Multiple positive solutions to mixed boundary value problems for singular ordinary differential equations on the whole line*

Yuji Liu

Department of Mathematics
Guangdong University of Business Studies
Guangzhou 510320, China
liuyuji888@sohu.com

Abstract. This paper is concerned with the mixed boundary value problem of the second order singular ordinary differential equation

$$\begin{cases} [\Phi(\rho(t)x'(t))]' + f(t,x(t),x'(t)) = 0, \quad t \in R, \\ \lim_{t\to-\infty} x(t) = \int_{-\infty}^{+\infty} g(s,x(s),x'(s))\,\mathrm{d}s, \\ \lim_{t\to+\infty} \rho(t)x'(t) = \int_{-\infty}^{+\infty} h(s,x(s),x'(s))\,\mathrm{d}s. \end{cases}$$

Sufficient conditions to guarantee the existence of at least one positive solution are established. The emphasis is put on the one-dimensional p-Laplacian term $[\Phi(\rho(t)x'(t))]'$ involved with the nonnegative function ρ satisfying $\int_{-\infty}^{+\infty} 1/\rho(s)\,\mathrm{d}s = +\infty$.

Keywords: Second order differential equation with quasi-Laplacian on the whole line, integral type boundary value problem, positive solution, fixed point theorem.

1 Introduction

The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev [1], motivated by the work of Bitsadze and Samarskii on nonlocal linear elliptic boundary problems [2]. Since then, more general nonlinear multi-point boundary value problems (BVPs) were studied by several authors, see the text books [2–4] and the survey papers [5,6] and the references cited therein.

In [7], a class of boundary value problems for the second order nonlinear ordinary differential equations on the whole line is studied. Two theorems are proved. The first theorem is established by the use of the Schauder theorem and concerns the existence of solutions, while the second theorem is concerned with the existence and uniqueness of

solutions and is derived by the Banach contraction principle. To the author's knowledge, there is no paper concerned with the existence of positive solutions of the boundary value problems to the second order differential equations

$$\big[\rho(t)\Phi\big(x'(t)\big)\big]' + f\big(t, x(t), x'(t)\big) = 0, \quad t \in (-\infty, +\infty) =: R.$$

In [8], Bianconi and Papalini investigate the existence of solutions of the following boundary value problem:

$$\begin{cases} [\Phi(x'(t))]' + a(t, x(t))b(x(t), x'(t)) = 0, \quad t \in R, \\ \lim_{t \to -\infty} x(t) =: x(-\infty) = 0, \\ \lim_{t \to +\infty} x(t) =: x(+\infty) = 1, \end{cases} \tag{1}$$

where Φ is a monotone function which generalizes the one-dimensional p-Laplacian operator. Some criteria for the existence and non-existence of solutions of BVP (1) are established.

In [9, 10], Avramescu and Vladimirescu study the following boundary value problem:

$$\begin{cases} x''(t) + 2f(t)x'(t) + x(t) + g(t, x(t)) = 0, \quad t \in R, \\ \lim_{t \to \pm\infty} x(t) =: x(\pm\infty) = 0, \\ \lim_{t \to \pm\infty} x'(t) =: x'(\pm\infty) = 0, \end{cases} \tag{2}$$

where f and g are given functions. The existence of solutions of BVP (2) is obtained.

In [11], Avramescu and Vladimirescu study the following boundary value problem:

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, \quad t \in R, \\ \lim_{t \to -\infty} x(t) = \lim_{t \to +\infty} x(t), \\ \lim_{t \to -\infty} x'(t) = \lim_{t \to +\infty} x'(t), \end{cases} \tag{3}$$

under adequate hypothesis and using the Bohnenblust–Karlin fixed point theorem, the existence of solutions of BVP (3) is established.

Cabada and Cid [12] prove the solvability of the boundary value problem on the whole line

$$\begin{cases} [\Phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in R, \\ \lim_{t \to -\infty} x(t) = -1, \\ \lim_{t \to +\infty} x(t) = 1, \end{cases} \tag{4}$$

where f is a continuous function, $\Phi : (-a, a) \to R$ is a homeomorphism with $a \in (0, +\infty)$, i.e., Φ is singular.

Calamai [13] and Marcelli, Papalini [14] discuss the solvability of the following strongly nonlinear BVP:

$$\begin{cases} [a(x(t))\Phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in R, \\ \lim_{t \to -\infty} x(t) = \alpha, \\ \lim_{t \to +\infty} x(t) = \beta, \end{cases} \tag{5}$$

where $\alpha < \beta$, Φ is a general increasing homeomorphism with bounded domain (singular Φ-Laplacian), a is a positive, continuous function and f is a Carathéodory nonlinear function. Conditions for the existence and non-existence of heteroclinic solutions are given in terms of the behavior of $y \to f(t, x, y)$ and $y \to \Phi(y)$ as $y \to 0$, and of $t \to f(t, x, y)$ as $|t| \to +\infty$. The approach is based on fixed point techniques suitably combined to the method of upper and lower solutions.

Motivated by mentioned papers, we consider the mixed boundary value problem for second order singular differential equation on the whole line with quasi-Laplacian operator

$$\begin{cases} [\Phi(\rho(t)x'(t))]' + f(t, x(t), x'(t)) = 0, & t \in R, \\ \lim_{t \to -\infty} x(t) = \int_{-\infty}^{+\infty} g(s, x(s), x'(s)) \, \mathrm{d}s, \\ \lim_{t \to +\infty} \rho(t)x'(t) = \int_{-\infty}^{+\infty} h(s, x(s), x'(s)) \, \mathrm{d}s, \end{cases} \tag{6}$$

where

- f, g, h defined on $R \times [0, +\infty) \times [0, +\infty)$ are nonnegative Carathéodory functions,
- $\rho \in C^0(R, [0, \infty))$ with $\rho(t) > 0$ for all $t \neq 0$ satisfying conditions:

$$\int\limits_{-\infty}^{0} \frac{1}{\rho(s)} \, \mathrm{d}s < +\infty, \qquad \int\limits_{0}^{+\infty} \frac{1}{\rho(s)} \, \mathrm{d}s = +\infty.$$

- $\Phi : R \to R$ is called a quasi-Laplacian if it is nonsingular (i.e., surjective), and satisfies $\Phi \in C^1(R)$ with $\Phi'(y) > 0$ for all $y \neq 0$, $\Phi(0) = 0$ and its inverse function denoted by Φ^{-1}.

The purpose is to establish sufficient conditions for the existence of at least one positive solution of BVP (6). The results in this paper generalize and improve some known ones since the quasi-Laplacian term $[\Phi(\rho(t)x'(t))]'$ involved with the nonnegative function ρ that may satisfy $\rho(0) = 0$. It is easy to see that p-Laplacian $\phi_p(x) = |x|^{p-2}x$ with $p > 1$ is a quasi-Laplacian and $\Phi(x) = x^3/(1 + x^2)$ is also a quasi-Laplacian. BVP (6) is called a mixed boundary value problem.

The remainder of this paper is organized as follows: the preliminary results are given in Section 2, the main results are presented in Section 3.

2 Preliminary results

In this section, we present some background definitions in Banach spaces, state an important fixed point theorem and given some preliminary results.

Definition 1. Let X be a real Banach space. A nonempty convex closed subset P of X is called a cone in X if $ax \in P$ for all $x \in P$ and $a \geq 0$ and $x \in X$ and $-x \in X$ imply $x = 0$.

Definition 2. A map $\alpha : P \to [0, +\infty)$ is a nonnegative continuous convex functional map provided α is nonnegative, continuous and satisfies

$$\alpha\big(tx + (1-t)y\big) \le t\alpha(x) + (1-t)\alpha(y),$$

for all $x, y \in P$ and $t \in [0, 1]$.

