) = (\pm\sqrt{\beta(\xi - 1)}, \pm\sqrt{\beta(\xi - 1)}) = (\pm\sqrt{\beta(z_{12} + k_1)},$ $\pm\sqrt{\beta(z_{12} + k_1)})$. Then the y-coordinates $y_{02,03}$ are inserted into Eq. (21) instead of the reference point r_1:

$$\begin{aligned}
\dot{x} &= -\sigma x + \sigma y, \\
\dot{y} &= \xi x - y - xz + k_2(y_0 - y), \\
\dot{z} &= xy - \beta z
\end{aligned} \tag{23}$$

to stabilize the UFPs $(x_{02,03}, y_{02,03}, z_{02,03})$. In Eq. (23) the y_0 denotes either y_{02} or y_{03}. Linearizing Eq. (23) about this fixed point we obtain the corresponding characteristic equation:

$$\begin{aligned}
&\lambda^3 + (\sigma + \beta + 1 + k_2)\lambda^2 + \left[\beta(\sigma + \xi) + k_2(\sigma + \beta) \right]\lambda \\
&+ 2\sigma\beta(\xi - 1) + k_2\sigma\beta = 0,
\end{aligned}$$

which for $k_2 = 0$ and $\xi = \rho$ coincides with Eq. (20), as expected. The Routh–Hurwitz criteria provide the following necessary and sufficient condition of stability of this point for $k > 0$:

$$\begin{aligned}
&\xi(k_2 - \sigma + \beta + 1) + \sigma(\sigma + \beta + 3) \\
&+ k_2\left(\frac{\sigma^2}{\beta} + 2\sigma + \frac{\sigma}{\beta} + \beta + 1 \right) + k_2^2\left(\frac{\sigma}{\beta} + 1 \right) > 0.
\end{aligned} \tag{24}$$

Let us consider the 1st term only in the inequality (24), since it contains a negative component $-\sigma$. If

$$k_2 > \sigma - \beta - 1 \approx 6.33, \tag{25}$$

then the fixed point is stable for all $\xi > 0$. This is a very rough estimation (the upper limit) of the stabilization threshold. However this threshold is conveniently independent on ξ. Taking into account the 2nd term in the inequality (24), we find that depending on ξ the stabilization can be achieved at essentially lower feedback coefficients

$$k_2 > \sigma - \beta - 1 - \sigma\frac{\sigma + \beta + 3}{\xi} \approx 0.73$$

at $\xi = 28$. However for very large ξ the threshold approaches the value given by condition (25). The 3rd term in the inequality (24) further diminishes the stabilization threshold, e.g. to $k_2 \approx 0.22$ at $\xi = 28$. The 4th term, which is quadratic with respect to k, for small k makes only very small correction.

The two-step stabilization of the UFP (x_{02}, y_{02}, z_{02}) in the Lorenz system is demonstrated in Fig. 4 for two slightly different initial conditions. Similar results are obtained for the UFP (x_{03}, y_{03}, z_{03}).
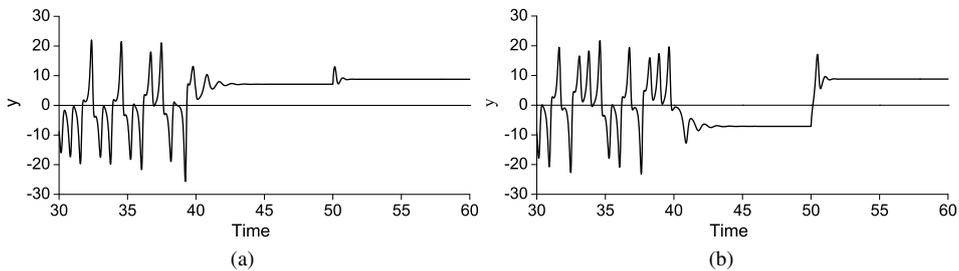

Fig. 4. Two-step stabilization of the spiral UFP in the Lorenz system given by Eqs. (22), (23). The first step is switched on at $t = 40$, the second step is applied at $t = 50$. The parameters are $\sigma = 10$, $\beta = 8/3$, $k_1 = k_2 = 10$, $r_1 = 0$. The z-coordinate of the artificial SFP $z_{12} = z_{13} = 19$, the extracted unknown parameter $\xi = z_{12} + k_1 + 1 = 30$, the stabilized UFP $(8.79, 8.79, 29.0)$. Initial conditions: (a) $x(0) = 0.10$, $y(0) = z(0) = 0$; (b) $x(0) = 0.11$, $y(0) = z(0) = 0$.