Definition 3. An operator $T : X \to X$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Definition 4. Let $\tau(t) = \int_{-\infty}^{t} 1/\rho(s)\,\mathrm{d}s$. A map $F : R \times R \times R \to R$ is called a Carathéodory function if it satisfies the following properties:

 (i) $t \to f(t, (1 + \tau(t))x, 1/\rho(t))y)$ is measurable for any $x, y \in R$;
 (ii) $(x, y) \to f(t, (1 + \tau(t))x, 1/\rho(t))y)$ is continuous for a.e. $t \in R$;
 (iii) for each $r > 0$, there exists nonnegative function $\phi_r \in L^1(R)$ such that $|x|, |y| \le r$ implies

$$\left| \left(t, \big(1 + \tau(t)\big)x, \frac{1}{\rho(t)}y\right) \right| \le \phi_r(t), \quad \text{a.e. } t \in R.$$

Lemma 1. (*See* [4]). *Let X be a real Banach space, P be a cone of X, Ω_1, Ω_2 be two nonempty bounded open sets of P with $0 \in \Omega_1 \subseteq \overline{\Omega_1} \subseteq \Omega_2$. Suppose that $T : \overline{\Omega_2} \to P$ is a completely continuous operator, and that*:

 (E1) $Tx \ne \lambda x$ *for all $\lambda \in [0, 1)$ and $x \in \partial\Omega_1$;*
 (E2) $Tx \ne \lambda x$ *for all $\lambda \in (1, +\infty)$ and $x \in \partial\Omega_2$;*
 (E3) $\inf\{\|Tx\| \colon x \in \partial\Omega_1\} > 0$.

Then T has at least one fixed point $x \in \overline{\Omega_2} \setminus \Omega_1$.

Let $\tau = \tau(t) = \int_{-\infty}^{t} 1/\rho(s)\,\mathrm{d}s$. Choose

$$X = \left\{ x : R \to R \colon x \in C^0(R),\ \rho x' \in C^0(R) \text{ and the limits exist and are finite} \right.$$

$$\left. \lim_{t \to -\infty} \frac{x(t)}{1 + \tau(t)},\ \lim_{t \to +\infty} \frac{x(t)}{1 + \tau(t)},\ \lim_{t \to -\infty} \rho(t)x'(t),\ \lim_{t \to +\infty} \rho(t)x'(t) \right\}.$$

For $x \in X$, define the norm of x by

$$\|x\| = \max\left\{ \sup_{t \in R} \frac{|x(t)|}{1 + \tau(t)},\ \sup_{t \in R} \rho(t)\big|x'(t)\big| \right\}.$$

One can prove that X is a Banach space with the norm $\|x\|$ for $x \in X$.

Choose $k > 0$ such that $\int_{-\infty}^{-k} 1/\rho(s)\,\mathrm{d}s < 1$. Denote

$$\mu = \frac{\int_{-\infty}^{-k} \frac{1}{\rho(s)}\,\mathrm{d}s}{1 + \int_{-\infty}^{k} \frac{1}{\rho(s)}\,\mathrm{d}s}.$$

Let

$$P = \left\{ x \in X\colon x(t) \geq 0,\ x'(t) \geq 0 \text{ for all } t \in R, \right.$$
$$\left. \min_{t \in [-k,k]} \frac{x(t)}{(1 + \tau(t))} \geq \mu \sup_{t \in R} \frac{x(t)}{(1 + \tau(t))} \right\}.$$

Define the operator T on X by

$$(Tx)(t) = \int_{-\infty}^{+\infty} g\big(r, x(r), x'(r)\big)\,\mathrm{d}r$$
$$+ \int_{-\infty}^{t} \frac{1}{\rho(s)} \Phi^{-1}\left(\Phi\left(\int_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\,\mathrm{d}u \right) + \int_{s}^{+\infty} f\big(u, x(u), x'(u)\big)\,\mathrm{d}u \right)\mathrm{d}s$$

for $x \in X$.

Lemma 2. *The following properties hold*:

(i) *for $x \in X$, it holds that*

$$\begin{cases} [\Phi(\rho(t)(Tx)'(t))]' + f(t, x(t), x'(t)) = 0, & t \in R, \\ \lim_{t \to -\infty} (Tx)(t) = \int_{-\infty}^{+\infty} g(u, x(u), x'(u))\,\mathrm{d}u, \\ \lim_{t \to +\infty} \rho(t)(Tx)'(t) = \int_{-\infty}^{+\infty} h(s, x(s), x'(s))\,\mathrm{d}s; \end{cases} \tag{7}$$

(ii) $T : P \to P$ *is well defined*;

(iii) $T : P \to P$ *is completely continuous*;

(iv) $x \in P$ *is a positive solution of BVP* (6) *if and only if x is a fixed point of T in P.*

Proof. (i) Since $x \in X$, f, g, h are Carathéodory functions, then

$$\|x\| = \max\left\{ \sup_{t \in R} \frac{|x(t)|}{1 + \tau(t)},\ \sup_{t \in R} \rho(t)\big|x'(t)\big| \right\} = r < +\infty, \tag{8}$$

and

$$\int_{-\infty}^{+\infty} f\big(u, x(u), x'(u)\big)\,\mathrm{d}u,\quad \int_{-\infty}^{+\infty} g\Big(u, x(u), x'(u)\Big)\,\mathrm{d}u,\quad \int_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\,\mathrm{d}u \tag{9}$$

converge. Then $(Tx)(t)$ is defined for all $t \in R$. From the definition of T, by direct computation, we get (7).

(ii) For $x \in P \subset X$, since f, g, h are Carathéodory functions, then (8) holds and the integrals in (9) converge. From $\lim_{t \to -\infty} \tau(t) = 0$, then $Tx \in C^0(R)$ and there exist the limits:

$$\lim_{t \to -\infty} \frac{(Tx)(t)}{1 + \tau(t)} = \int\limits_{-\infty}^{+\infty} g\big(u, x(u), x'(u)\big) \, \mathrm{d}u,$$

$$\lim_{t \to +\infty} \frac{(Tx)(t)}{1 + \tau(t)} = \lim_{t \to +\infty} \rho(t)(Tx)'(t) = \int\limits_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big) \, \mathrm{d}u,$$

and

$$\rho(t)(Tx)'(t)$$
$$= \Phi^{-1}\bigg(\Phi\bigg(\int\limits_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big) \, \mathrm{d}u\bigg) + \int\limits_{t}^{+\infty} f\big(u, x(u), x'(u)\big) \, \mathrm{d}u\bigg). \qquad (10)$$

It is easy to see that then $t \to \rho(t)(Tx)'$ is continuous on R and there exist the limits

$$\lim_{t \to -\infty} \rho(t)(Tx)'(t)$$
$$= \Phi^{-1}\bigg(\Phi\bigg(\int\limits_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big) \, \mathrm{d}u\bigg) + \int\limits_{-\infty}^{+\infty} f\big(u, x(u), x'(u)\big) \, \mathrm{d}u\bigg),$$

$$\lim_{t \to +\infty} \rho(t)(Tx)'(t) = \int\limits_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big) \, \mathrm{d}u.$$

It follows that $Tx \in X$. Since ρ, f, g, and h are nonnegative, then

$$Tx(t) \geq 0 \quad \text{for all } t \in R. \qquad (11)$$

Now, we prove that $(Tx)(t)$ is concave with respect to

$$\tau = \tau(t) = \int\limits_{-\infty}^{t} \frac{1}{\rho(s)} \, \mathrm{d}s.$$

It is easy to see that $\tau \in C(R, (0, +\infty))$ and

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{\rho(t)} > 0.$$

Thus

$$\frac{\mathrm{d}(Tx)}{\mathrm{d}t} = \frac{\mathrm{d}(Tx)}{\mathrm{d}\tau} \frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathrm{d}(Tx)}{\mathrm{d}\tau} \frac{1}{\rho(t)}. \qquad (12)$$

It follows that

$$\Phi\left(\rho(t)\frac{\mathrm{d}(Tx)}{\mathrm{d}t}\right) = \Phi\left(\frac{\mathrm{d}(Tx)}{\mathrm{d}\tau}\right).$$

Hence

$$\left[\Phi\left(\rho(t)\frac{\mathrm{d}(Tx)}{\mathrm{d}t}\right)\right]' = \Phi'\left(\frac{\mathrm{d}(Tx)}{\mathrm{d}\tau}\right)\frac{\mathrm{d}^2(Tx)}{\mathrm{d}\tau^2}\frac{\mathrm{d}\tau}{\mathrm{d}t}.$$

So

$$\frac{\mathrm{d}^2(Tx)}{\mathrm{d}\tau^2} = \frac{[\Phi(\rho(t)\frac{\mathrm{d}(Tx)}{\mathrm{d}t})]'}{\Phi'(\frac{\mathrm{d}(Tx)}{\mathrm{d}\tau})\frac{\mathrm{d}\tau}{\mathrm{d}t}}.$$

Since $[\Phi(\rho(t)(Tx)'(t))]' \le 0$, $\Phi'(y) > 0$ ($y \ne 0$) and $\mathrm{d}\tau/\mathrm{d}t > 0$, we get that $\mathrm{d}^2(Tx)/\mathrm{d}\tau^2 \le 0$. Hence $(Tx)(t)$ is concave with respect to τ.

We need to prove that

$$\min_{t\in[-k,k]} \frac{(Tx)(t)}{1+\tau(t)} \ge \mu \sup_{t\in R} \frac{(Tx)(t)}{1+\tau(t)}. \tag{13}$$

Since $\mathrm{d}\tau/\mathrm{d}t > 0$ for all $t \in R$, there exists the inverse function of $\tau = \tau(t)$. Denote the inverse function of $\tau = \tau(t)$ by $t = t(\tau)$.

Suppose that there exists $t_0 \in R$ such that $\sup_{t\in R}(Tx)(t)/(1+\tau(t)) = (Tx)(t_0)/(1+\tau(t_0))$. One sees

$$\lim_{t\to+\infty}\rho(t)(Tx)'(t) = \int_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\,\mathrm{d}u \ge 0.$$

Since

$$\big[\Phi\big(\rho(t)(Tx)'(t)\big)\big]' = -f\big(t, x(t), x'(t)\big) \le 0,$$

then $\rho(t)(Tx)'(t) \ge 0$ for all $t \in R$. Hence $(Tx)(t)$ is increasing on R. It is easy to show that $(Tx)(t) \ge 0$ for all $t \in R$. Then

$$\min_{t\in[-k,k]} \frac{(Tx)(t)}{1+\tau(t)}$$

$$\ge \frac{(Tx)(-k)}{1+\tau(k)} = \frac{1}{1+\tau(k)}(Tx)\big(t(\tau(-k))\big)$$

$$= \frac{1}{1+\tau(k)}(Tx)\left(t\left(\frac{1+\tau(t_0)-\tau(-k)}{1+\tau(t_0)}\frac{\tau(-k)}{1+\tau(t_0)-\tau(-k)} + \frac{\tau(-k)}{1+\tau(t_0)}\tau(t_0)\right)\right)$$

$$\ge \frac{1}{1+\tau(k)}\left[\frac{1+\tau(t_0)-\tau(-k)}{1+\tau(t_0)}(Tx)\left(t\left(\frac{\tau(-k)}{1+\tau(t_0)-\tau(-k)}\right)\right)\right.$$

$$\left. + \frac{\tau(-k)}{1+\tau(t_0)}(Tx)\big(t(\tau(t_0))\big)\right]$$

$$\ge \frac{1}{1+\tau(k)}\int_{-\infty}^{-k}\frac{1}{\rho(s)}\,\mathrm{d}s\frac{1}{1+\tau(t_0)}(Tx)(t_0) > \mu\sup_{t\in R}\frac{(Tx)(t)}{1+\tau(t)}.$$

We get (13). If $\sup_{t\in R}(Tx)(t)/(1+\tau(t)) = \lim_{t\to\pm\infty}(Tx)(t)/(1+\tau(t))$, we choose $t_0 \in R$. Similarly to above discussion, we get

$$\min_{t\in[-k,k]} \frac{(Tx)(t)}{1+\tau(t)} \geq \mu\frac{(Tx)(t_0)}{1+\tau(t_0)}.$$

Let $t \to \pm\infty$, we get (13).

From (11) and (13), we see $Tx \in P$. Hence $T : P \to P$ is well defined.

(iii) Now we prove that T is completely continuous. The following five steps are needed.

Step 1. We prove that the function $T : X \to X$ is continuous.

Let $\{x_n\} \in X$ with $x_n \to x_0$ as $n \to \infty$. Then

$$\sup_{n=0,1,2,\ldots} \|x_n\| = \sup_{n=0,1,2,\ldots} \max\left\{\sup_{t\in R} \frac{|(Tx_n)(t)|}{1+\tau(t)}, \sup_{t\in R}\rho(t)\big|(Tx_n)'(t)\big|\right\} =: r < +\infty.$$

Hence there exists $\phi_r \in L^1(R)$ such that

$$0 \leq f\big(t, x_n(t), x_n'(t)\big) = f\left(t, \big(1+\tau(t)\big)\frac{x_n(t)}{1+\tau(t)}, \frac{1}{\rho(t)}\rho(t)x_n'(t)\right) \leq \phi_r(t),$$

$$0 \leq g\big(t, x_n(t), x_n'(t)\big) = g\left(t, \big(1+\tau(t)\big)\frac{x_n(t)}{1+\tau(t)}, \frac{1}{\rho(t)}\rho(t)x_n'(t)\right) \leq \phi_r(t),$$

$$0 \leq h\big(t, x_n(t), x_n'(t)\big) = h\left(t, \big(1+\tau(t)\big)\frac{x_n(t)}{1+\tau(t)}, \frac{1}{\rho(t)}\rho(t)x_n'(t)\right) \leq \phi_r(t).$$

From

$$(Tx_n)(t)$$
$$= \int_{-\infty}^{+\infty} g\big(u, x_n(u), x_n'(u)\big)\,\mathrm{d}u$$
$$+ \int_{-\infty}^{t} \frac{1}{\rho(s)}\Phi^{-1}\left(\Phi\left(\int_{-\infty}^{+\infty} h\big(u, x_n(u), x_n'(u)\big)\,\mathrm{d}u\right) + \int_{s}^{+\infty} f\big(u, x_n(u), x_n'(u)\big)\,\mathrm{d}u\right)\mathrm{d}s,$$

for $n = 0, 1, 2, \ldots$, we need to prove that $Tx_n \to Tx_0$ as $n \to +\infty$. One sees that

$$\left|\frac{(Tx_n)(t)}{1+\tau(t)} - \frac{(Tx_0)(t)}{1+\tau(t)}\right|$$
$$\leq \frac{|\int_{-\infty}^{+\infty} g(u, x_n(u), x_n'(u))\,\mathrm{d}u - \int_{-\infty}^{+\infty} g(u, x_0(u), x_0'(u))\,\mathrm{d}u|}{1+\tau(t)}$$
$$+ \left|\frac{\int_{-\infty}^{t} \frac{1}{\rho(s)}\Phi^{-1}(\Phi(\int_{-\infty}^{+\infty}h(u, x_n(u), x_n'(u))\,\mathrm{d}u) + \int_{s}^{+\infty}f(u, x_n(u), x_n'(u))\,\mathrm{d}u)\,\mathrm{d}s}{1+\tau(t)}\right.$$
$$\left. - \frac{\int_{-\infty}^{t} \frac{1}{\rho(s)}\Phi^{-1}(\Phi(\int_{-\infty}^{+\infty}h(u, x_0(u), x_0'(u))\,\mathrm{d}u) + \int_{s}^{+\infty}f(u, x_0(u), x_0'(u))\,\mathrm{d}u)\,\mathrm{d}s}{1+\tau(t)}\right|.$$