The same two-step method can be used to stabilize the saddle-type UFP at the origin $(0, 0, 0)$. The coordinates of the UFP are known (they do not depend on ξ) in this specific case. One may think that the proportional feedback method can be applied directly. However, the feedback coefficient k, required to make this UFP stable, essentially depends on the unknown parameter ξ. Linearizing Eq. (23) around the origin we obtain one negative eigenvalue immediately, $\lambda_3 = -\beta$, independent on k_2. Two remaining eigenvalues are easily found from the second order characteristic equation:

$$\lambda^2 + (\sigma + k_2 + 1)\lambda + \sigma(k_2 + 1 - \xi) = 0,$$

$\lambda_{1,2} = -(\sigma + k_2 + 1)/2 \pm \sqrt{(\sigma + k_2 + 1)^2/4 - \sigma(k_2 + 1 - \xi)}$. The both eigenvalues $\lambda_{1,2}$ are negative only if

$$k_2 > \xi - 1. \tag{26}$$

Here the parameter ξ is unknown. Therefore, it should be found from the first step, given by Eq. (22), and then used in condition (26) and in Eq. (23) (with $y_0 = y_{01} = 0$) to stabilize the UFP $(x_{01}, y_{01}, z_{01}) = (0, 0, 0)$.

Numerical results of stabilizing the saddle-type UFP are presented in Fig. 5, again for two different initial conditions.
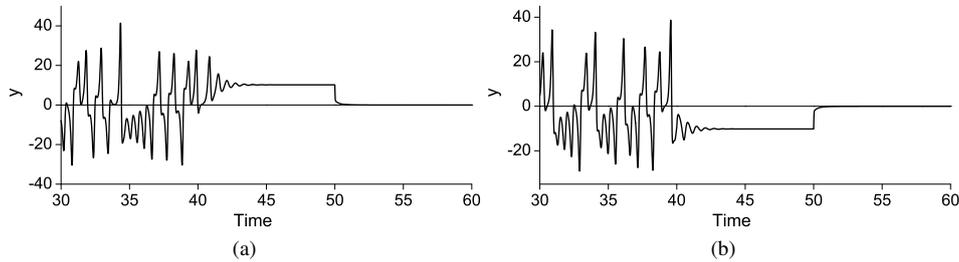
Fig. 5. Two-step stabilization of the saddle UFP in the Lorenz system given by Eqs. (22), (23). The first step is switched on at $t = 40$, the second step is applied at $t = 50$. The parameters are $\sigma = 10$, $\beta = 8/3$, $k_1 = 10$, $r_1 = 0$. The z-coordinate of the artificial SFP $z_{12} = z_{13} = 39$, the extracted unknown parameter $\xi = z_{12} + k_1 + 1 = 50$, $k_2 = 60$. The stabilized UFP $(0, 0, 0)$. Initial conditions: (a) $x(0) = 0.10$, $y(0) = z(0) = 0$; (b) $x(0) = 0.11$, $y(0) = z(0) = 0$.

6 Concluding remarks

We have suggested simple proportional feedback technique for stabilizing uncertain UFPs of dynamical systems. The method involves either one or two step algorithm of stabilization. It makes use of either natural or of artificially created SFPs in order to find the hidden coordinates of the UFP. Two simple mathematical examples have been presented and four different physical examples have been investigated. Specifically, the mechanical pendulum, the autonomous Duffing damped oscillator, the self-excited van der Pol oscillator, and the chaotic Lorenz system with either unknown external forces or unknown control parameters are considered analytically and numerically.

Moreover, different physical examples, presented here, demonstrate the performance of the method not only in the case of saddle-type UFPs, but also in the case of node/spiral UFPs as well. From this point of view the described technique has an advantage, compared to the methods employing stable [4,7] and unstable [5–8] tracking filters, which are suitable to stabilize either unstable nodes/spirals or saddles, respectively. In the case of stabilizing unstable spirals in the chaotic Lorenz system [7,12,13] the proposed technique is simpler, both mathematically and physically, than the conventional stable filter method [4,7,14–16].

The described proportional feedback method reminds the two-step control algorithm used to stabilize unstable periodic orbits (UPOs) by means of unstable delayed feedback controller [17]. However, in [17] the two-step technique was used simply to improve the global control performance, namely to enlarge the control domain of the UPOs in the parameter space. In the present paper the two-step algorithm is an essential procedure. It is necessary to find the unknown coordinates of the UFPs, in the case the system does not possess any natural SFP. The typical examples are the saddle-type Lagrange point L2 [5], the node/spiral UFP of the van der Pol oscillator, the saddle-type UFP at the origin $(0, 0, 0)$ and the two symmetric non-zero spiral-type UFPs in the chaotic Lorenz system.

In addition to the Lorenz system, we have checked the technique for some other autonomous chaotic systems, namely the Rossler equations [18], the Vilnius oscillator [19], and the autonomous Duffing–Holmes type oscillator [20].

Finally, we emphasize, that the control terms, $k(\varphi_{02} - \varphi)$, $k(x_{02} - x)$, $k_2(\xi - x)$, and $k_2(y_0 - y)$ in all the examples converge to zero, as the goal fixed points are stabilized, i.e. the stabilization is achieved with vanishing small perturbations. The feedback does not change the system, but changes its stability properties.

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