Since

$$
\left| \Phi\left(\int_{-\infty}^{+\infty} h\bigl(u, x_n(u), x_n'(u)\bigr)\,\mathrm{d}u \right) + \int_{s}^{+\infty} f\bigl(u, x_n(u), x_n'(u)\bigr)\,\mathrm{d}u \right.
$$

$$
\left. - \left(\Phi\left(\int_{-\infty}^{+\infty} h\bigl(u, x_0(u), x_0'(u)\bigr)\,\mathrm{d}u \right) + \int_{s}^{+\infty} f\bigl(u, x_0(u), x_0'(u)\bigr)\,\mathrm{d}u \right) \right|
$$

$$
\leq 2\Phi\left(\int_{-\infty}^{+\infty} \phi_r(s)\,\mathrm{d}s \right) + 2\int_{-\infty}^{+\infty} \phi_r(s)\,\mathrm{d}s =: r_0 < +\infty, \quad n = 0, 1, 2, \ldots,
$$

and Φ^{-1} is uniformly continuous on $[0, r_0]$, then there exists $M > 0$ (independent of t and n) such that

$$
\left| \Phi^{-1}\left(\Phi\left(\int_{-\infty}^{+\infty} h\bigl(u, x_n(u), x_n'(u)\bigr)\,\mathrm{d}u \right) + \int_{s}^{+\infty} f\bigl(u, x_n(u), x_n'(u)\bigr)\,\mathrm{d}u \right) \right.
$$

$$
\left. - \Phi^{-1}\left(\Phi\left(\int_{-\infty}^{+\infty} h\bigl(u, x_0(u), x_0'(u)\bigr)\,\mathrm{d}u \right) + \int_{s}^{+\infty} f\bigl(u, x_0(u), x_0'(u)\bigr)\,\mathrm{d}u \right) \right| \leq M,
$$

$n = 1, 2, 3, \ldots.$

So

$$
\left| \frac{(Tx_n)(t)}{1 + \tau(t)} - \frac{(Tx_0)(t)}{1 + \tau(t)} \right|
$$

$$
\leq \frac{\int_{-\infty}^{+\infty} |g(u, x_n(u), x_n'(u)) - g(u, x_0(u), x_0'(u))|\,\mathrm{d}u}{1 + \tau(t)} + \left| \frac{\int_{-\infty}^{t} \frac{1}{\rho(s)}\,\mathrm{d}s\, M}{1 + \tau(t)} \right|
$$

$$
\leq 2\int_{-\infty}^{+\infty} \phi_r(s)\,\mathrm{d}s + M, \quad n = 1, 2, 3, \ldots, \ t \in R.
$$

Since f, g, h are Carathéodory functions, by Lebesgue dominated convergence theorem, we have

$$
\lim_{n \to +\infty} \sup_{t \in R} \left| \frac{(Tx_n)(t)}{1 + \tau(t)} - \frac{(Tx_0)(t)}{1 + \tau(t)} \right| = 0. \tag{14}
$$

Similarly we get

$$
\left| \rho(t)(Tx_n)'(t) - \rho(t)(Tx_0)'(t) \right|
$$

$$
= \left| \Phi^{-1}\left(\Phi\left(\int_{-\infty}^{+\infty} h\bigl(u, x_n(u), x_n'(u)\bigr)\,\mathrm{d}u \right) + \int_{t}^{+\infty} f\bigl(u, x_n(u), x_n'(u)\bigr)\,\mathrm{d}u \right) \right.
$$

$$- \Phi^{-1}\left(\Phi\left(\int_{-\infty}^{+\infty} h\bigl(u, x_0(u), x_0'(u)\bigr)\,\mathrm{d}u\right) + \int_{t}^{+\infty} f\bigl(u, x_0(u), x_0'(u)\bigr)\,\mathrm{d}u\right)\biggr|,$$

and so

$$\lim_{n\to+\infty} \sup_{t\in R} \bigl|\rho(t)(Tx_n)'(t) - \rho(t)(Tx_0)'(t)\bigr| = 0. \tag{15}$$

Hence (14) and (15) imply that $Tx_n \to Tx_0$ as $n \to +\infty$. Then T is continuous.

Step 2. We show that T is maps bounded subsets into bounded sets. Let $D \subseteq X$ be bounded. Then there exists $M > 0$ such that $D \subseteq \{x \in X \colon \|x\| \le M\}$. So there exists $\phi_M \in L^1(R)$ such that

$$0 \le f\bigl(t, x(t), x'(t)\bigr) = f\left(t, \bigl(1+\tau(t)\bigr)\frac{x(t)}{1+\tau(t)}, \frac{1}{\rho(t)}\rho(t)x'(t)\right) \le \phi_M(t),$$

$$0 \le g\bigl(t, x(t), x'(t)\bigr) = g\left(t, \bigl(1+\tau(t)\bigr)\frac{x(t)}{1+\tau(t)}, \frac{1}{\rho(t)}\rho(t)x'(t)\right) \le \phi_M(t),$$

$$0 \le h\bigl(t, x(t), x'(t)\bigr) = h\left(t, \bigl(1+\tau(t)\bigr)\frac{x(t)}{1+\tau(t)}, \frac{1}{\rho(t)}\rho(t)x'(t)\right) \le \phi_M(t).$$

So

$$\begin{aligned}
\frac{|(Tx)(t)|}{1+\tau(t)} \\
= \frac{\int_{-\infty}^{+\infty} g(u, x(u), x'(u))\,\mathrm{d}u}{1+\tau(t)} \\
\quad + \frac{\int_{-\infty}^{t}\frac{1}{\rho(s)}\Phi^{-1}(\Phi(\int_{-\infty}^{+\infty} h(u, x(u), x'(u))\,\mathrm{d}u) + \int_{s}^{+\infty} f(u, x(u), x'(u))\,\mathrm{d}u)\,\mathrm{d}s}{1+\tau(t)} \\
\le \int_{-\infty}^{+\infty} \phi_M(r)\,\mathrm{d}r + \frac{\int_{-\infty}^{t}\frac{1}{\rho(s)}\,\mathrm{d}s\,\Phi^{-1}(\Phi(\int_{-\infty}^{+\infty}\phi_M(r)\,\mathrm{d}r) + \int_{-\infty}^{+\infty}\phi_M(s)\,\mathrm{d}s)}{1+\tau(t)} \\
\le \int_{-\infty}^{+\infty}\phi_M(r)\,\mathrm{d}r + \Phi^{-1}\left(\Phi\left(\int_{-\infty}^{+\infty}\phi_M(r)\,\mathrm{d}r\right) + \int_{-\infty}^{+\infty}\phi_M(s)\,\mathrm{d}s\right) =: M_1.
\end{aligned}$$

Furthermore, we have

$$\rho(t)\bigl|(Tx)'(t)\bigr| = \left|\Phi^{-1}\left(\Phi\left(\int_{-\infty}^{+\infty} h\bigl(u, x(u), x'(u)\bigr)\,\mathrm{d}u\right) + \int_{t}^{\infty} f\bigl(u, x(u), x'(u)\bigr)\,\mathrm{d}u\right)\right|$$

$$\le \Phi^{-1}\left(\Phi\left(\int_{-\infty}^{\infty}\phi_M(r)\,\mathrm{d}r\right) + \int_{-\infty}^{+\infty}\phi_M(s)\,\mathrm{d}s\right) =: M_2.$$

Then

$$\|(Tx)\| = \max\left\{\sup_{t\in R}\frac{|(Tx)(t)|}{1+\tau(t)},\ \sup_{t\in R}\rho(t)\big|(Tx)'(t)\big|\right\} < \infty.$$

So $\{Tx\colon x\in D\}$ is bounded.

Step 3. We prove that both $\{Tx/(1+\tau(t))\colon x\in D\}$ and $\{\rho(t)(Tx)'\colon x\in D\}$ are equicontinuous on each subinterval of R.

Let $D\subset\{x\in X\colon \|x\|\le M\}$. For any $K>0$, $t_1,t_2\in[-K,K]$ with $t_1\le t_2$ and $x\in X$, since f,g,h are Carathéodory functions, then there exists $\phi_M\in L^1(R)$ such that

$$\big|f\big(t,x(t),x'(t)\big)\big| \le \phi_M(t), \quad t\in R,$$
$$\big|g\big(t,x(t),x'(t)\big)\big| \le \phi_M(t), \quad t\in R,$$
$$\big|h\big(t,x(t),x'(t)\big)\big| \le \phi_M(t), \quad t\in R.$$

Then

$$\left|\Phi\left(\int_{-\infty}^{+\infty} h\big(u,x(u),x'(u)\big)\,\mathrm{d}u\right) + \int_t^{+\infty} f\big(u,x(u),x'(u)\big)\,\mathrm{d}u\right|$$

$$\le \Phi\left(\int_{-\infty}^{\infty}\phi_M(r)\,\mathrm{d}r\right) + \int_{-\infty}^{+\infty}\phi_M(s)\,\mathrm{d}s =: r.$$

Since $\Phi^{-1}(s)$ is uniformly continuous on $[0,r]$, then for each $\epsilon>0$ there exists $\mu>0$ such that $|s_1-s_2|<\mu$ with $s_1,s_2\in[0,r]$ implies that $|\Phi^{-1}(s_1)-\Phi^{-1}(s_2)|<\epsilon$.

Since

$$\big|\Phi\big(\rho(t_1)(Tx)'(t_1)\big) - \Phi\big(\rho(t_2)(Tx)'(t_2)\big)\big|$$

$$= \left|\int_{t_2}^{t_1} f\big(u,x(u),x'(u)\big)\,\mathrm{d}u\right| \le \int_{t_1}^{t_2}\phi_M(r)\,\mathrm{d}r \to 0 \quad \text{uniformly as } t_1\to t_2,$$

then there exists $\sigma>0$ such that $|t_2-t_1|<\sigma$ implies that $|\Phi(\rho(t_1)(Tx)'(t_1)) - \Phi(\rho(t_2)(Tx)'(t_2))|<\mu$. Thus $|t_1-t_2|<\sigma$ implies that

$$\big|\rho(t_1)(Tx)'(t_1) - \rho(t_2)(Tx)'(t_2)\big|$$
$$= \big|\Phi^{-1}\big(\Phi\big(\rho(t_1)(Tx)'(t_1)\big)\big) - \Phi^{-1}\big(\Phi\big(\rho(t_2)(Tx)'(t_2)\big)\big)\big| < \epsilon. \tag{16}$$

Furthermore, we have

$$\left|\frac{(Tx)(t_1)}{1+\tau(t_1)} - \frac{(Tx)(t_2)}{1+\tau(t_2)}\right|$$

$$\le \left|\frac{\int_{-\infty}^{+\infty} g(u,x(u),x'(u))\,\mathrm{d}u}{1+\tau(t_1)} - \frac{\int_{-\infty}^{+\infty} g(u,x(u),x'(u))\,\mathrm{d}u}{1+\tau(t_2)}\right|$$

$$+ \left|\frac{\int_{-\infty}^{t_1}\frac{1}{\rho(s)}\Phi^{-1}(\Phi(\int_{-\infty}^{+\infty} h(u,x(u),x'(u))\,\mathrm{d}u) + \int_s^{+\infty} f(u,x(u),x'(u))\,\mathrm{d}u)\,\mathrm{d}s}{1+\tau(t_1)}\right.$$

$$- \frac{\int_{-\infty}^{t_2} \frac{1}{\rho(s)} \Phi^{-1}(\Phi(\int_{-\infty}^{+\infty} h(u, x(u), x'(u)) \, du) + \int_s^{+\infty} f(u, x(u), x'(u)) \, du) \, ds}{1 + \tau(t_2)} \Bigg|$$

$$\leq \int_{-\infty}^{+\infty} \phi_M(r) \, dr |\tau(t_1) - \tau(t_2)|$$

$$+ \frac{\int_{t_1}^{t_2} \frac{1}{\rho(s)} \Phi^{-1}(\Phi(\int_{-\infty}^{+\infty} h(u, x(u), x'(u)) \, du) + \int_s^{+\infty} f(u, x(u), x'(u)) \, du) \, ds}{1 + \tau(t_1)}$$

$$+ \int_{-\infty}^{t_2} \frac{1}{\rho(s)} \Phi^{-1} \left(\Phi \left(\int_{-\infty}^{+\infty} h(u, x(u), x'(u)) \, du \right) + \int_s^{+\infty} f(u, x(u), x'(u)) \, du \right) ds$$

$$\times \left| \frac{1}{1 + \tau(t_1)} - \frac{1}{1 + \tau(t_2)} \right|$$

$$\leq \int_{-\infty}^{+\infty} \phi_M(r) \, dr |\tau(t_1) - \tau(t_2)|$$

$$+ \int_{t_1}^{t_2} \frac{1}{\rho(s)} \, ds \Phi^{-1} \left(\Phi \left(\int_{-\infty}^{+\infty} \phi_M(r) \, dr \right) + \int_{-\infty}^{+\infty} \phi_M(r) \, dr \right)$$

$$+ \int_{-\infty}^{t_2} \frac{1}{\rho(s)} \, ds \Phi^{-1} \left(\Phi \left(\int_{-\infty}^{+\infty} \phi_M(r) \, dr \right) + \int_{-\infty}^{+\infty} \phi_M(r) \, dr \right) \frac{|\tau(t_1) - \tau(t_2)|}{[1 + \tau(t_1)][1 + \tau(t_2)]}$$

$$\leq \int_{-\infty}^{+\infty} \phi_M(r) \, dr |\tau(t_1) - \tau(t_2)|$$

$$+ \int_{t_1}^{t_2} \frac{1}{\rho(s)} \, ds \Phi^{-1} \left(\Phi \left(\int_{-\infty}^{+\infty} \phi_M(r) \, dr \right) + \int_{-\infty}^{+\infty} \phi_M(r) \, dr \right)$$

$$+ \Phi^{-1} \left(\Phi \left(\int_{-\infty}^{+\infty} \phi_M(r) \, dr \right) + \int_{-\infty}^{+\infty} \phi_M(r) \, dr \right) |\tau(t_1) - \tau(t_2)| \to 0$$

uniformly as $t_1 \to t_2$ on $[-K, K]$.

Then there exists $\sigma_2 > 0$ such that $|t_1 - t_2| < \sigma_2$ implies

$$\left| \frac{(Tx)(t_1)}{1 + \tau(t_1)} - \frac{(Tx)(t_2)}{1 + \tau(t_2)} \right| < \epsilon. \tag{17}$$

Then (16) and (17) imply both $\{Tx/(1 + \tau(t)) : x \in D\}$ and $\{\rho(t)(Tx)' : x \in D\}$ are equi-continuous on $[-K, K]$. So both $\{Tx/(1 + \tau(t)) : x \in D\}$ and $\{\rho(t)(Tx)' : x \in D\}$ are equi-continuous on each subinterval on R.

Step 4. We show that both $\{Tx/(1+\tau(t))\colon\ x\ \in\ D\}$ and $\{\rho(t)(Tx)'\colon\ x\ \in\ D\}$ are equiconvergent at $+\infty$ and $-\infty$ respectively.

Since

$$\left| \Phi\big(\rho(t)(Tx)'(t)\big) - \int\limits_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\,\mathrm{d}u \right|$$

$$= \left| \int\limits_{t}^{\infty} f\big(u, x(u), x'(u)\big)\,\mathrm{d}u \right| \le \int\limits_{t}^{\infty} \phi_M(r)\,\mathrm{d}r \to 0$$

uniformly as $t \to +\infty$, we get similarly that

$$\left| \rho(t)(Tx)'(t) - \Phi^{-1}\!\left(\int\limits_{-\infty}^{+\infty} h\big(r, x(r), x'(r)\big)\,\mathrm{d}r \right) \right| \to 0 \quad \text{uniformly as } t \to +\infty.$$

In fact, for any $\epsilon > 0$, there exists $\sigma_1 > 0$ such that $|s_1 - s_2| < \sigma_1$ implies that $|\Phi^{-1}(s_1) - \Phi^{-1}(s_2)| < \epsilon/2$. So there exists $T_{1,\epsilon} > 0$ such that $t > T_{1,\epsilon}$ implies that

$$\left| \Phi\big(\rho(t)(Tx)'(t)\big) - \int\limits_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\,\mathrm{d}u \right| < \sigma_1.$$

Hence

$$\left| \rho(t)(Tx)'(t) - \Phi^{-1}\!\left(\int\limits_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\,\mathrm{d}u \right) \right|$$

$$= \left| \Phi^{-1}\big(\Phi\big(\rho(t)(Tx)'(t)\big)\big) - \Phi^{-1}\!\left(\int\limits_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\,\mathrm{d}u \right) \right| < \frac{\epsilon}{2}, \quad t > T_{1,\epsilon}.$$

Then

$$\left| \rho(t)(Tx)'(t) - \Phi^{-1}\!\left(\int\limits_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\,\mathrm{d}u \right) \right| < \epsilon, \quad t > T_{1,\epsilon}. \qquad (18)$$

Furthermore, we have

$$\left| \frac{(Tx)(t)}{1+\tau(t)} - \int\limits_{-\infty}^{+\infty} h\big(r, x(r), x'(r)\big)\,\mathrm{d}r \right|$$

$$= \left| \frac{1}{1+\tau(t)} \int\limits_{-\infty}^{+\infty} g\big(u, x(u), x'(u)\big)\,\mathrm{d}u \right.$$

$$\left. + \frac{\int_{-\infty}^{t} \frac{1}{\rho(s)} \Phi^{-1}(\Phi(\int_{-\infty}^{+\infty} h\big(r, x(r), x'(r)\big)\,\mathrm{d}r) + \int_{s}^{+\infty} f\big(u, x(u), x'(u)\big)\,\mathrm{d}u)\,\mathrm{d}s}{1+\tau(t)} \right.$$

$$
\left. - \int\limits_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\, \mathrm{d}u \right|
$$

$$
\leq \frac{\int_{-\infty}^{+\infty} \phi_m(r)\, \mathrm{d}r}{1 + \tau(t)}
$$

$$
+ \left| \frac{\int_{-\infty}^{t} \frac{1}{\rho(s)} \Phi^{-1}(\Phi(\int_{-\infty}^{+\infty} h(u, x(u), x'(u))\, \mathrm{d}u) + \int_{s}^{+\infty} f(u, x(u), x'(u))\, \mathrm{d}u)\, \mathrm{d}s}{1 + \tau(t)} \right.
$$

$$
\left. - \frac{\int_{-\infty}^{+\infty} h(u, x(u), x'(u))\, \mathrm{d}u + \int_{-\infty}^{t} \frac{1}{\rho(s)}\, \mathrm{d}s\, \Phi^{-1}(\Phi(\int_{-\infty}^{+\infty} h(u, x(u), x'(u))\, \mathrm{d}u))}{1 + \tau(t)} \right|
$$

$$
\leq 2\frac{\int_{-\infty}^{+\infty} \phi_M(r)\, \mathrm{d}r}{1 + \tau(t)}
$$

$$
+ \left| \frac{\int_{-\infty}^{t} \frac{1}{\rho(s)} \Phi^{-1}(\Phi(\int_{-\infty}^{+\infty} h(u, x(u), x'(u))\, \mathrm{d}u) + \int_{s}^{+\infty} f(u, x(u), x'(u))\, \mathrm{d}u)\, \mathrm{d}s}{1 + \tau(t)} \right.
$$

$$
\left. - \frac{\int_{-\infty}^{t} \frac{1}{\rho(s)}\, \mathrm{d}s\, \Phi^{-1}(\Phi(\int_{-\infty}^{+\infty} h(u, x(u), x'(u))\, \mathrm{d}u))}{1 + \tau(t)} \right|.
$$

Since

$$
\left| \Phi\left(\int\limits_{-\infty}^{+\infty} h\big(r, x(r), x'(r)\big)\, \mathrm{d}r \right) + \int\limits_{t}^{+\infty} f\big(u, x(u), x'(u)\big)\, \mathrm{d}u \right|
$$

$$
\leq \Phi\left(\int\limits_{-\infty}^{+\infty} \phi_M(s)\, \mathrm{d}s \right) + \int\limits_{-\infty}^{+\infty} \phi_M(s)\, \mathrm{d}s =: r,
$$

together with

$$
\Phi\left(\int\limits_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\, \mathrm{d}u \right) + \int\limits_{t}^{+\infty} f\big(u, x(u), x'(u)\big)\, \mathrm{d}u - \Phi\left(\int\limits_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\, \mathrm{d}u \right)
$$

$$
= \int\limits_{t}^{+\infty} f\big(u, x(u), x'(u)\big)\, \mathrm{d}u \to 0
$$

uniformly as $t \to +\infty$, and Φ^{-1} is uniformly continuous on $[0, r]$, then

$$
\left| \frac{(Tx)(t)}{1 + \tau(t)} - \int\limits_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\, \mathrm{d}u \right| \to 0 \quad \text{uniformly as } t \to +\infty.
$$

Then there exists $T_{2,\epsilon} > 0$ such that

$$\left| \frac{(Tx)(t)}{1 + \tau(t)} - \int_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big) \, \mathrm{d}u \right| < \epsilon, \quad t > T_{2,\epsilon}. \tag{19}$$

So (18) and (19) imply that both $\{\rho(t)(Tx)'\colon x \in D\}$ and $\{Tx/(1 + \tau(t))\colon x \in D\}$ are equiconvergent at $+\infty$.

Similarly we can prove that both $\{Tx/(1 + \tau(t))\colon x \in D\}$ and $\{\rho(t)(Tx)'\colon x \in D\}$ are equiconvergent at $-\infty$. The details are omitted.

Therefore, the operator $T : P \to P$ is completely continuous. The proof is complete. $\square$

(iv) It is easy to see that $x \in P$ is a positive solution of BVP (6) if and only if x is a fixed point of T in P. The proof is complete.

3 Main theorems

Choose $k > 0$ sufficiently large such that $\tau(-k) = \int_{-\infty}^{-k} 1/\rho(s) \, \mathrm{d}s < 1$. Let

$$\mu = \frac{\int_{-\infty}^{-k} \frac{1}{\rho(s)} \, \mathrm{d}s}{1 + \int_{-\infty}^{k} \frac{1}{\rho(s)} \, \mathrm{d}s}.$$

Theorem 1. *Suppose that $\phi \in L^1(R)$ is nonnegative, $L_1 < L_2$ and $a < b$ are positive numbers. Denote*

$$M_0 = \inf\left\{ M > 0 \colon \frac{\int_{-k}^{k} \frac{1}{\rho(s)} \Phi^{-1}(M \int_{s}^{k} \phi(r) \, \mathrm{d}r) \, \mathrm{d}s}{1 + \tau(k)} \geq a \right\},$$

$$W_0 = \inf\left\{ W > 0 \colon \Phi^{-1}\left(W \int_{-\infty}^{+\infty} \phi(r) \, \mathrm{d}r \right) \geq L_1 \right\},$$

$$E_0 = \sup\left\{ E \colon E \int_{-\infty}^{+\infty} \phi(r) \, \mathrm{d}r + \Phi^{-1}\left(\Phi\left(E \int_{-\infty}^{+\infty} \phi(r) \, \mathrm{d}r \right) + E \int_{-\infty}^{+\infty} \phi(r) \, \mathrm{d}r \right) \leq b, \right.$$

$$\left. \Phi^{-1}\left(\Phi\left(E \int_{-\infty}^{+\infty} \phi(r) \, \mathrm{d}r \right) + E \int_{-\infty}^{+\infty} \phi(r) \, \mathrm{d}r \right) \leq L_2 \right\}.$$

If

$$f\left(t, \big(1 + \tau(t)\big)x, \frac{1}{\rho(t)}y \right) \geq M_0 \phi(t), \quad t \in [-k, k], \ \ x \in [\mu a, a], \ \ y \in [0, L_1],$$

$$f\left(t, \big(1 + \tau(t)\big)x, \frac{1}{\rho(t)}y \right) \geq W_0 \phi(t), \quad t \in R, \ \ x \in [0, a], \ \ y \in [0, L_1],$$

$$f\left(t, \big(1+\tau(t)\big)x, \frac{1}{\rho(t)}y\right) \le E_0\phi(t), \quad t \in R, \ \ x \in [0,b], \ \ y \in [0, L_2],$$

$$g\left(t, \big(1+\tau(t)\big)x, \frac{1}{\rho(t)}y\right) \le E_0\phi(t), \quad t \in R, \ \ x \in [0,b], \ \ y \in [0, L_2],$$

$$h\left(t, \big(1+\tau(t)\big)x, \frac{1}{\rho(t)}y\right) \le E_0\phi(t), \quad t \in R, \ \ x \in [0,b], \ \ y \in [0, L_2],$$

then BVP (6) *has at least one positive solution* x *satisfying*

$$a \le \sup_{t \in R} \frac{x(t)}{1 + \tau(t)} \le b, \quad 0 < \sup_{t \in R} \rho(t)x'(t) \le L_2 \tag{20}$$

or

$$0 < \sup_{t \in R} \frac{x(t)}{1 + \tau(t)} \le b, \quad L_1 \le \sup_{t \in R} \rho(t)x'(t) \le L_2. \tag{21}$$

Proof. Let X, P and the operator T be as in Section 2. By Lemma 2, $T : P \to P$ is completely continuous, and x is a positive solution of BVP (6) if and only if x is a fixed point of T in P.

Define

$$\alpha(x) = \sup_{t \in R} \frac{x(t)}{1 + \tau(t)}, \quad \beta(x) = \sup_{t \in R} \rho(t)|x'(t)|, \quad x \in X,$$
$$\Omega_1 = \big\{x \in P \colon \alpha(x) < a, \ \beta(x) < L_1\big\},$$
$$\Omega_2 = \big\{x \in P \colon \alpha(x) < b, \ \beta(x) < L_2\big\}.$$

It is easy to see that α and β are convex functionals and Ω_1 and Ω_2 are nonempty bounded open subsets of P.

Let

$$C_1 = \big\{x \in P \colon \alpha(x) = a, \ \beta(x) \le L_1\big\},$$
$$D_1 = \big\{x \in P \colon \alpha(x) \le a, \ \beta(x) = L_1\big\},$$
$$C_2 = \big\{x \in P \colon \alpha(x) = b, \ \beta(x) \le L_2\big\},$$
$$D_2 = \big\{x \in P \colon \alpha(x) \le b, \ \beta(x) = L_2\big\}.$$

Then

$$\partial\Omega_1 \subseteq C_1 \cup D_1, \quad \partial\Omega_2 \subseteq C_2 \cup D_2.$$

For $x \in C_1$, we have

$$\mu a \le \frac{x(t)}{1 + \tau(t)} \le a, \quad t \in [-k, k], \ \ 0 \le \rho(t)x'(t) \le L_1.$$

Then

$$f\left(t, \big(1+\tau(t)\big)x, \frac{1}{\rho(t)}y\right) \ge M_0\phi(t), \quad t \in [-k, k], \ \ x \in [\mu a, a], \ \ y \in [0, L_1].$$

From (7), we see that $\rho(t)(Tx)'(t) \geq 0$ for all $t \in R$. So we have $(Tx)(t) \geq 0$ for all $t \in R$. Then

$\alpha(Tx)$

$= \sup\limits_{t \in R} \left[\dfrac{\int_{-\infty}^{+\infty} g(u, x(u), x'(u))\,\mathrm{d}u}{1 + \tau(t)} \right.$

$\left. + \dfrac{\int_{-\infty}^{t} \frac{1}{\rho(s)} \Phi^{-1}(\Phi(\int_{-\infty}^{+\infty} h(u, x(u), x'(u))\,\mathrm{d}u) + \int_{s}^{+\infty} f(u, x(u), x'(u))\,\mathrm{d}u)\,\mathrm{d}s}{1 + \tau(t)} \right]$

$\geq \dfrac{\int_{-\infty}^{+\infty} g(u, x(u), x'(u))\,\mathrm{d}u}{1 + \tau(k)}$

$\quad + \dfrac{\int_{-\infty}^{k} \frac{1}{\rho(s)} \Phi^{-1}(\Phi(\int_{-\infty}^{+\infty} h(u, x(u), x'(u))\,\mathrm{d}u) + \int_{s}^{+\infty} f(u, x(u), x'(u))\,\mathrm{d}u)\,\mathrm{d}s}{1 + \tau(k)}$

$\geq \dfrac{\int_{-k}^{k} g(u, x(u), x'(u))\,\mathrm{d}u}{1 + \tau(k)}$

$\quad + \dfrac{\int_{-k}^{k} \frac{1}{\rho(s)} \Phi^{-1}(\Phi(\int_{-k}^{k} h(u, x(u), x'(u))\,\mathrm{d}u) + \int_{s}^{k} f(u, x(u), x'(u))\,\mathrm{d}u)\,\mathrm{d}s}{1 + \tau(k)}$

$\geq \dfrac{\int_{-k}^{k} \frac{1}{\rho(s)} \Phi^{-1}(\int_{s}^{k} f(u, x(u), x'(u))\,\mathrm{d}u)\,\mathrm{d}s}{1 + \tau(k)} \geq \dfrac{\int_{-k}^{k} \frac{1}{\rho(s)} \Phi^{-1}(M_0 \int_{s}^{k} \phi(r)\,\mathrm{d}r)\,\mathrm{d}s}{1 + \tau(k)} \geq a.$

For $x \in D_1$, we have

$$0 \leq \frac{x(t)}{1 + \tau(t)} \leq a, \quad 0 \leq \rho(t)x'(t) \leq L_1, \;\; t \in R.$$

Then

$$f\left(t, \left(1 + \tau(t)\right)x, \frac{1}{\rho(t)}y\right) \geq W_0 \phi(t), \quad t \in R, \;\; x \in [0, a], \;\; y \in [0, L_1].$$

From (7), we get $\rho(t)(Tx)'(t) \geq 0$ and $(Tx)(t) \geq 0$ for all $t \in R$. So

$\beta(Tx) = \sup\limits_{t \to R} \rho(t)\big|(Tx)'(t)\big|$

$\geq \sup\limits_{t \in R} \Phi^{-1}\left(\Phi\left(\int_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\,\mathrm{d}u\right) + \int_{t}^{+\infty} f\big(u, x(u), x'(u)\big)\,\mathrm{d}u\right)$

$= \Phi^{-1}\left(\int_{-\infty}^{+\infty} f\big(u, x(u), x'(u)\big)\,\mathrm{d}u\right) \geq \Phi^{-1}\left(\int_{-\infty}^{+\infty} \phi(r)W_0\,\mathrm{d}r\right) \geq L_1.$

We claim that

$$Tx \neq \lambda x \quad \text{for all } \lambda \in [0, 1) \text{ and } x \in \partial\Omega_1. \tag{22}$$

In fact, if $Tx = \lambda x$ for some $\lambda \in [0,1)$ and $x \in \partial\Omega_1$, then either $x \in C_1$ or $x \in D_1$.

If $x \in C_1$, we get from above discussion that $\alpha(Tx) \geq a$. On the other hand, we have $\alpha(Tx) = \lambda\alpha(x) < \alpha(x) = a$, a contradiction.

If $x \in D_1$, from above discussion, we have $\beta(Tx) \geq L_1$. On the other hand, we have $\beta(Tx) = \lambda\beta(x) < \beta(x) = L_1$, a contradiction too.

For $x \in C_2$, we have

$$0 \leq \frac{x(t)}{1+\tau(t)} \leq b, \quad 0 \leq \rho(t)x'(t) \leq L_2, \ t \in R.$$

Then

$$f\left(t, \left(1+\tau(t)\right)x, \frac{1}{\rho(t)}y\right) \leq E_0\phi(t), \quad t \in R, \ x \in [0,b], \ y \in [0,L_2],$$

$$g\left(t, \left(1+\tau(t)\right)x, \frac{1}{\rho(t)}y\right) \leq E_0\phi(t), \quad t \in R, \ x \in [0,b], \ y \in [0,L_2],$$

$$h\left(t, \left(1+\tau(t)\right)x, \frac{1}{\rho(t)}y\right) \leq E_0\phi(t), \quad t \in R, \ x \in [0,b], \ y \in [0,L_2].$$

So

$$\alpha(Tx)$$

$$= \sup_{t \in R} \frac{(Tx)(t)}{1+\tau(t)}$$

$$= \sup_{t \in R}\left[\frac{\int_{-\infty}^{+\infty} g(u, x(u), x'(u))\,\mathrm{d}u}{1+\tau(t)}\right.$$

$$\left.+ \frac{\int_{-\infty}^{t}\frac{1}{\rho(s)}\Phi^{-1}(\Phi(\int_{-\infty}^{+\infty}h(u,x(u),x'(u))\,\mathrm{d}u)+\int_{s}^{+\infty}f(u,x(u),x'(u))\,\mathrm{d}u)\,\mathrm{d}s}{1+\tau(t)}\right]$$

$$\leq \sup_{t \in R}\left[\int_{-\infty}^{+\infty} g(u, x(u), x'(u))\,\mathrm{d}u\right.$$

$$\left.+ \frac{\int_{-\infty}^{t}\frac{1}{\rho(s)}\,\mathrm{d}s\,\Phi^{-1}(\Phi(\int_{-\infty}^{+\infty}h(u,x(u),x'(u))\,\mathrm{d}u)+\int_{-\infty}^{+\infty}f(u,x(u),x'(u))\,\mathrm{d}u)}{1+\tau(t)}\right]$$

$$\leq \int_{-\infty}^{+\infty} g(u, x(u), x'(u))\,\mathrm{d}u$$

$$+ \Phi^{-1}\left(\Phi\left(\int_{-\infty}^{+\infty} h(u, x(u), x'(u))\,\mathrm{d}u\right) + \int_{-\infty}^{+\infty} f(u, x(u), x'(u))\,\mathrm{d}u\right)$$

$$\leq E_0\int_{-\infty}^{+\infty}\phi(r)\,\mathrm{d}r + \Phi^{-1}\left(\Phi\left(E_0\int_{-\infty}^{+\infty}\phi(r)\,\mathrm{d}r\right) + E_0\int_{-\infty}^{+\infty}\phi(r)\,\mathrm{d}r\right) \leq b.$$

For $x \in D_2$, we have

$$
\begin{aligned}
\beta(Tx) &= \sup_{t \in R} \rho(t)\big|(Tx)'(t)\big| \\
&= \sup_{t \in R} \Phi^{-1}\bigg(\Phi\bigg(\int_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\,\mathrm{d}u\bigg) + \int_{t}^{+\infty} f\big(u, x(u), x'(u)\big)\,\mathrm{d}u\bigg) \\
&\leq \Phi^{-1}\bigg(\Phi\bigg(\int_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\,\mathrm{d}u\bigg) + \int_{-\infty}^{+\infty} f\big(u, x(u), x'(u)\big)\,\mathrm{d}u\bigg) \\
&\leq \Phi^{-1}\bigg(\Phi\bigg(E_0 \int_{-\infty}^{+\infty} \phi(r)\,\mathrm{d}r\bigg) + E_0 \int_{-\infty}^{+\infty} \phi(r)\,\mathrm{d}r\bigg) \leq L_2.
\end{aligned}
$$

We claim that

$$
Tx \neq \lambda x \quad \text{for all } \lambda \in (1, +\infty) \text{ and } x \in \partial\Omega_2. \tag{23}
$$

In fact, if $Tx = \lambda x$ for some $\lambda \in (1, +\infty)$ and $x \in \partial\Omega_2$, then either $x \in C_2$ or $x \in D_2$.

If $x \in C_2$, we get from above discussion that $\alpha(Tx) \leq b$. On the other hand, we have $\alpha(Tx) = \lambda\alpha(x) > \alpha(x) = b$, a contradiction.

If $x \in D_2$, from above discussion, we have $\beta(Tx) \leq L_2$. On the other hand, we have $\beta(Tx) = \lambda\beta(x) > \beta(x) = L_2$, a contradiction too.

For $x \in \partial\Omega_1 = \Omega_1 = \{x \in P\colon \alpha(x) < a, \beta(x) < L_1\}$, we have either $\alpha(x) = a > 0$ or $\beta(x) = L_1 > 0$. Then

$$
\mu a \leq \frac{x(u)}{1 + \tau(u)} \leq a, \quad 0 \leq \rho(u)x'(u) \leq L_1.
$$

Since

$$
\begin{aligned}
(Tx)(t) &= \int_{-\infty}^{+\infty} g\big(r, x(r), x'(r)\big)\,\mathrm{d}r \\
&\quad + \int_{-\infty}^{t} \frac{1}{\rho(s)} \Phi^{-1}\bigg(\Phi\bigg(\int_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\,\mathrm{d}u\bigg) + \int_{s}^{+\infty} f\big(u, x(u), x'(u)\big)\,\mathrm{d}u\bigg)\mathrm{d}s,
\end{aligned}
$$

$$
\rho(t)(Tx)'(t) = \Phi^{-1}\bigg(\Phi\bigg(\int_{-\infty}^{+\infty} h\big(u, x(u), x'(u)\big)\,\mathrm{d}u\bigg) + \int_{t}^{+\infty} f\big(u, x(u), x'(u)\big)\,\mathrm{d}u\bigg)
$$

and

$$
\|Tx\| = \max\bigg\{\sup_{t \in R} \frac{|(Tx)(t)|}{1 + \tau(t)}, \sup_{t \in R} \rho(t)\big|(Tx)'(t)\big|\bigg\},
$$

together with

$$f\left(t, \left(1 + \tau(t)\right)x, \frac{1}{\rho(t)}y\right) \geq M_0\phi(t), \quad t \in [-k, k], \ \ x \in [\mu a, a], \ \ y \in [0, L_1],$$

we have

$$\sup_{t \in R} \rho(t)\big|(Tx)'(t)\big| \geq \Phi^{-1}\left(\int\limits_{-\infty}^{+\infty} f\big(u, x(u), x'(u)\big)\,\mathrm{d}u\right)$$

$$\geq \Phi^{-1}\left(\int\limits_{-k}^{k} f\left(u, \left(1 + \tau(u)\right)\frac{x(u)}{1 + \tau(u)}, \frac{1}{\rho(u)}\rho(u)x'(u)\right)\mathrm{d}u\right)$$

$$\geq \Phi^{-1}\left(M_0 \int\limits_{-k}^{k} \phi(u)\,\mathrm{d}u\right).$$

It is easy to see that

$$\inf\big\{\|Tx\|\colon x \in \partial\Omega_1\big\} > 0. \tag{24}$$

It follows from (22), (23), (24) and Lemma 1 that T has at least one fixed point $x \in \overline{\Omega_2} \setminus \Omega_1$. So BVP (6) has at least one positive solution x such that $x \in \overline{\Omega_2} \setminus \Omega_1$ which imply that x satisfies (20) and (21). The proof is complete. $\square$

Acknowledgments

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